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BERNSTEIN-TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS REPRESENTED BY POWER SERIES

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Let $R\in(0,+\infty]$ and $\mathbb{D}_R=\{z\in\mathbb{C}\colon |z|<R\}$. Denote by $\mathcal{A}_R$ the class of all functions f analytic in $\mathbb{D}_R$ such that $f(z)\not\equiv 0$. For any function $f\in\mathcal{A}_R$, let $M(r,f)=\max\{|f(z)|\colon |z|=r\}$ be the maximum modulus, $K(r,f)=rM(r,f')/M(r,f)$, and $\mu(r,f)=\max\{|a_n(f)|r^n\colon n\ge 0\}$ be the maximal term of the Maclaurin series of the function f, where $a_n(f)$ denotes the n-th coefficient of this series. Suppose that Φ is a continuous function on $[a,\ln R)$ such that for every $x\in\mathbb{R}$ we have $x\sigma-\Phi(\sigma)\to-\infty$ as $\sigma\uparrow\ln R$, and let $\widetilde{\Phi}(x)=\max\{x\sigma-\Phi(\sigma)\colon \sigma\in D_\Phi\}$ be the Young conjugate function of Φ, $\varphi(x)=\widetilde{\Phi}'_+(x)$ for all $x\in\mathbb{R}$, and $\Gamma(x)=(\widetilde{\Phi}(x)-\ln x)/x$ for all sufficiently large x. Put

$$\Delta=\varlimsup_{x\to+\infty}\frac{\ln x}{\Phi(\varphi(x))},\ t(f)=\varlimsup_{r\uparrow R}\frac{\ln\mu(r,f)}{\Phi(\ln r)},\ k(f)=\varlimsup_{r\uparrow R}\frac{K(r,f)}{\overline{\Phi}^{-1}(\ln r)},\ k_1(f)=\varlimsup_{r\uparrow R}\frac{K(r,f)}{\Gamma^{-1}(\ln r)},$$

where $f\in\mathcal{A}_R$. We prove the following results:

(a) for any function $f\in\mathcal{A}_R$ such that $t(f)\le 1$, the inequality $k_1(f)\le 1$ holds;

(b) for an arbitrary positive sequence (r_n) increasing to R, there exists a function $f\in\mathcal{A}_R$ such that $t(f)=1$ and $\varlimsup\limits_{n\to+\infty}K(r_n,f)/\Gamma^{-1}(\ln r_n)=1$;

(c) for any function $f\in\mathcal{A}_R$ such that $t(f)\le 1$, the inequality $k(f)\le 1+\Delta$ holds;

(d) there exists a function $f\in\mathcal{A}_R$ such that $t(f)=1$ and $k(f)=1+\Delta$.

1. Introduction. Let $R\in(0,+\infty]$ and $\mathbb{D}_R=\{z\in\mathbb{C}\colon |z|<R\}$. By $\mathcal{A}_R$ we denote the class of all functions f analytic in $\mathbb{D}_R$ such that $f(z)\not\equiv 0$. For any function $f\in\mathcal{A}_R$ and all $r\in(0,R)$, we put

$$M(r,f)=\max\{|f(z)|\colon |z|=r\},\quad K(r,f)=r\frac{M(r,f')}{M(r,f)}.$$

If P is a polynomial of degree n, then for every $r>0$ by Bernstein's classical inequality, we have $K(r,P)\le n$. An analogue of this inequality for transcendental entire functions was also obtained by S. Bernstein [1, p. 76].

Theorem A ([1])**.** *Let ρ and T be positive numbers. If $f\in\mathcal{A}_{+\infty}$ is a function of order ρ and type T, i.e.*

$$\varlimsup_{r\to+\infty}\frac{\ln M(r,f)}{Tr^\rho}=1,$$

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then

$$\varlimsup_{r\to+\infty}\frac{K(r,f)}{eT\rho r^{\rho}}\le 1. \tag{1}$$

T. Kövari [2] proved that inequality (1) is sharp.

Theorem B ([2]). *Let ρ and T be positive numbers. Then there exists a function $f\in\mathcal{A}_{+\infty}$ of order ρ and type T such that*

$$\varlimsup_{r\to+\infty}\frac{K(r,f)}{eT\rho r^{\rho}}=1.$$

Analogues of the results of S. Bernstein and T. Kövari for wide classes of analytic functions were established in [3]–[7]. In order to formulate those of the obtained results that relate directly to the classes $\mathcal{A}_R$, we introduce some notations and definitions.

Suppose that $A\in(-\infty,+\infty]$. By Ω_A we denote the class of all functions $\Phi\colon D_\Phi\to\mathbb{R}$ such that D_Φ is an interval of the form $[a,A)$, Φ is continuous on D_Φ, and the following condition

$$\forall x\in\mathbb{R}:\quad \lim_{\sigma\uparrow A}(x\sigma-\Phi(\sigma))=-\infty \tag{2}$$

holds. It is easy to see that in the case when $A<+\infty$ condition (2) is equivalent to the condition $\Phi(\sigma)\to+\infty$ as $\sigma\to A-0$, and in the case when $A=+\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma\to+\infty$ as $\sigma\to+\infty$.

Let Ω'_A be the class of all functions $\Phi\in\Omega_A$ such that Φ is a continuously differentiable function on D_Φ and Φ' is an increasing function on D_Φ.

If $\Phi\in\Omega_A$, then let $\widetilde{\Phi}$ be the Young conjugate function of Φ, i.e.

$$\widetilde{\Phi}(x)=\max\{x\sigma-\Phi(\sigma)\colon \sigma\in D_\Phi\},\quad x\in\mathbb{R}.$$

Properties of Young conjugate functions are well known. Some of these properties are given in the following lemma (see, for example, [8]).

Lemma 1. *Let $A\in(-\infty,+\infty]$, $\Phi\in\Omega_A$, and $\varphi(x)=\max\{\sigma\in D_\Phi\colon x\sigma-\Phi(\sigma)=\widetilde{\Phi}(x)\}$ for all $x\in\mathbb{R}$. Then the following statements are true:*

(i) *φ is a non-decreasing function on $\mathbb{R}$;*

(ii) *φ is a continuous from the right function on $\mathbb{R}$;*

(iii) *$\varphi(x)\to A$ as $x\to+\infty$;*

(iv) *the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at each point $x\in\mathbb{R}$;*

(v) *if $x_0=\inf\{x>0\colon \Phi(\varphi(x))>0\}$, then $\widetilde{\Phi}(x)/x$ increases to A on $(x_0,+\infty)$;*

(vi) *the function $\Phi(\varphi(x))$ is non-decreasing on $[0,+\infty)$.*

Suppose that $\Phi\in\Omega_A$, and let $\varphi(x)$ and x_0 be defined by Φ as in Lemma 1. Put

$$\Delta_\Phi=\varlimsup_{x\to+\infty}\frac{\ln x}{\Phi(\varphi(x))}.$$

For all $x>x_0$, we set $\overline{\Phi}(x)=\widetilde{\Phi}(x)/x$. By Lemma 1, the function $\overline{\Phi}$ is continuous increasing to A on $(x_0,+\infty)$. So, if $A_0=\overline{\Phi}(x_0+0)$, then the inverse function $\overline{\Phi}^{-1}$ is continuous increasing to $+\infty$ on the interval (A_0,A). We will assume that $\overline{\Phi}^{\,-1}(\sigma)=+\infty$ for all $\sigma\in[A,+\infty]$. Put

$$\beta(\sigma)=\ln\overline{\Phi}^{-1}(\sigma)/\overline{\Phi}^{-1}(\sigma),\quad \sigma\in(A_0,A). \tag{3}$$

We also put $b = \max\{x_0, e\}$ and consider the function

$$\Gamma(x) = \overline{\Phi}(x) - \frac{\ln x}{x}, \quad x \in (b, +\infty). \tag{4}$$

This function is continuous increasing to A on $(b, +\infty)$. Therefore, if $B = \Gamma(b-0)$, then the inverse function Γ^{-1} is continuous increasing to $+\infty$ on (B, A). It is easy to see that $\overline{\Phi}^{-1}(\sigma + \beta(\sigma)) > \Gamma^{-1}(\sigma) > \overline{\Phi}^{-1}(\sigma)$ for all $\sigma < A$ sufficiently close to A.

Note also that if $\Phi \in \Omega'_A$, $D_\Phi = [a, A)$ and $c = \Phi'(a)$, then the function $\sigma = \varphi(x)$, $x \in [c, +\infty)$, is the inverse of the function $x = \Phi'(\sigma)$. In addition, in this case

$$\Delta_\Phi = \varlimsup_{\sigma \uparrow A} \frac{\ln \Phi'(\sigma)}{\Phi(\sigma)}.$$

Let $R \in (0, +\infty]$ and $f \in \mathcal{A}_R$. We expand the function f into a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{5}$$

and put

$$\mu(r, f) = \max\{|a_n| r^n \colon n \in \mathbb{N}_0\}, \quad G(r, f) = \sum_{n=0}^{\infty} |a_n| r^n$$

for all $r \in [0, R)$, where $\mathbb{N}_0$ denotes the set of all non-negative integers.

If $\Phi \in \Omega_{\ln R}$, and $f \in \mathcal{A}_R$ is a function of the form (5), then let

$$t_\Phi(f) = \varlimsup_{r \uparrow R} \frac{\ln \mu(r, f)}{\Phi(\ln r)}, \quad T_\Phi(f) = \varlimsup_{r \uparrow R} \frac{\ln M(r, f)}{\Phi(\ln r)}, \quad \mathcal{T}_\Phi(f) = \varlimsup_{r \uparrow R} \frac{\ln G(r, f)}{\Phi(\ln r)};$$

$$k_\Phi(f) = \varlimsup_{r \uparrow R} \frac{K(r, f)}{\overline{\Phi}^{-1}(\ln r)}, \quad k_{1,\Phi}(f) = \varlimsup_{r \uparrow R} \frac{K(r, f)}{\Gamma^{-1}(\ln r)}, \quad k_{2,\Phi}(f) = \varlimsup_{r \uparrow R} \frac{K(r, f)}{\overline{\Phi}^{-1}(\ln r + \beta(\ln r))}.$$

It is clear that $t_\Phi(f) \le T_\Phi(f) \le \mathcal{T}_\Phi(f)$ and $k_{2,\Phi}(f) \le k_{1,\Phi}(f) \le k_\Phi(f)$.

Let $\Phi \in \Omega_{\ln R}$. It is proved in [8] that for any function $f \in \mathcal{A}_R$ of the form (5) we have

$$t_\Phi(f) = \varlimsup_{n \to \infty} \frac{n}{\overline{\Phi}^{\,-1}\left(\frac{1}{n} \ln \frac{1}{|a_n|}\right)}. \tag{6}$$

In addition, in [8] (see also [9, 10]) it is shown that the condition $\Delta_\Phi = 0$ is necessary and sufficient for $\mathcal{T}_\Phi(f) = t_\Phi(f)$ holds for any function $f \in \mathcal{A}_R$ of the form (5).

The following result of M. M. Sheremeta [3] is a generalization of Theorem A to the case of entire functions of arbitrary growth.

Theorem C ([3])**.** *Let $\Phi \in \Omega'_{+\infty}$. Then for any function $f \in \mathcal{A}_{+\infty}$ such that $T_\Phi(f) = 1$, we have $k_{2,\Phi}(f) \le 1$.*

A detailed analysis of the proof of Theorem C, which was proposed by M. M. Sheremeta in [3], shows that for functions analytic in a disk, we can prove a complete analogue of Theorem C (see also [4]). At the same time, in order to establish the inequality $k_{2,\Phi}(f) \le 1$ it is enough to require the fulfillment of the condition $t_\Phi(f) \le 1$ instead of the stronger condition $T_\Phi(f) = 1$. In other words, the following result is true.

Theorem D. *Let $R \in (0, +\infty]$ and $\Phi' \in \Omega_{\ln R}$. Then for any function $f \in \mathcal{A}_R$ of the form* (5) *such that $t_\Phi(f) \le 1$, we have $k_{2,\Phi}(f) \le 1$.*

We note that if $R < +\infty$, then for functions Φ from the class $\Omega_{\ln R}$ we can have a situation in which $\ln r+\beta(\ln r) \geq \ln R$ for all $r \in [r_1, R)$ with some $r_1 \in (0, R)$ (for example, if $\Phi(\ln r) \leq -1 - \ln\ln(R/r)$ for all $r \in [r_2, R)$ with some $r_2 \in (0, R)$). Then $\overline{\Phi}^{-1}(\ln r + \beta(\ln r)) = +\infty$ for all $r \in [r_1, R)$. Therefore, if $f \in \mathcal{A}_R$, then Theorem D does not give any information about the behavior of the quantity $K(r, f)$ as $r \uparrow R$. This gap is filled to some extent by the following two theorems.

Theorem E ([5])**.** *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. Then for any function $f \in \mathcal{A}_R$ of the form* (5) *such that $\mathcal{T}_\Phi(f) \leq 1$, we have $k_\Phi(f) \leq 1$.*

Theorem F ([5])**.** *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. Then there exists a function $f \in \mathcal{A}_R$ of the form* (5) *such that $\mathcal{T}_\Phi(f) = T_\Phi(f) = t_\Phi(f) = 1$ and $k_\Phi(f) = 1$.*

As we mentioned above, in the case when $\Delta_\Phi = 0$ we have $\mathcal{T}_\Phi(f) = T_\Phi(f) = t_\Phi(f)$. So, in this case, the condition $\mathcal{T}_\Phi(f) \leq 1$ in Theorem E can be replaced by each of the conditions $T_\Phi(f) \leq 1$ or $t_\Phi(f) \leq 1$.

Note that using results from [11] (see also [12, 13]), it is easy to prove the existence of functions $\Phi \in \Omega_{+\infty}$ and $f \in \mathcal{A}_{+\infty}$ such that $T_\Phi(f) = 1$, but $\mathcal{T}_\Phi(f) = +\infty$. However, despite this, we have the following result.

Theorem G ([7])**.** *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. Then for any function $f \in \mathcal{A}_R$ of the form* (5) *such that $T_\Phi(f) \leq 1$, we have $k_\Phi(f) \leq C_1$, where $C_1 < 1.1276$ is an absolute constant.*

In connection with the formulated results, the following question arises: *does there exist an absolute constant $C_2 > 1$ such that for any functions $\Phi \in \Omega_{\ln R}$ and $f \in \mathcal{A}_R$ with $t_\Phi(f) \leq 1$, we have $k_\Phi(f) \leq C_2$?* A negative answer to this question follows from the following two theorems (see Corollary 2 below), which give a sharp estimate from above on the behavior of the quantity $K(r, f)$ as $r \uparrow R$ under the condition $t_\Phi(f) \leq 1$.

Theorem 1. *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. Then for any function $f \in \mathcal{A}_R$ of the form* (5) *such that $t_\Phi(f) \leq 1$, we have $k_{1,\Phi}(f) \leq 1$.*

Theorem 2. *Let $R \in (0, +\infty]$, $\Phi \in \Omega_{\ln R}$, and let $(r_n)_{n\in\mathbb{N}_0}$ be a positive sequence increasing to R. Then there exists a function $f \in \mathcal{A}_R$ of the form* (5) *such that $t_\Phi(f) = 1$ and*

$$\varlimsup_{n\to+\infty} \frac{K(r_n, f)}{\Gamma^{-1}(\ln r_n)} = 1. \tag{7}$$

2. Auxiliary results. In this section, we give some lemmas that we will need to prove Theorems 1 and 2, as well as their consequences.

Lemma 2 ([8])**.** *Let $R \in (0, +\infty]$, $\Phi \in \Omega_{\ln R}$, and let $f \in \mathcal{A}_R$ be a function of the form* (5)*. Then the following statements are equivalent:*

(i) there exists $r_0 \in (0, R)$ such that $\ln \mu(r, f) \leq \Phi(\ln r)$ for all $r \in [r_0, R)$;

(ii) there exists $n_0 \in \mathbb{N}_0$ such that $\ln |a_n| \leq -\widetilde{\Phi}(n)$ for all integers $n \geq n_0$.

The following lemma is actually proved in [6] (see the proof of Theorem 2 in [6]).

Lemma 3 ([6])**.** *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and let Γ be the function defined by* (4)*. Then there exist a function $\Theta \in \Omega_A$ and a number $y_0 > 0$ such that $\overline{\Theta}(y) = \Gamma(y)$ for all $y \geq y_0$.*

In the following two lemmas, we assume that $\varphi(x)$ and x_0 are defined by a given function $\Phi \in \Omega_A$ as in Lemma 1.

Lemma 4 ([14]). *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0 + 0)$, and $y(\sigma) = \varphi(\overline{\Phi}^{\,-1}(\sigma))$ for all $\sigma \in (\sigma_0, A)$. Then for every $\delta \in (0, 1)$ and each $\sigma \in (\sigma_0, A)$ we have*

$$\overline{\Phi}^{-1}\left(\sigma + \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}\right) \le \frac{\overline{\Phi}^{-1}(\sigma)}{1-\delta}.$$

Lemma 5. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $c \in \mathbb{R}$ be a fixed number, and let $h(c) = \varphi(c)$ and*

$$h(x) = \frac{\widetilde{\Phi}(x) - \widetilde{\Phi}(c)}{x - c}, \quad x > c.$$

Then the function h has the following properties:

(i) h is continuous non-decreasing on $[c, +\infty)$ and $h(x) \le \varphi(x)$ for all $x \ge c$;

(ii) if $d = \inf\{x > c\colon \varphi(x) > \varphi(c)\}$, then h is increasing on $[d, +\infty)$;

(iii) if $c > x_0$, then $\overline{\Phi}(x) < h(x)$ for all $x > c$;

(iv) $\overline{\Phi}^{-1}(h(x)) \sim x$ at $x \to +\infty$.

Proof. By Lemma 1 the function $\widetilde{\Phi}$ is convex on $\mathbb{R}$ and we have $\varphi(x) = \widetilde{\Phi}'_+(x)$ for all $x \in \mathbb{R}$. Taking into account this fact, it is easy to justify (i) and (ii).

Let $c > x_0$ and $x > c$. Since

$$h(x) = \overline{\Phi}(x) + \frac{c(\overline{\Phi}(x) - \overline{\Phi}(c))}{x - c}, \tag{8}$$

we obtain $\overline{\Phi}(x) < h(x)$. Next, we note that

$$\overline{\Phi}(x) - \overline{\Phi}(c) = o(\Phi(\varphi(x))), \quad x \to +\infty. \tag{9}$$

In fact, since by Lemma 1 we have $\overline{\Phi}(x) \to A$ and $\Phi(\varphi(x)) \to +\infty$ for $x \to +\infty$, relation (9) is obvious in the case $A < +\infty$. If $A = +\infty$, then we get $\overline{\Phi}(x) \le \varphi(x) = o(\Phi(\varphi(x)))$ as $x \to +\infty$, and this aslo implies (9).

Taking into account (8) and (9), as well as using the notation $\sigma = \overline{\Phi}(x)$, we obtain

$$h(x) = \overline{\Phi}(x) + o\left(\frac{\Phi(\varphi(x))}{x}\right) = \sigma + o\left(\frac{\Phi(\varphi(\overline{\Phi}^{-1}(\sigma)))}{\overline{\Phi}^{-1}(\sigma)}\right), \quad x \to +\infty.$$

Then by Lemma 4 we have $\overline{\Phi}^{-1}(h(x)) \sim \overline{\Phi}^{-1}(\sigma) = x$ as $x \to +\infty$. □

Lemma 6. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and let Γ be the function defined by (4). Then*

$$\varlimsup_{\sigma \uparrow A} \frac{\Gamma^{-1}(\sigma)}{\overline{\Phi}^{-1}(\sigma)} = 1 + \Delta_\Phi. \tag{10}$$

Proof. Denote the left part in (10) by Δ_0 and first of all prove the inequality $\Delta_0 \le 1 + \Delta_\Phi$. Suppose the contrary, that is, there exist a number $p > \Delta_\Phi$ and a sequence $(\sigma_n)_{n \in \mathbb{N}_0}$ increasing to A such that $\Gamma^{-1}(\sigma_n) \ge (1 + p)\overline{\Phi}^{-1}(\sigma_n)$ for all $n \in \mathbb{N}_0$. Fixing some $q \in (\Delta_\Phi, p)$, for all $n \ge n_0$ we obtain

$$\frac{\ln \Gamma^{-1}(\sigma_n)}{\Gamma^{-1}(\sigma_n)} \le \frac{1}{1+q} \cdot \frac{\ln \overline{\Phi}^{-1}(\sigma_n)}{\overline{\Phi}^{-1}(\sigma_n)} =$$

$$= \frac{1}{1+q} \cdot \frac{\ln \overline{\Phi}^{-1}(\sigma_n)}{\Phi(\varphi(\overline{\Phi}^{-1}(\sigma_n)))} \cdot \frac{\Phi(\varphi(\overline{\Phi}^{-1}(\sigma_n)))}{\overline{\Phi}^{-1}(\sigma_n)} \le \frac{q}{1+q} \cdot \frac{\Phi(\varphi(\overline{\Phi}^{-1}(\sigma_n)))}{\overline{\Phi}^{-1}(\sigma_n)}.$$

Therefore, using Lemma 4 with $\delta = q/(1+q)$, for all $n \ge n_0$ we have

$$\Gamma^{-1}(\sigma_n) = \overline{\Phi}^{-1}\Big(\sigma_n + \frac{\ln \Gamma^{-1}(\sigma_n)}{\Gamma^{-1}(\sigma_n)}\Big) \le$$

$$\le \overline{\Phi}^{-1}\Big(\sigma_n + \frac{\delta\Phi(\varphi(\overline{\Phi}^{-1}(\sigma_n)))}{\overline{\Phi}^{-1}(\sigma_n)}\Big) \le \frac{\overline{\Phi}^{-1}(\sigma_n)}{1-\delta} = (1+q)\overline{\Phi}^{-1}(\sigma_n),$$

but this contradicts the inequality $\Gamma^{-1}(\sigma_n) \ge (1+p)\overline{\Phi}^{-1}(\sigma_n)$.

Now we prove that $\Delta_0 \ge 1 + \Delta_\Phi$. Suppose the contrary again. Then there exist numbers $\delta \in (0, \Delta_\Phi)$ and $b < A$ such that for all $\sigma \in (b, A)$ we have $\Gamma^{-1}(\sigma) \le (1+\delta)\overline{\Phi}^{-1}(\sigma)$. Setting $\sigma = \Gamma(x)$, for all sufficiently large x we obtain $x \le (1+\delta)\overline{\Phi}^{-1}(\Gamma(x))$, and therefore

$$\overline{\Phi}\left(\frac{x}{1+\delta}\right) \le \Gamma(x) = \overline{\Phi}(x) - \frac{\ln x}{x}.$$

Since, as it is easy to see, $\overline{\Phi}'_+(x) = \Phi(\varphi(x))/x^2$ for all $x > x_0$, and by Lemma 1 the function $\alpha(x) = \Phi(\varphi(x))$ is non-decreasing on $[0, +\infty)$, for all sufficiently large x we obtain

$$\frac{\ln x}{x} \le \overline{\Phi}(x) - \overline{\Phi}\left(\frac{x}{1+\delta}\right) = \int_{x/(1+\delta)}^{x} \frac{\Phi(\varphi(t))}{t^2} dt \le \Phi(\varphi(x)) \int_{x/(1+\delta)}^{x} \frac{dt}{t^2} = \frac{\delta}{x}\Phi(\varphi(x)).$$

This implies that $\Delta_\Phi \le \delta$, but it is impossible because $\delta \in (0, \Delta_\Phi)$. □

Lemma 7. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and let β be the function defined by* (3)*. Then*

$$\varlimsup_{\sigma\uparrow A} \frac{\overline{\Phi}^{-1}(\sigma+\beta(\sigma))}{\overline{\Phi}^{-1}(\sigma)} = \begin{cases} +\infty, & \text{if } \Delta_\Phi \ge 1; \\ 1/(1-\Delta_\Phi), & \text{if } \Delta_\Phi < 1. \end{cases} \tag{11}$$

Proof. We denote the left-hand and right-hand side parts in (11) by Δ_0 and δ_0, respectively, and first of all we prove the inequality $\Delta_0 \le \delta_0$. We assume that $\Delta_\Phi < 1$, since otherwise this inequality is trivial. Let $\Delta \in (\Delta_\Phi, 1)$ be a fixed number. The definition of Δ_Φ implies the existence of a number $\sigma_1 < A$ such that

$$\ln \overline{\Phi}^{-1}(\sigma) \le \Delta\Phi(\varphi(\overline{\Phi}^{-1}(\sigma))), \quad \sigma \in [\sigma_1, A).$$

Therefore, using Lemma 4, for some $\sigma_2 < A$ and all $\sigma \in [\sigma_2, A)$ we obtain

$$\overline{\Phi}^{-1}(\sigma+\beta(\sigma)) = \overline{\Phi}^{-1}\left(\sigma + \frac{\ln \overline{\Phi}^{-1}(\sigma)}{\overline{\Phi}^{-1}(\sigma)}\right) \le \overline{\Phi}^{-1}\left(\sigma + \Delta\frac{\Phi(\varphi(\overline{\Phi}^{-1}(\sigma)))}{\overline{\Phi}^{-1}(\sigma)}\right) \le \frac{\overline{\Phi}^{-1}(\sigma)}{1-\Delta}.$$

Since $\Delta \in (\Delta_\Phi, 1)$ is arbitrary, this implies the inequality $\Delta_0 \le \delta_0$.

Now we prove that $\Delta_0 \ge \delta_0$. Assume that $\Delta_0 < \delta_0$. Then, fixing an arbitrary $\delta \in (\Delta_0, \delta_0)$, for some $\sigma_3 < A$ we have

$$\overline{\Phi}^{-1}(\sigma+\beta(\sigma)) \le \delta\overline{\Phi}^{-1}(\sigma), \quad \sigma \in [\sigma_3, A). \tag{12}$$

Put $\Delta = (\delta-1)/\delta$. Note that then $\Delta < \Delta_\Phi$. In addition, for some $\sigma_4 < A$ and all $\sigma \in [\sigma_4, A)$ by (12) and Lemma 4 we obtain

$$\frac{\ln \overline{\Phi}^{-1}(\sigma)}{\overline{\Phi}^{-1}(\sigma)} = \beta(\sigma) \le \overline{\Phi}(\delta\overline{\Phi}^{-1}(\sigma)) - \sigma = \overline{\Phi}(\delta\overline{\Phi}^{-1}(\sigma)) - \overline{\Phi}(\overline{\Phi}^{-1}(\sigma)) = \int_{\overline{\Phi}^{-1}(\sigma)}^{\delta\overline{\Phi}^{-1}(\sigma)} \frac{\Phi(\varphi(x))}{x^2} dx \le$$

$$\leq \Phi(\varphi(\delta\overline{\Phi}^{-1}(\sigma)))\int_{\overline{\Phi}^{-1}(\sigma)}^{\delta\overline{\Phi}^{-1}(\sigma)}\frac{dx}{x^2} = \Phi(\varphi(\delta\overline{\Phi}^{-1}(\sigma)))\frac{\Delta}{\overline{\Phi}^{-1}(\sigma)}.$$

This implies that $\ln\overline{\Phi}^{-1}(\sigma) \leq \Delta\Phi(\varphi(\delta\overline{\Phi}^{-1}(\sigma)))$ for all $\sigma \in [\sigma_4, A)$. Then, as it is easy to see, $\Delta_\Phi \leq \Delta$, but this contradicts the inequality $\Delta < \Delta_\Phi$. □

For a function $f \in \mathcal{A}_R$ of the form (5) and for each $r \in (0, R)$, let $\nu(r, f)$ denote the central index of this function, i.e. $\nu(r, f) = \max\{n \in \mathbb{N}_0 \colon |a_n|r^n = \mu(r, f)\}$. As it is known, $\nu(r, f) = r(\ln\mu(r, f))'_+$ for all $r \in (0, R)$. In addition, we have the following lemma (see [15]).

Lemma 8. *Let $R \in (0, +\infty]$, $(n_k)_{k\in\mathbb{N}_0}$ be an increasing sequence of non-negative integers, and $(c_k)_{k\in\mathbb{N}_0}$ be a positive sequence increasing to R. If a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ is such that $a_{n_0} \neq 0$, $a_n = 0$ for each integer $n < n_0$, and for all $k \in \mathbb{N}_0$ the following relations*

$$|a_{n_{k+1}}| = |a_{n_0}|\prod_{j=0}^{k} c_j^{n_j - n_{j+1}}, \qquad |a_n| \leq |a_{n_k}|c_k^{n_k - n} \quad (n \in (n_k, n_{k+1}))$$

hold, then the function f given by (5) belongs to the class $\mathcal{A}_R$ and we have:

(i) $\nu(r, f) = n_0$ for all $r \in (0, c_0)$;

(ii) $\nu(r, f) = n_{k+1}$ for all $r \in [c_k, c_{k+1})$ and $k \in \mathbb{N}_0$.

3. Proof of Theorems.

Proof of Theorem 1. Let $R \in (0, +\infty]$, $\Phi \in \Omega_{\ln R}$, and let $f \in \mathcal{A}_R$ be a function of the form (5) such that $t_\Phi(f) \leq 1$. Let also Γ be the function defined above by (4).

We fix an arbitrary $p > 1$ and choose some $q \in (1, p)$. Since $t_\Phi(f) \leq 1$, for some $r_1 \in (0, R)$ and all $r \in [r_1, R)$ we have $\ln\mu(r, f) \leq q\Phi(\ln r)$. Applying Lemma 2 with $q\Phi$ instead of Φ, for some $n_1 \in \mathbb{N}_0$ and all integers $n \geq n_1$ we obtain $\ln|a_n| \leq -q\widetilde{\Phi}(n/q)$. Therefore, for some integer $n_2 \geq n_1$ we have

$$\ln(n|a_n|) \leq \ln n - q\widetilde{\Phi}(n/q) \leq q(\ln(n/q) - \widetilde{\Phi}(n/q)) = -n\Gamma(n/q), \quad n \geq n_2. \tag{13}$$

According to Lemma 3, there exist a function $\Theta \in \Omega_{\ln R}$ and a number $r_2 \in (0, R)$ such that for all $r \in [r_2, R)$ we have $\overline{\Theta}^{\,-1}(\ln r) = \Gamma^{-1}(\ln r) > 0$. Put $\theta(x) = \widetilde{\Theta}'_+(x)$ for all $x \in \mathbb{R}$, and let $\gamma(r) = \Theta(\theta(\overline{\Theta}^{\,-1}(\ln r)))$ and $n_0 = n_0(r) = [p\Gamma^{-1}(\ln r)] + 1$ for all $r \in [r_2, R)$. Setting $\delta = (p - q)/p$ and using Lemma 4 with Θ instead of Φ, for all $r \in [r_2, R)$ we get

$$\Gamma^{-1}\Big(\ln r + \frac{\delta\gamma(r)}{\overline{\Gamma}^{\,-1}(\ln r)}\Big) = \overline{\Theta}^{\,-1}\Big(\ln r + \frac{\delta\Theta(\theta(\overline{\Theta}^{\,-1}(\ln r)))}{\overline{\Theta}^{\,-1}(\ln r)}\Big) \leq \frac{\overline{\Theta}^{\,-1}(\ln r)}{1 - \delta} = \frac{p}{q}\Gamma^{-1}(\ln r) \leq \frac{n_0}{q}.$$

This implies that

$$\Gamma(n_0/q) - \ln r \geq \delta\gamma(r)/\overline{\Gamma}^{\,-1}(\ln r), \quad r \in [r_2, R). \tag{14}$$

We choose a point $r_3 \in [r_2, R)$ such that the inequalities $n_0(r_3) \geq n_2$ and $\gamma(r_3) > 0$ are satisfied. Using (13) and (14), for each fixed $r \in [r_3, R)$ we obtain

$$\sum_{n>n_0} n|a_n|r^n \leq \sum_{n>n_0}\frac{1}{e^{n(\Gamma(n/q)-\ln r)}} \leq \sum_{n>n_0}\frac{1}{e^{n(\Gamma(n_0/q)-\ln r)}} = \sum_{n>n_0}\frac{1}{e^{n\delta\gamma(r)/\overline{\Gamma}^{\,-1}(\ln r)}} =$$
$$= \frac{1}{e^{n_0\delta\gamma(r)/\overline{\Gamma}^{\,-1}(\ln r)}\big(e^{\delta\gamma(r)/\overline{\Gamma}^{\,-1}(\ln r)} - 1\big)} \leq \frac{\overline{\Gamma}^{\,-1}(\ln r)}{e^{p\delta\gamma(r)}\delta\gamma(r)}. \tag{15}$$

Assuming that $r \in [r_3, R)$ is fixed, we set $P(z) = \sum_{n\le n_0} a_n z^n$ for all $z \in \mathbb{C}$. Using Berstein's inequality for polynomials and estimate (15), we obtain

$$rM(r,f') \le rM(r,P') + \sum_{n>n_0} n|a_n|r^n \le n_0 M(r,P) + \sum_{n>n_0} n|a_n|r^n \le$$
$$\le n_0\Big(M(r,f) + \sum_{n>n_0} |a_n|r^n\Big) + \sum_{n>n_0} n|a_n|r^n \le n_0 M(r,f) + 2\sum_{n>n_0} n|a_n|r^n \le$$
$$\le ([p\Gamma^{-1}(\ln r)]+1)M(r,f) + \frac{2\overline{\Gamma}^{\,-1}(\ln r)}{e^{p\delta\gamma(r)}\delta\gamma(r)}.$$

Since $\gamma(r) \to +\infty$ as $r \uparrow R$, this shows that $k_{1,\Phi}(f) \le p$. Finally, since $p > 1$ is arbitrary, we have $k_{1,\Phi}(f) \le 1$. □

Proof of Theorem 2. Let $R \in (0, +\infty]$, $\Phi \in \Omega_{\ln R}$, Γ be the function defined above by (4), and let $(r_n)_{n\in\mathbb{N}_0}$ be a positive sequence increasing to R.

By Lemma 3 there exists a function $\Theta \in \Omega_{\ln R}$ such that $\overline{\Theta}(x) = \Gamma(x)$ for all sufficiently large $x \in \mathbb{R}$. Put $\theta(x) = \widetilde{\Theta}'_+(x)$ for all $x \in \mathbb{R}$ and let $x_0 = \inf\{x > 0\colon \Theta(\theta(x)) > 0\}$. For every fixed $a \in \mathbb{R}$, we set $h_a(a) = \theta(a)$ and let $h_a(x) = \frac{\widetilde{\Theta}(x)-\widetilde{\Theta}(a)}{x-a}$, $x > a$. Also we put $d(a) = \inf\{x > a\colon \theta(x) > \theta(a)\}$. According to Lemma 5, the restriction of the function h_a to the interval $[d(a), +\infty)$ is a continuous increasing to $+\infty$ function. We denote the inverse function to this restriction by h_a^{-1}.

Let $(\delta_k)_{k\in\mathbb{N}_0}$ be a fixed sequence of points in the interval $(1, 2)$ decreasing to 1. Using Lemma 5, it is easy to justify the existence of a subsequence $(\rho_k)_{k\in\mathbb{N}_0}$ of the sequence $(r_n)_{n\in\mathbb{N}_0}$ and an increasing sequence $(n_k)_{k\in\mathbb{N}_0}$ with $n_0 = [x_0] + 1$ such that for all $k \in \mathbb{N}_0$ we have:

(a) $d(n_k) \le \ln\rho_k$;

(b) $n_{k+1} = \big[h_{n_k}^{-1}(\ln\rho_k)\big]$;

(c) $h_{n_k}(x) \le \overline{\Theta}(\delta_k x)$ if $x \ge n_{k+1}$;

(d) $(k+1)^2 n_k \le \exp\left(\Theta(\theta(2\,n_{k+1}))/8\right)$;

(e) $n_{k+1} \ge 4n_k$.

For every $k \in \mathbb{N}_0$, we set $c_k = \exp(h_{n_k}(n_{k+1}))$. Since

$$\overline{\Theta}(n_{k+1}) < h_{n_k}(n_{k+1}) \le \overline{\Theta}(\delta_k n_{k+1}) < \overline{\Theta}(2n_{k+1}) < \overline{\Theta}(n_{k+2}), \quad k \in \mathbb{N}_0, \tag{16}$$

and by Lemma 1 we have $\overline{\Theta}(x) \to \ln R$ as $x \to +\infty$, the sequence $(c_k)_{k\in\mathbb{N}_0}$ is positive and increasing to R. In addition, since $\overline{\Theta}'_+(x) = \Theta(\theta(x))/x^2$ for all $x \in \mathbb{R}$ and the function $\Theta(\theta(x))$ is nondecreasing on $[0, +\infty)$, for every $k \in \mathbb{N}_0$ by (16) and (e) we obtain

$$\ln c_{k+1} - \ln c_k > \overline{\Theta}(n_{k+2}) - \overline{\Theta}(2n_{k+1}) = \int_{2n_{k+1}}^{n_{k+2}} \frac{\Theta(\theta(x))}{x^2}dx \ge \Theta(\theta(2n_{k+1}))\int_{2n_{k+1}}^{n_{k+2}} \frac{dx}{x^2} =$$
$$= \Theta(\theta(2n_{k+1}))\left(\frac{1}{2n_{k+1}} - \frac{1}{n_{k+2}}\right) \ge \frac{1}{4n_{k+1}}\Theta(\theta(2n_{k+1})). \tag{17}$$

Let $b_{n_k} = \exp(-\widetilde{\Theta}(n_k))$ for all $k \in \mathbb{N}_0$. Since

$$\ln c_k = h_{n_k}(n_{k+1}) = \frac{\widetilde{\Theta}(n_{k+1}) - \widetilde{\Theta}(n_k)}{n_{k+1} - n_k} = \frac{\ln b_{n_k} - \ln b_{n_{k+1}}}{n_{k+1} - n_k},$$

we have $b_{n_{k+1}} = b_{n_k} c_k^{n_k - n_{k+1}}$, and therefore, as it is easy to see,

$$b_{n_{k+1}} = b_{n_0} \prod_{j=0}^{k} c_j^{n_j - n_{j+1}}, \quad k \in \mathbb{N}_0.$$

Consider the power series $g_0(z) = \sum_{k=0}^{\infty} b_{n_k} z^{n_k}$. According to Lemma 8, the function g_0 belongs to the class $\mathcal{A}_R$, and $\mu(r, g_0) = b_{n_{k+1}} r^{n_{k+1}}$ for all $r \in [c_k, c_{k+1}]$ and $k \in \mathbb{N}_0$. In addition, by Lemma 2 there exists $s_0 \in (0, R)$ such that $\ln \mu(r, g_0) \le \Theta(\ln r)$ for all $r \in [s_0, R)$.

Let $n \in \mathbb{N}_0$. We put $b_n = 0$ if $n < n_0$, and let $b_n = b_{n_k} c_k^{n_k - n}$ if $n \in (n_k, n_{k+1})$ for some $k \in \mathbb{N}_0$. Consider the series $g(z) = \sum_{n=0}^{\infty} b_n z^n$. According to Lemma 8, the function g belongs to the class $\mathcal{A}_R$, and $\mu(r, g) = \mu(r, g_0)$ for all $r \in [0, R)$. Therefore, $\ln \mu(r, g) \le \Theta(\ln r)$ for all $r \in [s_0, R)$. Then, by Lemma 2, there exists $p_0 \in \mathbb{N}_0$ such that $\ln b_n \le -\widetilde{\Theta}(n)$ for all integers $n \ge p_0$.

Further, we note that if $n, k \in \mathbb{N}_0$, $k \ge 1$ and $n \le n_k$, then

$$b_n c_k^n = b_n c_{k-1}^n \left(\frac{c_k}{c_{k-1}}\right)^n \le \mu(c_{k-1}, g) \left(\frac{c_k}{c_{k-1}}\right)^n = b_{n_k} c_{k-1}^{n_k} \left(\frac{c_k}{c_{k-1}}\right)^n =$$
$$= b_{n_k} c_k^{n_k} \left(\frac{c_{k-1}}{c_k}\right)^{n_k - n} = \mu(c_k, g) \left(\frac{c_{k-1}}{c_k}\right)^{n_k - n}.$$

Therefore, using (17), (e) and (d), we obtain

$$\sum_{n \le n_{k-1}} b_n c_k^n \le \mu(c_k, g) \sum_{n \le n_{k-1}} \left(\frac{c_{k-1}}{c_k}\right)^{n_k - n} \le \mu(c_k, g) n_{k-1} \left(\frac{c_{k-1}}{c_k}\right)^{n_k - n_{k-1}} \le$$
$$\le \mu(c_k, g) \frac{n_{k-1}}{e^{\Theta(\theta(2n_k))/8}} \le \frac{\mu(c_k, g)}{(k+1)^2}. \tag{18}$$

Put $m_k = n_{k+1} - [\frac{n_{k+1}}{k+2}]$, $k \in \mathbb{N}_0$. Using (e), we see that $n_k < m_k < n_{k+1}$ ($\forall\, k \in \mathbb{N}_0$).

Let $n \in \mathbb{N}_0$. We put $a_n = b_n / n$ if $n \in (m_{2p}, n_{2p+1}]$ for some $p \in \mathbb{N}_0$, and let $a_n = 0$ otherwise. Consider series (5) and note that it can be written in the form

$$f(z) = \sum_{p=0}^{\infty} \sum_{n=m_{2p}+1}^{n_{2p+1}} a_n z^n.$$

It is obvious that f is a function from the class $\mathcal{A}_R$. We will prove that $t_\Phi(f) = 1$ and (7) holds.

Using (4), for all sufficiently large $n \in \mathbb{N}_0$ we have

$$\ln a_n \le \ln(b_n/n) \le -\widetilde{\Theta}(n) - \ln n = -n\Gamma(n) - \ln n = -n\overline{\Phi}(n).$$

In addition, if $p \in \mathbb{N}_0$ is large enough and $n = n_{2p+1}$, then $\ln a_n = \ln(b_n/n) = -n\overline{\Phi}(n)$. Therefore, to establish the equality $t_\Phi(f) = 1$, it is enough to use formula (6).

Let $n \in \mathbb{N}_0$ and $r \in (0, R)$ be fixed numbers. We put

$$A_N(r) = \sum_{n \le N} n a_n r^n, \quad B_N(r) = \sum_{n > N} n a_n r^n, \quad C_N(r) = \sum_{n \le N} a_n r^n, \quad D_N(r) = \sum_{n > N} a_n r^n.$$

Since, as it is easy to see, $A_N(r) D_N(r) \le C_N(r) B_N(r)$, and $C_N(r) > 0$ if $N > m_0$, we get

$$K(r, f) = \frac{A_N(r) + B_N(r)}{C_N(r) + D_N(r)} \ge \frac{A_N(r)}{C_N(r)}, \quad N > m_0. \tag{19}$$

If $l \in \mathbb{N}_0$ and $l \to \infty$, we have

$$A_l := \sum_{n \le n_{2l+1}} n a_n c_{2l}^n \ge \sum_{n=m_{2l}+1}^{n_{2l+1}} b_n c_{2l}^n = (n_{2l+1} - m_{2l}) \mu(c_{2l}, g) = (1 + o(1)) \frac{n_{2l+1}}{2l} \mu(c_{2l}, g); \tag{20}$$

in addition, using (18), we obtain

$$C_l := \sum_{n\le n_{2l+1}} a_n c_{2l}^n \le \sum_{n\le n_{2l-1}} b_n c_{2l}^n + \sum_{n=m_{2l}+1}^{n_{2l+1}} \frac{b_n}{n} c_{2l}^n \le \frac{\mu(c_{2l},g)}{(2l+1)^2} + \mu(c_{2l},g)\frac{n_{2l+1}-m_{2l}}{m_{2l}} \le \\ \le \mu(c_{2l},g)\left(\frac{1}{(2l+1)^2} + \frac{n_{2l+1}}{(2l+2)m_{2l}}\right) = (1+o(1))\frac{\mu(c_{2l},g)}{2l}. \tag{21}$$

Using (19), (20) and (21), we have $K(c_{2l},f) \ge A_l/C_l \ge (1+o(1))n_{2l+1}$ as $l\to\infty$. Since $a_n \ge 0$ for each $n\in\mathbb{N}_0$, for all $r\in(0,R)$ we get $K(r,f) = r(\ln M(r,f))'_+$. Hence, the function $K(r,f)$ is non-decreasing on $(0,R)$, and therefore by (b) and (c) we have

$$K(\varrho_{2l},f) \ge K(c_{2l},f) \ge (1+o(1))\delta_{2l}n_{2l+1} \ge (1+o(1))\overline{\Theta}^{-1}\left(h_{n_{2l}}(n_{2l+1})\right) = \\ = (1+o(1))\overline{\Theta}^{-1}\left(\ln \varrho_{2l}\right) = (1+o(1))\Gamma^{-1}\left(\ln\varrho_{2l}\right), \quad l\to\infty. \tag{22}$$

Since $(\rho_k)_{k\in\mathbb{N}_0}$ is a subsequence of the sequence $(r_n)_{n\in\mathbb{N}_0}$ and according to Theorem 1 we have $k_{1,\Phi}(f)\le 1$, from (22) we see that (7) is satisfied. □

4. Some consequences. The following statement is a direct consequence of Theorem 1 and Lemma 6.

Corollary 1. *Let $R\in(0,+\infty]$ and $\Phi\in\Omega_{\ln R}$. Then for any function $f\in\mathcal{A}_R$ of the form* (5) *such that $t_\Phi(f)\le 1$, we have $k_\Phi(f)\le 1+\Delta_\Phi$.*

Let $\Phi\in\Omega_{\ln R}$ and let Γ be the function defined above by (4). According to Lemma 6, there exists a positive sequence $(r_n)_{n\in\mathbb{N}_0}$ increasing to R such that $\Gamma^{-1}(\ln r_n)/\overline{\Phi}^{-1}(\ln r_n)\to 1+\Delta_\Phi$ as $n\to\infty$. Theorem 2, applied precisely with this sequence, and Theorem 1 immediately imply the following statement, which shows that the inequality $k_\Phi(f)\le 1+\Delta_\Phi$ from Corollary 1 is sharp.

Corollary 2. *Let $R\in(0,+\infty]$ and $\Phi\in\Omega_{\ln R}$. Then there exists a function $f\in\mathcal{A}_R$ of the form* (5) *such that $t_\Phi(f)=1$ and $k_\Phi(f)=1+\Delta_\Phi$.*

Note that under the conditions $\Delta_\Phi\in(0,1)$ and $t_\Phi(f)\le 1$, Theorem B together with Lemma 7 allow us to obtain only the following inequality $k_\Phi(f)\le 1/(1-\Delta_\Phi)$, which is not sharp. Nevertheless, as it turns out, in the case when $R=+\infty$ the estimate $k_{2,\Phi}(f)\le 1$ from Theorem B is sharp, that is, for any function $\Phi\in\Omega_{+\infty}$ there exists a function $f\in\mathcal{A}_{+\infty}$ of the form (5) such that $t_\Phi(f)=1$ and $k_{2,\Phi}(f)=1$. Moreover, we have the following statement.

Corollary 3. *Let $\Phi\in\Omega_{+\infty}$ and let ε be a positive continuous non-increasing function on $[t_0,+\infty)$ such that $\int_{t_0}^{+\infty}\varepsilon(t)dt<+\infty$. Then there exists a function $f\in\mathcal{A}_{+\infty}$ of the form* (5) *such that $t_\Phi(f)=1$ and*

$$\varlimsup_{r\to+\infty}\frac{K(r,f)}{\overline{\Phi}^{-1}(\ln r+\varepsilon(\ln\overline{\Phi}^{-1}(\ln r)))}\ge 1.$$

In fact, using the classical Borel-Nevanlinna theorem (see, for example, [16, p. 90]), by the conditions of Corollary 3 we have $\overline{\Phi}^{-1}(\ln r+\varepsilon(\ln\overline{\Phi}^{-1}(\ln r)))\sim\overline{\Phi}^{-1}(\ln r)$ as $r\to+\infty$ outside some exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure. It is clear that then there exists a positive sequence $(r_n)_{n\in\mathbb{N}_0}$ increasing to $+\infty$ such that

$$\overline{\Phi}^{-1}(\ln r_n+\varepsilon(\ln\overline{\Phi}^{-1}(\ln r_n)))\sim\Gamma^{-1}(\ln r_n),\quad n\to\infty.$$

It remains to apply Theorem 2 with exactly this sequence $(r_n)_{n\in\mathbb{N}_0}$.

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