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# A MAXIMAL RIESZ-KANTOROVICH THEOREM WITH APPLICATIONS TO MARKETS WITH AN ARBITRARY COMMODITY SET

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By analyzing proofs of the classical Riesz-Kantorovich theorem, the Mazón-Segura de León theorem on abstract Uryson operators and the Pliev-Ramdane theorem on C-bounded orthogonally additive operators on Riesz spaces, we find the most general (to our point of view) algebraic structure, which we call a complementary space, for which the theorem can be generalized with a similar proof. By a complementary space we mean a PO-set $G$ with a least element 0 such that every order interval $[0, e]$ of $G$ with $e \neq 0$ is a Boolean algebra with respect to the induced order. There are natural examples of complementary spaces: Boolean rings, Riesz spaces with the lateral order. Moreover, the disjoint union of complementary spaces is a complementary space. Our main result asserts that, the set of all additive (in certain sense) functions from a complementary space to a Dedekind complete Riesz space admits a natural Dedekind complete Riesz space structure, described by formulas which are close to the classical Riesz-Kantorovich ones. This theorem generalizes the above mentioned Mazón and Segura de León and Pliev-Ramdane theorems. In the final section, we construct a model of market with an arbitrary commodity set, connected to a complementary space.

**1. Introduction.** The classical Riesz-Kantorovich theorem [1, Theorem 1.18] asserts that if $E$ and $F$ are Riesz spaces with $F$ Dedekind complete then the vector space $L_b(E, F)$ of all order bounded linear operators $T\colon E \to F$ is a Riesz space with respect to the natural order (that is, $S \leq T$ if and only if $(T - S)(E^+) \subseteq F^+$). The second part of the Riesz-Kantorovich theorem provides formulas for the lattice operations on $L_b(E, F)$.

Later an analogue of the Riesz-Kantorovich theorem was obtained by Mazón and Segura de León in [6] for a wide class of mappings $T\colon E \to F$, the vector space $U(E, F)$ of all order bounded orthogonally additive operators (referred also as *abstract Uryson operators*). Then Pliev and Ramdane generalized the Mazón-Segura de León theorem to much more wider class of laterally-to-order bounded (in other terminology, C-bounded) orthogonally additive operators in [10, Theorem 3.6].

Below we familiarly use some standard information on Riesz spaces and positive operators on them from the frames of Aliprantis-Burkinshaw's book [1], and on general lattices, Boolean algebras and boolean rings [5]. We use the special notation $\bigsqcup_{i=1}^n x_i$ and $x_1 \sqcup \ldots \sqcup x_n$ for a disjoint sum of elements $x_i$ of a Riesz space. So once it is written, we automatically assume that $x_i \perp x_j$ as $i \neq j$ (recall that $x \perp y$ means $|x| \wedge |y| = 0$ in a Riesz space).

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Recall that a function $T\colon E \to F$ acting between Riesz space $E$ and $F$ is called an *orthogonally additive operator* if for every $x, y \in E$ with $x \perp y$ one has $T(x+y) = T(x)+T(y)$. The order $\le$ on the set $\mathcal{OAO}(E,F)$ of all orthogonally additive operators $T\colon E \to F$ is different from the defined above order on $L_b(E,F)$ (notice that $L_b(E,F)$ formally is a vector subspace of $\mathcal{OAO}(E,F)$) and defined by setting $S \le T$ if and only if $(T-S)(E) \subseteq F^+$. The two distinct orders are denoted by the same symbol $\le$, which in most cases does not lead to a misunderstanding (it is easy to see that the only linear operator $T\colon E \to F$ which is positive, that is, $T \ge 0$, as an orthogonally additive operator is zero). See [9] for a survey on orthogonally additive operators.

One of the ideas of the present paper is to find, as general as possible, a domain set $E$ with a suitable structure such that for the set $\mathcal{A}(E,F)$ of all (in a certain sense) additive operators the Riesz-Kantorovich theorem remains valid. Such a domain set we call a complementary space below. One of our main results generalizes the Riesz-Kantorovich theorem to complementary spaces.

**2. Complementary spaces.** Looking ahead, we announce that the most valuable examples of complementary spaces are, on the one hand, Boolean rings and, on the other hand, Riesz spaces with the lateral order, which is a much less known object in mathematics than Boolean rings. This is why we begin with an information on the lateral order.

**2.1. The lateral order on a Riesz space.** Every Riesz space (= vector lattice) $E$ is naturally endowed with a partial order which differs from the given order $\le$. We set $x \sqsubseteq y$ provided $x$ is a *fragment* (= component) of $y$, that is, $x \perp y - x$. The set of all fragments of a given element $e \in E$ is denoted by $\mathfrak{F}_e$. Obviously, if $e = x + y$ then the following three conditions are equivalent: $x \sqsubseteq e$, $y \sqsubseteq e$ and $x \perp y$. Hence if $e = \bigsqcup_{k=1}^m x_k$ then $(x_k)_{k=1}^m$ are disjoint fragments of $e$. By [7, Proposition 3.4] (see also [1, Theorem 3.15]), $\mathfrak{F}_e$ is a Boolean algebra with respect to the lateral order for every $e \in E \setminus \{0\}$. We refer the readers to [7] and [8] for a detailed information on the lateral order.

**2.2. Main definitions and notation.** As usual, the supremum (or infimum) of a two-point subset $\{x, y\}$ of a PO-set (= partially ordered set) $(G, \preceq)$ in the lattice theory is denoted by $x \vee y$ (respectively, by $x \wedge y$). This is not convenient in our setting, because one of the most important examples of complementary spaces is a Riesz space $G$ with the lateral order $\sqsubseteq$, and the expressions $x \vee y$, $x \wedge y$ in $G$ mean the supremum and infimum with respect to the given order $\le$ on $G$. So for suprema and infima we will use the same notation as for the lateral order on Riesz spaces:

$$\boldsymbol{\bigcup} A := \preceq\text{-}\sup A, \quad \boldsymbol{\bigcap} A := \preceq\text{-}\inf A \ \text{ for } A \subseteq G;$$
$$x \boldsymbol{\cup} y := \preceq\text{-}\sup\{x, y\}, \quad x \boldsymbol{\cap} y := \preceq\text{-}\inf\{x, y\} \ \text{ for } x, y \in G.$$

We use the symbols in bold to distinguish them from the symbols of union and intersection.

**Definition 1.** Let $(G, \preceq)$ be a PO-set with a least element 0 and $x, y, z \in G$. Elements $y, z$ of $G$ are said to be *separated* (write $y \dagger z$), if $y \boldsymbol{\cap} z = 0$. If $y \dagger z$ and $x = y \boldsymbol{\cup} z$ then we say that $x$ is a *direct sum* of $y$ and $z$ and write $x = y \oplus z$.

In other words, $x \dagger y$ means that for every $t \in E$ the conditions $t \preceq x$ and $t \preceq y$ imply $t = 0$.

In the sequel, once $x \oplus y$ is written, we understand that $x \dagger y$ and the supremum $x \oplus y$ exists.

Given a PO-set$(G, \preceq)$ with a least element 0 and $g \in G$, we set $\mathfrak{F}_g := \{x \in G : x \preceq g\}$.

**Definition 2.** A PO-set $(G, \preceq)$ with a least element 0 is called a *complementary space*, if the following conditions hold:

$(CS1)$ for every $x, y \in G$ with $y \preceq x$ there exists $z \in G$ such that $x = y \oplus z$;

$(CS2)$ for every $x, y, u, v \in G$ with $x \oplus y = u \oplus v$ there exist $a, b, c, d \in G$ such that $x = a \oplus b$, $y = c \oplus d$, $u = a \oplus c$ and $v = b \oplus d$.

**Remark 1.** (1) (CS1) does not imply (CS2). The following set of subsets of the set $\Omega = \{1, 2, 3, 4\}$ possesses (CS1) with respect to the inclusion relation and does not possess (CS2):

$$G = \big\{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \Omega\big\}.$$

(2) (CS2) does not imply (CS1): every linearly ordered set consisting of, at least, 3 elements satisfies (CS2) and does not satisfy (CS1).

One can show that, a PO-set $(G, \preceq)$ with a least element possessing (CS1) may have two distinct elements $z_1 \neq z_2$ in $G$ for certain $y \preceq x$ such that $x = y \oplus z_1 = y \oplus z_2$. However, in a complementary space the complement is unique, as the following assertion confirms.

**Proposition 1.** *Let $(G, \preceq)$ be a PO-set with the least element 0 possessing (CS2). Let $e, r, s, t \in G$ satisfy $e = r \oplus s = r \oplus t$. Then $s = t$.*

*Proof.* Using (CS2) with $x = u = r$, $y = s$ and $v = t$, we conclude that there are $a, b, c, d \in G$ such that $r = x = a \oplus b = u = a \oplus c$, $s = y = c \oplus d$, $t = v = b \oplus d$. Therefore, $b \preceq a \oplus b = a \oplus c = r$ and $b \preceq b \oplus d = t$. So $r \dagger t$ implies $b = 0$.

Analogously, $c \preceq a \oplus c = a \oplus b = r$ and $c \preceq c \oplus d = s$ and hence $c = 0$. Thus, $s = c \oplus d = d = b \oplus d = t$. □

**Corollary 1.** *A PO-set $(G, \preceq)$ with the least element 0 is a complementary space if and only if $(G, \preceq)$ possesses (CS2) and the following strong version of (CS1):*

$(CS1')$ *for every $x, y \in G$ with $y \preceq x$ there exists a unique $z \in G$ such that $x = y \oplus z$.*

**Definition 3.** Let $(G, \preceq)$ be a complementary space, $x, y \in G$ and $y \preceq x$. The unique element $z$ of $G$ such that $x = y \oplus z$ is called the *direct difference* of $x$ and $y$, and is denoted by $x \ominus y$.

So if $(G, \preceq)$ is a complementary space then

$$(\forall x, y \in G)\ \ y \preceq x \ \Rightarrow\ x = y \oplus (x \ominus y). \tag{1}$$

Observe that, if for every nonzero element $e$ of $G$ with a least element 0 the order interval $\mathfrak{F}_e$ is a Boolean algebra then (CS1)–(CS2) hold true, and $G$ is a complementary space. Below we prove that the converse is also true.

**2.3. Important examples.** Below we provide important examples of complementary spaces.

**Example 1.** Any Boolean ring is a complementary space.

**Example 2.** Every subset $G$ of a Riesz space $E$ with the properties

(1) $(\forall x, y \in G)\big((y \sqsubseteq x) \Rightarrow (x - y \in G)\big)$;

(2) $(\forall x, y \in G)\,(x \sqcap y \in G)$

is a complementary space with respect to the lateral order $\sqsubseteq$ on $E$.

*Proof.* (CS1) easily follows from (1). We prove (CS2). Fix any $x, y, u, v \in G$ with $e := x \oplus y = u \oplus v$ in $G$, and so $e \in G$. By [7, Proposition 3.4], the vectors $a := x \sqcap u$, $b := x \sqcap v$, $c := y \sqcap u$ and $d := y \sqcap v$ are well defined in $E$, and by (2) belong to $G$. Again by [7, Proposition 3.4], $x = a \sqcup b$, $y = c \sqcup d$, $u = a \sqcup c$ and $v = b \sqcup d$. It remains to observe that the expressions $f \sqcup g$ and $f \oplus g$ mean the same in a Riesz space. $\square$

**Example 3.** Let $\Omega, Y$ be nonempty sets and $G$ be a set of functions $f \colon X \to Y$, where $X \subseteq \Omega$. Define a partial order $\preceq$ on $G$ by setting $g \preceq f$ provided $g$ is a restriction of $f$. Suppose that the empty function $\varnothing \colon \varnothing \to Y$ belongs to $G$, and the following implication holds: if $f, g \in E$ with $f \colon X \to Y$ and $g = f|_{X'}$ for some subset $X'$ of $X$ then $h = f|_{X \setminus X'} \in G$ as well. Then $(G, \preceq)$ is a complementary space.

Next construction provides lots of new examples of complementary spaces. Informally speaking, the disjoint union of complementary spaces with common zero element is a complementary space.

**Example 4.** Let $\big((G_i, \preceq_i)\big)_{i \in I}$ be a family of complementary spaces such that $G_i \cap G_j = \{0\}$ for all distinct $i, j \in I$, where $0$ is the least element of each $G_k$, $k \in I$. Set $G := \bigcup_{i \in I} G_i$ and define a partial order on $G$ by setting $x \preceq y$ provided there exists $i \in I$ such that $x, y \in G_i$ and $x \preceq_i y$. Obviously, $(G, \preceq)$ is a complementary space.

**2.4. Selected properties of complementary spaces.** Following [5], a Boolean algebra is a distributive lattice with complements. Axiom (CS2) requires much less than the notion of a distributive lattice. However, in couple with (CS1), it gives all a Boolean algebra needs, as the following theorem asserts.

**Theorem 1.** *For a PO-set $(G, \preceq)$ with a least element $0$ the following assertions are equivalent:*

*(1) $G$ is a complementary space;*

*(2) for every $e \in G \setminus \{0\}$ the subset $\mathfrak{F}_e$ is a Boolean algebra.*

The proof of implication (1) $\Rightarrow$ (2) is divided into several steps (lemmas), claiming certain properties of a complementary space.

**Lemma 1.** *Let $(G, \preceq)$ be a complementary space, $w, x, y \in G$, $x \dagger y$ and $w \preceq x \oplus y$. Then there exist a unique pair of elements $u \in \mathfrak{F}_x$ and $v \in \mathfrak{F}_y$ such that $w = u \oplus v$.*

*Proof.* By (1), $x \oplus y = w \oplus \big((x \oplus y) \ominus w\big)$. Choose by (CS2) $a, b, c, d \in G$ so that $x = a \oplus b$, $y = c \oplus d$ and $w = a \oplus c$. Then $u := a$ and $v := c$ are as desired.

To prove the uniqueness, assume $u' \in \mathfrak{F}_x$ and $v' \in \mathfrak{F}_y$ are such that $w = u' \oplus v'$ as well. Now using the equality $u \oplus v = u' \oplus v'$, choose $a', b', c', d' \in G$ so that $u = a' \oplus b'$, $v = c' \oplus d'$, $u' = a' \oplus c'$ and $v' = b' \oplus d'$. Since $b' \preceq u \preceq x$ and $b' \preceq v' \preceq y$, the separateness of $x$ and $y$ implies $b' = 0$. In a similar way, $c' \preceq u' \preceq x$ and $c' \preceq v' \preceq y$ yield $c' = 0$. This is why $u = u'$ and $v = v'$, that means the uniqueness. $\square$

**Lemma 2.** *Let $(G, \preceq)$ be a complementary space and $x, y, z$ be pairwise separate elements of $G$. If $x \oplus y$ exists then $(x \oplus y) \dagger z$.*

*Proof.* Let $w$ be any lower bound for the set $\{x \oplus y, z\}$. Using $w \preceq x \oplus y$, choose by Lemma 1 $u \in \mathfrak{F}_x$ and $v \in \mathfrak{F}_y$ so that $w = u \oplus v$. Then the conditions $u \preceq w \preceq z$, $u \preceq x$ and $z \dagger x$ imply $u = 0$. Likewise the conditions $v \preceq w \preceq z$, $u \preceq y$ and $z \dagger y$ imply $v = 0$. The latter two conclusions yield $w = 0$ and so $(x \oplus y) \dagger z$. □

**Lemma 3.** *Let $(G, \preceq)$ be a complementary space, $x, y, u, v, e \in G$ with $e = x \oplus y = u \oplus v$. Then there are unique elements $a, b, c, d \in G$ such that $x = a \oplus b$, $y = c \oplus d$, $u = a \oplus c$ and $v = b \oplus d$.*
*Moreover,* (1) *the elements $a, b, c, d$ are pairwise separated;* (2) $a = x \cap u$, $b = x \cap v$, $c = y \cap u$, $d = y \cap v$.

*Proof.* Choose by definition $a, b, c, d \in G$ such that $x = a \oplus b$, $y = c \oplus d$, $u = a \oplus c$ and $v = b \oplus d$. First we prove that $a, b, c, d$ are pairwise separated. One has $a \dagger b$, $c \dagger d$, $a \dagger c$ and $b \dagger d$ by definition. The rest of separateness conditions $a \dagger d$ and $b \dagger c$ easily follow from the condition $x \dagger (e - x)$, see (1).

Now we prove that $a = x \cap u$. Obviously, $a$ is a lower bound for $\{x, u\}$. Let $w$ be any lower bound for $\{x, u\}$. Since $w \preceq x = a \oplus b$, by Lemma 1, $w = w' \oplus w_b$ for some $w' \in \mathfrak{F}_a$ and $w_b \in \mathfrak{F}_b$. Observe that $w_b \preceq w \preceq a \oplus c$, $w_b \preceq b$ and $(a \oplus c) \dagger b$ by Lemma 2. Therefore, $w_b = 0$. Hence $w = w'$, which yields $w \preceq a$. So $a = x \cap u$ is proved.

The rest of formulas in (2) can be proved analogously. These formulas imply the uniqueness of $a, b, c, d$. □

Now using Lemma 3, we can rewrite Lemma 1 in more details.

**Corollary 2.** *Let $(G, \preceq)$ be a complementary space, $w, x, y \in G$, $x \dagger y$ and $w \preceq x \oplus y$. Then there exist a unique pair of elements $u \in \mathfrak{F}_x$ and $v \in \mathfrak{F}_y$ such that $w = u \oplus v$. More precisely, $w = (w \cap x) \oplus (w \cap y)$.*

*Proof.* Indeed, in the proof of Lemma 1 we got $w = a \oplus c$, and by Lemma 3 (2), $a = x \cap w$ and $c = y \cap w$. □

The following statement shows that $\mathfrak{F}_e$ is a lattice for every nonzero element $e$ of $G$.

**Lemma 4.** *Let $(G, \preceq)$ be a complementary space, $e \in E$ and $x, y \in \mathfrak{F}_e$. Then there exist $x \cap y$ and $x \cup y$ in $G$ (and hence, in $\mathfrak{F}_e$). Moreover,*

$$x \cup y = e \ominus \big((e \ominus x) \cap (e \ominus y)\big). \tag{2}$$

*Proof.* Taking into account the equalities $e = x \oplus (e \ominus x) = y \oplus (e \ominus y)$, choose $a, b, c, d \in G$ so that $x = a \oplus b$, $e \ominus x = c \oplus d$, $y = a \oplus c$ and $e \ominus y = b \oplus d$. By Lemma 3, $a = x \cap y$, $b = x \cap (e \ominus y)$, $c = y \cap (e \ominus x)$ and $d = (e \ominus x) \cap (e \ominus y)$. The existence of $x \cap y$ is obtained. To prove both the existence of $x \cup y$ and (2), we show that $e \ominus d = x \cup y$.

Using $x \preceq e = d \oplus (e \ominus d)$, choose by Lemma 1 $x' \in \mathfrak{F}_d$ and $x'' \in \mathfrak{F}_{e \ominus d}$ so that $x = x' \oplus x''$. Since $x \dagger (e \ominus x)$, $x' \preceq x$ and $d \preceq (e \ominus x)$, one has $x' \dagger d$, which together with $x' \preceq d$ implies $x' = 0$. Hence, $x = x'' \preceq e \ominus d$. Analogously, $y \preceq e \ominus d$ and so $e \ominus d$ is an upper bound for $\{x, y\}$ in $G$. Let $w \in G$ be any upper bound for $\{x, y\}$. Write $e \ominus d \leq e = x \oplus (e \ominus x) = x \oplus (c \oplus d)$ and use Lemma 3 to obtain $v' \in \mathfrak{F}_x$ and $v'' \in \mathfrak{F}_{c \oplus d}$ so that

$$e \ominus d = v' \oplus v''. \tag{3}$$

Using $v'' \preceq c \oplus d$, choose by Lemma 3 $v_c \in \mathfrak{F}_c$ and $v_d \in \mathfrak{F}_d$ so that $v'' = v_c \oplus v_d$. Then $v_d \preceq d$ and $v_d \preceq e \ominus d$ imply $v_d = 0$. Hence $v'' = v_c$ which yields $v'' \preceq c \preceq y$. Since $w$ is an upper bound for $\{x, y\}$, we obtain that $w$ is an upper bound for smaller elements $\{v', v''\}$. By (3), $e \ominus d \preceq w$ and so $e \ominus d = x \Cup y$. □

**Remark 2.** Let $(G, \preceq)$ be a complementary space, $e \in E$, $A \subseteq \mathfrak{F}_e$. Once $\bigcup A$ exists in $G$, since $e$ is an upper bound for $A$, one has $\bigcup A \in \mathfrak{F}_e$ and hence $\bigcup A$ is the same in $\mathfrak{F}_e$ and $G$. Likewise, if $\bigcap A$ exists in $G$ then $\bigcap A \in \mathfrak{F}_e$ and hence $\bigcap A$ is the same in $\mathfrak{F}_e$ and $G$.

Since $\mathfrak{F}_e$ by Lemma 4 is a lattice, [5, Lemma 1 of Section 1.5] yields the associativity of the lattice operations on $\mathfrak{F}_e$, including the direct sum $\oplus$.

**Corollary 3.** *Let $(G, \preceq)$ be a complementary space, $e \in E \setminus \{0\}$ and $x, y, z \in \mathfrak{F}_e$. Then $(x \Cup y) \Cup z = x \Cup (y \Cup z)$; $(x \Cap y) \Cap z = x \Cap (y \Cap z)$. If, moreover, $x \dagger y$, $y \dagger z$ and $z \dagger x$ then $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.*

Using the above associativity, we obtain some more properties.

**Corollary 4.** *Let $(G, \preceq)$ be a complementary space, $e \in E \setminus \{0\}$ and $x, y \in \mathfrak{F}_e$. Then*
(1) $e = (x \Cap y) \oplus \big(x \Cap (e \ominus y)\big) \oplus \big((e \ominus x) \Cap y\big) \oplus \big((e \ominus x) \Cap (e \ominus y)\big)$;
(2) $x \Cup y = (x \Cap y) \oplus \big(x \Cap (e \ominus y)\big) \oplus \big((e \ominus x) \Cap y\big)$.

*Proof.* (1) By Lemma 3 and the associativity of $\oplus$, $e = a \oplus b \oplus c \oplus d$, that is, (1) holds.
(2) follows from (1) and Lemma 4. □

Using induction and associativity, one can easily obtain the following generalization of Corollary 2 from two to an arbitrary finite number of summands.

**Corollary 5.** *Let $(G, \preceq)$ be a complementary space, $n \in \mathbb{N}$, $w \in G$ and $x_1, \ldots, x_n$ be pairwise separate elements of $G$ such that $w \preceq x_1 \oplus \ldots \oplus x_n$. Then there exists a unique collection of elements $y_i \in \mathfrak{F}_{x_i}$ for $i = 1, \ldots, n$ such that $w = y_1 \oplus \ldots \oplus y_n$. More precisely, $w = (w \Cap x_1) \oplus \ldots \oplus (w \Cap x_n)$.*

As a further consequence, we obtain a distributivity law for direct sums.

**Corollary 6.** *Let $(G, \preceq)$ be a complementary space, $n \in \mathbb{N}$, $w \in G$ and $x_1, \ldots, x_n$ be pairwise separate elements of $G$. If $w \Cap (x_1 \oplus \ldots \oplus x_n)$ exists in $G$ then*

$$w \Cap (x_1 \oplus \ldots \oplus x_n) = (w \Cap x_1) \oplus \ldots \oplus (w \Cap x_n).$$

*Proof.* Observe that $w \Cap (x_1 \oplus \ldots \oplus x_n) \preceq x_1 \oplus \ldots \oplus x_n$ and

$$\big(w \Cap (x_1 \oplus \ldots \oplus x_n)\big) \Cap x_k = w \Cap \big((x_1 \oplus \ldots \oplus x_n) \Cap x_k\big) = w \Cap x_k,$$

because $x_k \preceq x_1 \oplus \ldots \oplus x_n$ for all $k = 1, \ldots, n$. Then use Corollary 5. □

By induction and the associativity of the direct sum, one can obtain the following consequence of (CS2), which is an analogue of the Riesz decomposition property [1, Theorem 1.20].

**Corollary 7.** *Let $(G, \preceq)$ be a complementary space, $m, n \in \mathbb{N}$. Let $x_1, \ldots, x_m$ and $y_1, \ldots y_n$ be two finite collections of pairwise separate elements of $G$ such that*

$$x_1 \oplus x_2 \oplus \ldots \oplus x_m = y_1 \oplus y_2 \oplus \ldots \oplus y_n.$$

*Then there exists a collection $\{z_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ in $G$ such that*

$$(\forall i \in \{1, 2, \ldots, m\})\ x_i = z_{i,1} \oplus z_{i,2} \oplus \ldots \oplus z_{i,n},\ (\forall j \in \{1, 2, \ldots, n\})\ y_j = z_{1,j} \oplus z_{2,j} \oplus \ldots \oplus z_{m,j}.$$

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* Implication (2) $\Rightarrow$ (1) is obvious, so we prove (1) $\Rightarrow$ (2). Fix any $e \in G \setminus \{0\}$ and prove that $\mathfrak{F}_e$ is a Boolean algebra. By (CS1), it is enough to prove that $\mathfrak{F}_e$ is a distributive lattice. By Lemma 4, $\mathfrak{F}_e$ is a lattice. It remains to prove the distributivity. By [5, Theorem 9 of Section 1], it is enough to prove only one of the two distributivity laws. Fix any $x, y, z \in \mathfrak{F}_e$ and set

$$L := x \cap (y \cup z), \quad R := (x \cap y) \cup (x \cap z).$$

Our goal is to prove the equality $L = R$. The inequality $R \preceq L$ is clear, because $L$ is an upper bound for the two-point set $\{x \cap y, x \cap z\}$, and $R$ is the least upper bound for the same set. To show the converse inequality, observe that since $z \succeq x \cap z$, one has $e \ominus z \preceq e \ominus (x \cap z)$. Hence

$$x \cap y \cap (e \ominus z) \preceq x \cap y \cap \big(e \ominus (x \cap z)\big). \tag{4}$$

Analogously,

$$x \cap z \cap (e \ominus y) \preceq x \cap z \cap \big(e \ominus (x \cap y)\big). \tag{5}$$

Then

$$L = x \cap (y \cup z) \overset{4\,(2)}{=} x \cap \big((y \cap z) \oplus (y \cap (e \ominus z)) \oplus ((e \ominus y) \cap z)\big) =$$

$$\overset{6}{=} (x \cap y \cap z) \oplus \big(x \cap y \cap (e \ominus z)\big) \oplus \big(x \cap (e \ominus y) \cap z\big) \overset{(4),(5)}{\preceq}$$

$$\preceq (x \cap y \cap z) \oplus \big(x \cap y \cap \big(e \ominus (x \cap z)\big)\big) \oplus \big(x \cap \big(e \ominus (x \cap y)\big) \cap z\big) \overset{4\,(2)}{=} (x \cap y) \cup (x \cap z) = R.$$

Thus, $L \preceq R$, which together with $R \preceq L$ implies $L = R$ and the theorem is proved. $\square$

As partial cases, Theorem 1 gives characterizations of a Boolean algebra and a Boolean ring in terms of two axioms.

**Corollary 8.**

(1) *A PO set $(B, \preceq)$ with a least and a greatest elements is a Boolean algebra if and only if the following conditions hold:*

   (a) *for every $x, y \in B$ with $y \preceq x$ there exists $z \in B$ such that $x = y \oplus z$;*

   (b) *for every $x, y, u, v \in B$ with $x \oplus y = u \oplus v$ there exist $a, b, c, d \in B$ such that $x = a \oplus b$, $y = c \oplus d$, $u = a \oplus c$ and $v = b \oplus d$,*

   *where $w = w_1 \oplus w_2$ means that $\inf\{w_1, w_2\} = \min B$ and $\sup\{w_1, w_2\} = w$.*

(2) *A directed PO set $(B, \preceq)$ with a least element is a Boolean ring if and only if (a) and (b) hold.*

## 3. Vector charges on complementary spaces.

**Definition 4.** Let $(G, \preceq)$ be a complementary space and $F$ a Riesz space. A function $T\colon G \to F$ is called a *vector charge*, if for every elements $x, y, z$ of $E$ with $x = y \oplus z$ one has $Tx = Ty + Tz$. In the partial case where $F = \mathbb{R}$ we use the term *scalar charge* instead of "vector charge".

Since $x = 0 \oplus x$ for every $x \in G$, every vector charge sends zero of $G$ to zero of $F$.

If $G$ is a Riesz space with the lateral order then $\mathcal{A}(G, F)$ coincides with the set $\mathcal{OA}(G, F)$ of all orthogonally additive operators $T\colon G \to F$.

The set $\mathcal{A}(G, F)$ of all vector charges $T\colon G \to F$ is naturally endowed with the obvious vector space structure inspired by that of $F$.

**Definition 5.** Let $(G, \preceq)$ be a complementary space and $F$ a Riesz space. A vector charge $T \in \mathcal{A}(G, F)$ is said to be:

- *positive* (write $T \geq 0$), if $Tx \geq 0$ for all $x \in E$;
- *regular*, if $T$ equals a difference of two positive vector charges $T = T_1 - T_2$, $T_i \in \mathcal{A}(G, F)$, $i = 1, 2$. The set of all regular vector charges $T \in \mathcal{A}(E, F)$ is denoted by $\mathcal{A}_r(G, F)$;
- *C-bounded*, if for every $g \in G$ the set $T(\mathfrak{F}_g)$ is order bounded in $F$. The set of all C-bounded vector charges $T \in \mathcal{A}(G, F)$ is denoted by $\mathcal{A}_b(G, F)$.

Obviously, the vector space $\mathcal{A}(G, F)$ is an ordered vector space with respect to the order $S \leq T$ if and only if $T - S \geq 0$ for all $S, T \in \mathcal{A}(G, F)$, and the sets $\mathcal{A}_r(G, F)$ and $\mathcal{A}_b(G, F)$ are vector subspaces of $\mathcal{A}(G, F)$.

The following theorem in the main result of the paper.

**Theorem 2.** *Let $(G, \preceq)$ be a complementary space and $F$ a Dedekind complete Riesz space. Then the following assertions hold:*

(1) $\mathcal{A}_r(G, F) = \mathcal{A}_b(G, F)$.

(2) *$\mathcal{A}_b(G, F)$ is a Dedekind complete Riesz space and the lattice operations satisfy the following formulas for all $S, T \in \mathcal{A}_b(G, F)$ and $x \in G$:*

  *(a)* $(S \vee T)\, x = \sup \{Sy + Tz : y, z \in G \text{ with } x = y \oplus z\}$.

  *(b)* $(S \wedge T)\, x = \inf \{Sy + Tz : y, z \in G \text{ with } x = y \oplus z\}$.

  *(c)* $(T^+)x = \sup T(\mathfrak{F}_x)$.

  *(d)* $(T^-)x = -\inf T(\mathfrak{F}_x)$.

  *(e)* $|T|x = \sup \{Ty - Tz : y, z \in G \text{ with } x = y \oplus z\}$.

  *(f)* $|Tx| \leq |T|x$.

*Proof.* Observe that (1) is an easy consequence of (2). Indeed, the inclusion $\mathcal{A}_r(G, F) \subseteq \mathcal{A}_b(G, F)$ is obvious, and by (c) and (d), every element of $\mathcal{A}_b(G, F)$ is a difference of two positive vector charges.

Fix any $S, T \in \mathcal{A}_b(G, F)$ and prove the existence of $S \vee T$. For every $x \in G$ we set

$$Rx := \sup \{Sy + Tz : y, z \in G \text{ with } x = y \oplus z\} \tag{6}$$

(the supremum exists, because $S$ and $T$ are C-bounded and at least one decomposition exists $x = 0 \oplus x$) and prove that $R$ is a vector charge.

Fix any $u, v, w \in G$ with $w = u \oplus v$. Now fix any $v', v'' \in G$ with $v = v' \oplus v''$. Then for any $u', u'' \in G$ with $u = u' \oplus u''$ one has by Corollary 3

$$w = (u' \oplus u'') \oplus (v' \oplus v'') = (u' \oplus v') \oplus (u'' \oplus v'')$$

and by (6) we obtain

$$Su' + Tu'' + Sv' + Tv'' = S(u' \oplus v') + T(u'' \oplus v'') \leq Rw.$$

Since for any $u', u'' \in G$ with $u = u' \oplus u''$ we got $Su' + Tu'' \leq Rw - Sv' - Tv''$, one has by (6) $Ru \leq Rw - Sv' - Tv''$, that is, $Sv' + Tv'' \leq Rw - Ru$. By the arbitrariness of a decomposition $v = v' \oplus v''$, we obtain $Rv \leq Rw - Ru$, that is,

$$Ru + Rv \leq Rw. \tag{7}$$

Now we prove the converse inequality. Let $x, y \in G$ be any elements such that $w = x \oplus y$. Using that $w = u \oplus v$, choose by (CS2) $a, b, c, d \in G$ so that $x = a \oplus b$, $y = c \oplus d$, $u = a \oplus c$ and $v = b \oplus d$. Then

$$Sx + Ty = S(a \oplus b) + T(c \oplus d) = Sa + Tc + Sb + Td \leq Ru + Rv.$$

By the arbitrariness of $x, y \in G$ with $w = x \oplus y$, we obtain $Rw \leq Ru + Rv$. Taking into account (7), we deduce $Rw = Ru + Rv$, and so $R$ is a vector charge.

Moreover, $R \in \mathcal{A}_b(G, F)$. Indeed, given any $g \in G$, let the set $S(\mathfrak{F}_g)$ be order bounded by some $f_S \in F$ and the set $T(\mathfrak{F}_g)$ be order bounded by some $f_T \in F$. Then for any $x \in \mathfrak{F}_g$ and any $y, z \in G$ with $x = y \oplus z$ one has $y, z \in \mathfrak{F}_g$ and hence $Sy + Tz \leq f_S + f_T$, which yields $Rx \leq f_S + f_T$ by the arbitrariness of the decomposition $x = y \oplus z$.

Now we show that $R = S \vee T$ in $\mathcal{A}_b(G, F)$. Obviously, $R$ is an upper bound for $\{S, T\}$. Let $W \in \mathcal{A}_b(G, F)$ be any upper bound for $\{S, T\}$. Fix any $x \in G$. Then for every $y, z \in G$ with $x = y \oplus z$ one has $Sy + Tz \leq Wy + Wz = Wx$. By the arbitrariness of $y, z \in G$, we obtain $Rx \leq Wx$. Hence $R \leq W$ by the arbitrariness of $x \in G$. Therefore, $R = S \vee T$. Thus, $\mathcal{A}_b(G, F)$ is a Riesz space and (a) holds. It remains to show (b)–(f).

(b) For every $x \in G$ by [1, Theorem 1.3],

$$(S \wedge T)\, x = -\big((-S) \vee (-T)\big)\, x = -\sup\{-Sy - Tz :\ x = y \oplus z\} = \inf\{Sy + Tz :\ x = y \oplus z\}$$

and (b) is proved.

(c) For every $x \in G$ by (a) one has

$$(T^+)\, x = (T \vee 0)\, x = \sup\{Ty :\ y, z \in G \text{ with } x = y \oplus z\} = \sup\{Ty :\ y \preceq x\} = \sup T(\mathfrak{F}_x).$$

(d) easily follows from (c).

(e) can be proved using (a) similarly like we did prove (c).

(f) follows from (e). □

As a partial case of Theorem 2, where the complementary space $(G, \preceq)$ is an arbitrary Riesz space $E$ with the lateral order $\sqsubseteq$ on $E$, we obtain the Pliev-Ramdane [10, Theorem 3.6].

**4. Applications to a model of an economy with an arbitrary commodity set.** Consider a model of an economy with infinitely many commodities, which are exchanged, produced or consumed. Following classical approaches (see e.g. [2]), outputs of production are positively signed, and inputs are negatively signed. In the classical Arrow-Debreu model [3], the set of all commodity bundles is the finite dimensional Riesz space $E = \mathbb{R}^d$, where $d$ is the amount of goods. The rules for exchanging goods are given by a fixed vector $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{R}^d$. More precisely, $\frac{p_i}{p_j}$ is the amount of good $j$, which can be exchanged for a unit amount of good $i$ at prices $\mathbf{p}$. Now the *value* of a commodity vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is defined by the scalar product $(\mathbf{p}, \mathbf{x}) = \sum_{k=1}^{d} p_k x_k$, or, as the value of a suitable linear functional on $\mathbb{R}^d$. So the price space is the vector dual space $E' = \mathbb{R}^d$, which can be identified with $E$ in the finite dimensional case. The Riesz space structures of both commodity and price spaces are of great importance, because both commodity bundles and prices must be comparable with other commodity bundles and prices respectively. A *preference relation* on a set is a defined to be a linear order on the set. In a lot of natural cases, a preference relation $\boldsymbol{\leq}$ on a set $X$ is induced by a suitable *utility function* $u \colon X \to \mathbb{R}$ by setting $x \boldsymbol{\leq} y$ if and only if $u(x) \leq u(y)$ [2, Theorem 1.1.4]. There are also models considering infinite commodity sets, see e.g. [4].

In the present section, we offer some amendments to the Arrow-Debreu model. One is that we generalize functionals to scalar charges in place of prices. The reason is that, in real economics, prices often become cheaper for bigger amount of goods. So the wholesale prices are not linear functionals, but sometimes they are charges in the sense that the price for the union of two separate groups of goods equals the sum of their prices. There are also examples in economics, where the price per unit of a product increases with the quantity purchased. This phenomenon is known as *negative economies of scale* or *quantity-based price discrimination.* Another amendment is that we are going to consider a complementary space $G$ in place of a Riesz space $E$. Since a disjoint union of complementary spaces is a complementary space (and the same is false for Riesz spaces), it allows considering unions of markets to a larger market in the case, where goods from different markets are incompatible by prices, or by other valuable characteristics. Another superiority of such an approach: we can consider the positive cone $G = (\mathbb{R}^n)^+$ as a complementary space with the lateral order, however the same set is not a Riesz space.

Let $(G, \preceq)$ be a complementary space. Elements of $G$ are *commodity vectors*, and elements of $\mathcal{A}_b(G, \mathbb{R})$ are prices. The number $P(\mathbf{x})$ is called the value of a commodity vector $\mathbf{x}$ by a price $P \in \mathcal{A}_b(G, \mathbb{R})$. A price $P \in \mathcal{A}_b^+(G, \mathbb{R})$ is said to be *positive*. A price $P$ is called *strictly positive* (write $P \gg 0$) provided $P\mathbf{x} > 0$ for all $\mathbf{x} \in G \setminus \{0\}$. Finally, the complementary space $G$ with such an interpretation we call a *complementary commodity space*, or *CC-space* in short.

Now fix some commodity vector $\mathbf{g} \in G$ and some positive price $P \in \mathcal{A}_b^+(G, F)$.

**Definition 6.** The *budget set* for $P$ corresponding to a number $\lambda > 0$ is defined by setting

$$\mathcal{B}_\lambda(P) = \{\mathbf{x} \in G : P(\mathbf{x}) \le \lambda\}.$$

**Definition 7.** Given a preference relation $\boldsymbol{\le}$ on a CC-space $G$ and a subset $A$ of $G$, a commodity vector $\mathbf{x}_0 \in A$ such that $\mathbf{x} \boldsymbol{\le} \mathbf{x}_0$ for all $\mathbf{x} \in A$ is called the *demand vector* of $A$.

Since a preference relation is a linear order on $G$, the demand vector is unique, whenever exists.

Consider the following problem.

**Problem 1.** *Let $(G, \preceq)$ be a CC-space, $P \in \mathcal{A}_b^+(G, \mathbb{R})$ a given positive price and $\boldsymbol{\le}$ a preference relation on $G$. Characterize the set of all $\lambda > 0$ for which the budget set $\mathcal{B}_\lambda(P)$ has a demand vector.*

Below we solve this problem for the CC-space, which equals the positive cone of $\mathbb{R}^{\boldsymbol{d}}$ with the lateral order, and the preference relation equals the lexicographical order.

Recall that the lexicographical order $\le_n$ on $\mathbb{R}^d$, $d \in \mathbb{N}$ is defined recursively as follows: $\le_1$ coincides with the natural order on $\mathbb{R}$, and for every two elements $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$ we set $\mathbf{x} <_d \mathbf{y}$ if and only if either $x_1 < y_1$, or $x_1 = y_1$ and $(x_2, \ldots, x_d) <_{d-1} (y_2, \ldots, y_d)$. Then, as usual, $\mathbf{x} \le_d \mathbf{y}$ means that either $\mathbf{x} <_d \mathbf{y}$, or $\mathbf{x} = \mathbf{y}$.

Now consider the positive cone of $\mathbb{R}^d$, that is, $G = (\mathbb{R}^d)^+ = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \ge 0\}$, where by $\le$ we denote the usual lattice order on $\mathbb{R}^d$, where $\mathbf{x} \le \mathbf{y}$ means that $x_k \le y_k$ for all $k \in \{1, \ldots, d\}$. Then $G$ is a CC-space with respect to the lateral order $\mathbf{x} \sqsubseteq \mathbf{y}$ if and only if $\mathbf{x} \perp (\mathbf{y} - \mathbf{x})$, that is, $|\mathbf{x}| \wedge |\mathbf{y} - \mathbf{x}| = 0$.

Let $P \in \mathcal{A}_b^+(G, \mathbb{R})$ be any positive price and $(\mathbf{e}_k)_{k=1}^d$ be the unit vector basis of $\mathbb{R}^d$, that is, $\mathbf{e}_k = (\underbrace{0, \ldots 0}_{k-1}, 1, 0, \ldots 0)$. Observe that every commodity vector $\mathbf{x} = (x_1, \ldots, x_d) \in G$ is represented as the direct sum $\mathbf{x} = (x_1 \cdot \mathbf{e}_1) \oplus \ldots \oplus (x_d \cdot \mathbf{e}_d)$. Hence, the value of the commodity vector $\mathbf{x}$ equals $P(\mathbf{x}) = \sum_{k=1}^d P(x_k \cdot \mathbf{e}_k)$. Now define the *coordinate functions* $\varphi_k^P : \mathbb{R}^+ \to \mathbb{R}^+$ of a price $P$ by setting

$$\varphi_k^P(t) = P(t \cdot \mathbf{e}_k), \quad k \in \{1, \dots, d\}.$$

Actually, we have proved the following elementary statement.

**Proposition 2.** *The value of any commodity vector* $\mathbf{x} = (x_1, \dots, x_d) \in (\mathbb{R}^d)^+$ *under a price* $P \in \mathcal{A}_b^+((\mathbb{R}^d)^+, \mathbb{R})$ *equals*

$$P(\mathbf{x}) = \sum_{k=1}^d \varphi_k^P(x_k),$$

*where* $(\varphi_k^P)_{k=1}^d$ *are coordinate functions of* $P$.

The following proposition is easy to prove.

**Proposition 3.** *(1) A price* $P \in \mathcal{A}_b^+\big((\mathbb{R}^d)^+, \mathbb{R}\big)$ *is strictly positive if and only if all coordinate functions* $\varphi_k$ *of* $P$ *are strictly positive, that is, for every* $t > 0$ *one has* $\varphi_k(t) > 0$.

*(2) Functions* $\varphi_k \colon \mathbb{R}^+ \to \mathbb{R}$, $k \in \{1, \dots, d\}$ *are coordinate functions of some price* $P \in \mathcal{A}_b^+\big((\mathbb{R}^d)^+, \mathbb{R}\big)$ *if and only if* $\varphi_k(0) = 0$ *for all* $k$.

**Theorem 3.** *Let* $P \gg 0$ *be a price in* $\mathcal{A}_b\big((\mathbb{R}^d)^+, \mathbb{R}\big)$ *with coordinate functions* $(\varphi_k^P)_{k=1}^d$ *such that* $\varphi_1^P$ *is monotone and continuous. Then for every* $\lambda > 0$ *the following assertions are equivalent:* (1) *the budget set* $\mathcal{B}_\lambda(P)$ *has a demand vector;* (2) $\lambda \in (0, \mu)$, *where* $\mu := \sup \varphi_1^P(\mathbb{R}^+) \in [0, +\infty]$ *and* $\varphi_1^P$ *is the first coordinate function of* $P$.

*Moreover, the demand vector of* $\mathcal{B}_\lambda(P)$ *has the form* $\mathbf{x}_0 = (t_0, 0, \dots, 0)$ *in case of existence.*

*Proof.* Since $\varphi_1^P(0) = 0$, the monotonicity assumption on $\varphi_1^P$ means that it is (non-strictly) increasing.

(2) $\Rightarrow$ (1). Fix any $\lambda \in (0, \mu)$. By the continuity of $\varphi_1^P$, the set $\{t > 0 : \varphi_1^P(t) = \lambda\}$ is nonempty and compact, and so contains a greatest element $t_0$. Show that the bundle vector $\mathbf{x}_0 = (t_0, 0, \dots, 0)$ is the demand vector of $\mathcal{B}_\lambda(P)$. Assume, on the contrary, that there exists $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^+$ such that $\mathbf{x}_0 <_d \mathbf{x}$ and $P(\mathbf{x}) \le \lambda$. If $t_0 < x_1$ then $\lambda = \varphi_1^P(t_0) < \varphi_1^P(x_1) \le P(\mathbf{x})$ by the choice of $t_0$, a contradiction. Now let $t_0 = x_1$ and $(0, \dots, 0) <_{d-1} (x_2, \dots, x_d)$. Choose $k \in \{2, \dots, d\}$ so that $x_k > 0$. Since $P \gg 0$, the latter inequality implies $\varphi_k^P(x_k) > 0$. Then $P(\mathbf{x}) \ge \varphi_1^P(x_1) + \varphi_k^P(x_k) > \varphi_1^P(x_1) = \varphi_1^P(t_0) = \lambda$, a contradiction. Thus, $\mathbf{x}_0$ is the demand vector of $\mathcal{B}_\lambda(P)$.

(2) $\Rightarrow$ (1). Assume, on the contrary, that $\lambda \ge \mu$ (in particular, it follows that $\mu < \infty$). Let $\mathbf{x}_0 = (x_1^0, \dots, x_d^0)$ be the demand vector of $\mathcal{B}_\lambda(P)$. Consider cases.

*Case 1:* $x_2^0 = \ldots = x_d^0 = 0$. Then for $\mathbf{x}_1 := (x_1^0 + 1, 0, \dots, 0)$ we obtain $P(\mathbf{x}_1) = \varphi_1^P(x_1^0 + 1) \le \mu \le \lambda$ and $\mathbf{x}_0 <_d \mathbf{x}_1$, which contradicts the choice of $\mathbf{x}_0$.

*Case 2:* there exists $k > 1$ such that $x_k > 0$. Set $\varepsilon := \varphi_k^P(x_k)$. Since $P \gg 0$, one has $\varepsilon > 0$. Choose by the continuity of $\varphi_1^P$ a number $\delta > 0$ such that for every $t \ge 0$ if $|t - x_1^0| < \delta$ then $|\varphi_1^P(t) - \varphi_1^P(x_1^0)| < \varepsilon$. Now set $t_0 := x_1^0 + \delta/2$. Then $\varphi_1^P(x_1^0) \le \varphi_1^P(t_0) < \varphi_1^P(x_1^0) + \varepsilon$. Hence for $\mathbf{x}_1 := (t_0, x_2^0, \dots, x_{k-1}^0, 0, x_{k+1}^0, \dots, x_d^0)$ we obtain

$$P(\mathbf{x}_1) - P(\mathbf{x}_0) = \varphi_1^P(t_0) - \varphi_1^P(x_1^0) - \varphi_k^P(x_k^0) < \varepsilon - \varphi_k^P(x_k^0) = 0.$$

Therefore, $P(\mathbf{x}_1) \le P(\mathbf{x}_0) \le \lambda$. On the other hand, the inequality $x_1^0 < t_0$ yields $\mathbf{x}_0 <_d \mathbf{x}_1$, which contradicts the choice of $\mathbf{x}_0$.

Finally, the proof of implication (2) $\Rightarrow$ (1) contains an argument showing that any demand vector $\mathcal{B}_\lambda(P)$ has the form $\mathbf{x}_0 = (t_0, 0, \dots, 0)$. $\square$

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