УДК 517.98, 517.5

Ya. Mykytyuk, N. Sushchyk, D. Lukivska

ON THE DUAL SPACE OF A BANACH SPACE OF ENTIRE FUNCTIONS

Ya. Mykytyuk, N. Sushchyk, D. Lukivska. *On the dual space of a Banach space of entire functions*, Mat. Stud. **62** (2024), 155–167.

Let $\mathcal{L}_1$ denote the subspace of $L_1(\mathbb{R})$ consisting of the restrictions to $\mathbb{R}$ of entire functions of exponential type at most π, equipped with the $L_1(\mathbb{R})$-norm. In this paper, we describe the dual space $\mathcal{L}_1'$, showing that it is isomorphic to the Banach space $\mathrm{BMO}(\mathbb{Z})$ of sequences $x\colon \mathbb{Z} \to \mathbb{C}$ with bounded mean oscillation on $\mathbb{Z}$. This result is an analogue of Fefferman's classical description of the dual of the Hardy space $H_1(\mathbb{C}_+)$ of functions analytic in the upper half-plane. A central role in the construction of $\mathcal{L}_1'$ is played by the discrete Hilbert transform.

1. Introduction. Let $\mathcal{E}$ denote the linear space of all entire functions, and let $\mathcal{B}_\sigma$ $(\sigma > 0)$ be the subspace of functions $f \in \mathcal{E}$ such that

$$\sup_{x,y\in\mathbb{R}} |f(x+iy)|e^{-\sigma|y|} < \infty.$$

The space $\mathcal{B}_\sigma$ becomes a Banach space with the norm

$$\|f\|_{\mathcal{B}_\sigma} := \sup_{x,y\in\mathbb{R}} |f(x+iy)|e^{-\sigma|y|}, \quad f \in \mathcal{B}_\sigma.$$

Note that for every $a, \sigma > 0$, the linear mapping

$$(I_a f)(z) = f(az), \quad z \in \mathbb{C},$$

is a bijection from the linear space $\mathcal{B}_\sigma$ to $\mathcal{B}_{a\sigma}$. Therefore, to study all possible spaces $\mathcal{B}_\sigma$, $\sigma > 0$, it suffices to consider the space $\mathcal{B}_\pi$. Note that the entire functions $\sin \pi z$ and $\cos \pi z$ belong to the space $\mathcal{B}_\pi$.

Denote by $\mathcal{L}_p$ the subset of $L_p(\mathbb{R})$ consisting of the restrictions to $\mathbb{R}$ of functions in $\mathcal{B}_\pi$. Equipped with the $L_p(\mathbb{R})$-norm, $\mathcal{L}_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$ ([1]) and thus forms a Banach space. These spaces have been extensively studied, with most important results presented in the monograph by B. Ya. Levin ([1]). In particular, for $p \in (1, \infty)$, the space $\mathcal{L}_p$ is isomorphic to the Banach space $\ell_p := \ell_p(\mathbb{Z})$, with an isomorphism given by the linear mapping J defined by

$$(Jf)(n) := (-1)^n f(n), \quad n \in \mathbb{Z}. \tag{1}$$

The extreme spaces $\mathcal{L}_1$ and $\mathcal{L}_\infty$ are special and not isomorphic to the Banach spaces ℓ_1 and ℓ_∞, respectively; their descriptions in terms of the discrete Hilbert operator are suggested in [1,2]. In the monograph [1], there is no description of the dual space of $\mathcal{L}_1$, and we have not found it in the available literature. However, the question of the dual space of $\mathcal{L}_1$ is natural and analogous to the question of the dual space of the Hardy space $H_1(\mathbb{C}_+)$ of functions analytic in the upper half-plane ([5]). Indeed, the space $H_1(\mathbb{C}_+)$, like the space $\mathcal{L}_1$, can be

2020 *Mathematics Subject Classification:* 30H35,46B10, 46E10.
Keywords: Banach spaces; entire functions; discrete Hilbert transfor.
doi:10.30970/ms.62.2.155-167

identified with a closed subspace of $L_1(\mathbb{R})$. The well-known result of Fefferman ([5]) identifies the dual space $(H_1(\mathbb{C}_+))'$ with the space $\mathrm{BMO}(\mathbb{R})$ of functions of bounded mean oscillation on $\mathbb{R}$ via the Hilbert operator H. Likewise, we use the discrete Hilbert operator $\mathcal{H}$ to identify the dual space $\mathcal{L}_1'$ with the space $\mathrm{BMO}(\mathbb{Z})$ of sequences of bounded mean oscillation on $\mathbb{Z}$. In the case of the dual space $\mathcal{L}_1'$, the situation is similar. Here, the discrete Hilbert operator $\mathcal{H}$ naturally appears, acting on sequences $x\colon \mathbb{Z} \to \mathbb{C}$, along with the space $\mathrm{BMO}(\mathbb{Z})$ of sequences with bounded mean oscillation on $\mathbb{Z}$.

The purpose of this work is to study and describe the space $\mathcal{L}_1'$. The main result of this paper is the following theorem.

Theorem 1. *The space $\mathcal{L}_1'$ is isomorphic to the space* $\mathrm{BMO}(\mathbb{Z})$.

The paper is organized as follows. In Section 2, we study the action of the discrete Hilbert transform on the special Hilbert spaces G_+ and G_-. Section 3 explores the relationships between several auxiliary Banach spaces of sequences. In Section 4, we characterize the space $\mathrm{BMO}(\mathbb{Z})$ in terms of the discrete Hilbert transform. Finally, Section 5 presents the proof of Theorem 1, while the Appendix includes relevant definitions from Banach space theory.

2. The discrete Hilbert transform. There are several definitions of the discrete Hilbert transform acting in the spaces ℓ_p, $p \in (1, \infty)$. However, all of them describe operators that are variations of the operator introduced by D. Hilbert [2] and defined by the formula

$$(\mathcal{H}_0\, x)(n) := \frac{1}{\pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{x(n-k)}{k}, \quad n \in \mathbb{Z}, \quad x \in \ell_p.$$

In this work, we define the discrete Hilbert transform as the operator acting in ℓ_p according to the formula

$$(\mathcal{H}\, x)(n) := \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \frac{x(k)}{n-k+1/2}, \quad n \in \mathbb{Z}. \tag{2}$$

This operator is also called (see [2]) the Riesz-Titchmarsh operator. The advantage of the operator $\mathcal{H}$ over $\mathcal{H}_0$ is the existence of a continuous inverse operator in all spaces ℓ_p, $p \in (1, \infty)$. In fact, the following statement holds (see [2], [6]).

Proposition 1. *The operator $\mathcal{H}\colon \ell_p \to \ell_p$ for $p \in (1, \infty)$ is a linear homomorphism, and*

$$\mathcal{H}^2 = -S, \tag{3}$$

where S is the shift operator given by the formula $(Sx)(n) := x(n+1)$, $n \in \mathbb{Z}$. Moreover, the operator $\mathcal{H}\colon \ell_2 \to \ell_2$ is unitary.

Remark 1. To simplify notations, we denote norms in the spaces ℓ_p in the same way as norms in the spaces $L_p(\mathbb{R})$. Specifically, if $x \in \ell_p$ ($x \in \ell_\infty$), then

$$\|x\|_p := \left(\sum_{n\in\mathbb{Z}} |x(n)|^p\right)^{1/p}, \quad \|x\|_\infty := \sup_{n\in\mathbb{Z}} |x(n)|.$$

Let $\ell_{2,+}$ and $\ell_{2,-}$ denote the Hilbert spaces

$$\left\{x \in \mathbb{C}^{\mathbb{Z}} \colon \|x\|_{2,\pm} < \infty\right\}, \quad \|x\|_{2,\pm} := \left(\sum_{n\in\mathbb{Z}} (1+n^2)^{\pm 1} |x(n)|^2\right)^{1/2}.$$

It is clear that $\ell_{2,+} \subset \ell_1$, $\ell_\infty \subset \ell_{2,-}$. This implies that the sequence $d(j) \equiv 1$ ($j \in \mathbb{Z}$) belongs to the space $\ell_{2,-}$ and the functional $F_d(x) := \sum_{j\in\mathbb{Z}} x(j)$, $x \in \ell_1$, is continuous in the spaces $\ell_{2,+}$ and ℓ_1.

Let us consider the closed subspaces in $\ell_{2,+}$ and $\ell_{2,-}$

$$G_+ := \{x \in \ell_{2,+} \colon F_d(x) = 0\}, \quad G_- := \{y \in \ell_{2,-} \colon y(0) = 0\},$$

which are Hilbert spaces with the norms

$$\|x\|_+ := \|x\|_{2,+} \ (x \in G_+), \quad \|y\|_- := \|y\|_{2,-} \ (y \in G_-),$$

respectively. Denote by $(e_n)_{n\in\mathbb{Z}}$, $e_n = (e_n(1), \dots, e_n(j), \dots) = (\underbrace{0, \dots 0}_{n-1}, 1, 0, \dots)$ the standard basis in the space ℓ_2, i.e., $e_n(j) = 0$ if $j \neq n$, and $e_n(n) = 1$.

Denote by Φ the linear span of the set $\Phi_0 := \{e_j - e_{j+1} \colon j \in \mathbb{Z}\}$. Clearly,

$$\Phi_0 \subset \Phi \subset G_+ \subset \ell_1 \subset \ell_2.$$

Proposition 2. *The bilinear form* $G_+ \times G_- \ni (x, y) \mapsto \langle x, y\rangle := \sum_{j\in\mathbb{Z}} x(j)y(j)$ *is continuous and*

$$|\langle x, y\rangle| \le \|x\|_+ \|y\|_-, \quad x \in G_+, \quad y \in G_-. \tag{4}$$

Moreover:

(I) for every $y \in G_-$ *the formula* $F_y(x) := \langle x, y\rangle$, $x \in G_+$, *defines a functional* $F_y \in G'_+$, *in particular,* $\|F_y\| \le \|y\|_-$;

(II) if $y \in G_-$ *and* $\ker F_y \supset \Phi_0$, *then* $y = 0$;

(III) if $y \in G_- \setminus \{0\}$, *then* $F_y \neq 0$;

(IV) for every $F \in G'_+$, *there exists the unique* $y \in G_-$ *such that* $F = F_y$.

Proof. Continuity of the bilinear form $\langle\cdot,\cdot\rangle$ and the estimate (4) follow directly. This also implies (I).

Let us prove (II). If $y \in G_-$ and $\ker F_y \supset \Phi_0$, then

$$0 = F_y(e_j - e_{j+1}) = y(j) - y(j+1), \quad j \in \mathbb{Z}.$$

Therefore, $y = cd$, where $c \in \mathbb{C}$. Since $y \in G_-$, we have $0 = y(0) = c$, implying $y = 0$. The statement (II) yields (III).

Now we prove (IV). Let $F \in G'_+$. By the Hahn-Banach theorem, F can be extended to a functional $\widetilde{F} \in (\ell_{2,+})'$. By Riesz's theorem, there exists $u \in \ell_{2,-}$ such that

$$\widetilde{F}(x) = \sum_{j\in\mathbb{Z}} x(j)u(j), \quad x \in \ell_{2,+}.$$

Let $y = u - u(0)d$. Then $y \in G_-$ and for any $x \in G_+$ we have

$$F_y(x) = \sum_{j\in\mathbb{Z}} x(j)y(j) = \widetilde{F}(x) - u(0)F_d(x) = F(x),$$

thus $F = F_y$. To show uniqueness, assume $y_1 \in G_-$ such that $F = F_y = F_{y_1}$. Then $F_{y-y_1} = 0$. Hence, in view of (II), $y - y_1 = 0$, i.e. $y = y_1$. □

As shown below, the operator $\mathcal{H}$ maps G_+ into itself. However, the operator $\mathcal{H}$ does not act on the space G_-. Instead, there is a one-dimensional perturbation of $\mathcal{H}$, denoted $\mathring{\mathcal{H}}$, that maps $\ell_{2,-}$ (or G_-) into itself. This operator acts according to the formula

$$(\mathring{\mathcal{H}}x)(n) := \frac{1}{\pi} \sum_{k\in\mathbb{Z}} x(k) \left(\frac{1}{n - k + 1/2} + \frac{1}{k - 1/2} \right), \quad n \in \mathbb{Z}, \quad x \in \ell_{2,-}.$$

Theorem 2. *The operator* $\mathcal{H}$ *is a linear homeomorphism of the space* G_+ *onto itself, and* $\overset{\circ}{\mathcal{H}}$ *is a linear homeomorphism of the space* G_- *onto itself. Moreover,*

$$\langle \mathcal{H}^{-1}x, y\rangle = \langle x, \overset{\circ}{\mathcal{H}}y\rangle \quad (x \in G_+,\ y \in G_-), \tag{5}$$

$$\mathcal{H}^{-1}x = -\mathcal{H}S^{-1}x \quad (x \in G_+), \qquad (\overset{\circ}{\mathcal{H}})^{-1}y = -\overset{\circ}{\mathcal{H}}S^{-1}y \quad (y \in G_-). \tag{6}$$

First, we prove several auxiliary lemmas.

Lemma 1. *The set* Φ *is everywhere dense in* G_+*.*

Proof. Assume that Φ is not everywhere dense in G_+. Then there exists a nonzero functional $F \in G'_+$ such that $\ker F \supset \Phi$. Thus, $\ker F \supset \Phi_0$ and, according to points (IV) and (II) of Proposition 2, it follows that $F = 0$. This leads to a contradiction. Therefore, Φ is everywhere dense in G_+. □

Lemma 2. *Let* $x \in \ell_1$ *and* $\mathcal{H}x \in \ell_1$*. Then* $F_d(x) = F_d(\mathcal{H}x) = 0$*.*

Proof. Let $x \in \ell_1$ and $y = \mathcal{H}x \in \ell_1$. The functions

$$\psi(t) = \sum_{k\in\mathbb{Z}} x(k)e^{ikt}, \qquad \varphi(t) = \sum_{k\in\mathbb{Z}} y(k)e^{ikt}, \quad t \in [-\pi, \pi],$$

are continuous on the interval $[-\pi, \pi]$. It is easy to verify the equality

$$\varphi(t) = \psi(t)\theta(t), \quad t \in (-\pi, 0) \cup (0, \pi),$$

where

$$\theta(t) = \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{e^{imt}}{m + 1/2} = i\,\mathrm{sign}(t)\cdot e^{-it/2}.$$

Since the function θ is discontinuous at the origin, the above formula implies that $\phi(0) = \psi(0) = 0$, and thus $F_d(x) = \psi(0) = 0$, $F_d(y) = \phi(0) = 0$. □

Proof of Theorem 2. Let M be the operator acting in the space of sequences $\mathbb{C}^{\mathbb{Z}}$ defined by

$$(Mx)(k) := kx(k), \quad k \in \mathbb{Z}.$$

It is easy to see that the operators $(M \pm \frac{1}{4}I)$ homeomorphically map the space $\ell_{2,+}$ to ℓ_2 and

$$\mathcal{H}(M - \tfrac{1}{4}I)x - (M + \tfrac{1}{4}I)\mathcal{H}x = -\tfrac{1}{\pi}F_d(x)d = 0, \quad x \in G_+.$$

This implies that for every $x \in G_+$

$$\|\mathcal{H}x\|_{2,+} = \|(M + \tfrac{1}{4}I)^{-1}\mathcal{H}(M - \tfrac{1}{4}I)x\|_{2,+} \le c_1c_2\|x\|_+, \tag{7}$$

where $c_1 = \|(M + \frac{1}{4}I)^{-1}\|_{\ell_2\to\ell_{2,+}}$, $c_2 = \|M - \frac{1}{4}I\|_{\ell_{2,+}\to\ell_2}$. Here, we also take into account that $\|\mathcal{H}\|_{\ell_2\to\ell_2} = 1$. Thus, the operator $\mathcal{H}$ continuously acts from G_+ to $\ell_{2,+}$. Since

$$G_+ \subset \ell_{2,+} \subset \ell_1,$$

in view of Lemma 2, we obtain that $\mathcal{H}G_+ \subset G_+$. Thus, the operator $\mathcal{H}$ continuously maps G_+ to G_+. Applying the equality (3), we see that operator $\mathcal{H}\colon G_+ \to G_+$ has a continuous inverse operator $\mathcal{H}^{-1}$ and

$$\mathcal{H}^{-1} = -\mathcal{H}S^{-1}. \tag{8}$$

Now, let us consider the operator $\overset{\circ}{\mathcal{H}}$. It follows from the definition that

$$\overset{\circ}{\mathcal{H}}x = M\mathcal{H}(M - \tfrac{1}{2}I)^{-1}x, \quad x \in G_-.$$

The operator $(M - \frac{1}{2}I)^{-1}$ homeomorphically maps the space $\ell_{2,-}$ into the space ℓ_2, and therefore $\|\overset{\circ}{\mathcal{H}}x\|_- \le \widetilde{c}_1\widetilde{c}_2\|x\|_{2,-}$, where $\widetilde{c}_1 = \|M\|_{\ell_2\to\ell_{2,-}}$, $\widetilde{c}_2 = \|(M - \frac{1}{2}I)^{-1}\|_{\ell_{2,-}\to\ell_2}$. Since $(\overset{\circ}{\mathcal{H}}x)(0) = 0$ for $x \in \ell_{2,-}$, we conclude that the operator $\overset{\circ}{\mathcal{H}}\colon G_- \to G_-$ is continuous.

Let $\widetilde{G}_-$ be the set of finitely supported sequences $y \in G_-$ and note that $\widetilde{G}_-$ is everywhere dense in G_-. Let $x \in \Phi$, $y \in \widetilde{G}_-$. Since $F_d(x) = 0$, we have

$$\langle x, \overset{\circ}{\mathcal{H}} y\rangle = \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x(n)y(k)\Big(\frac{1}{n-k+1/2} + \frac{1}{k-1/2}\Big) = \\ = \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{x(n)y(k)}{n-k+1/2} = -\langle S^{-1}\mathcal{H}x, y\rangle.$$

Hence, in view of (8), we get

$$\langle x, \overset{\circ}{\mathcal{H}} y\rangle = \langle \mathcal{H}^{-1}x, y\rangle, \quad x \in \Phi, \quad y \in \widetilde{G}_-. \tag{9}$$

Since the bilinear form $\langle\cdot,\cdot\rangle$ is continuous on $G_+\times G_-$, and both $\overset{\circ}{\mathcal{H}}$ and $\mathcal{H}^{-1}$ are continuous in the spaces G_- and G_+, respectively, and taking into account that Φ is everywhere dense in G_+ and $\widetilde{G}_-$ is everywhere dense in G_-, from (9), we obtain the equality

$$\langle x, \overset{\circ}{\mathcal{H}} y\rangle = \langle \mathcal{H}^{-1}x, y\rangle, \quad x \in G_+, \quad y \in G_-.$$

Thus, applying (3), we have

$$\langle x, (\overset{\circ}{\mathcal{H}})^2 y\rangle = \langle \mathcal{H}^{-2}x, y\rangle = -\langle S^{-1}x, y\rangle = -\langle x, Sy\rangle, \quad x \in G_+, \quad y \in G_-.$$

Therefore, taking into account point (IV) of Proposition 2, we have $(\overset{\circ}{\mathcal{H}})^2 = -S$. The operator S homeomorphically maps the space G_- (G_+) onto itself, so the operator $\overset{\circ}{\mathcal{H}}\colon G_- \to G_-$ is a linear homeomorphism, and $(\overset{\circ}{\mathcal{H}})^{-1} = -\overset{\circ}{\mathcal{H}} S^{-1}$. □

3. The special spaces X and Y. Let us consider the Banach spaces

$$X_0 := \{x \in \ell_1 \colon F_d(x) = 0\}, \quad X_1 := \{x \in \ell_1 \colon \mathcal{H}^{-1}x \in X_0\}, \\ Y_0 := \{y \in \ell_\infty \colon y(0) = 0\}, \quad Y_1 := \{y \in G_- \colon (\overset{\circ}{\mathcal{H}})^{-1}y \in Y_0\}, \tag{10}$$

which are equipped with the norms

$$\|x\|_{X_0} := \|x\|_1,\ x \in X_0; \quad \|x\|_{X_1} := \|\mathcal{H}^{-1}x\|_{X_0},\ x \in X_1, \\ \|y\|_{Y_0} := \|y\|_\infty,\ y \in Y_0; \quad \|y\|_{Y_1} := \|(\overset{\circ}{\mathcal{H}})^{-1}y\|_{Y_0},\ y \in Y_1. \tag{11}$$

As we can see, there is a close connection between the spaces X_j and Y_j.

Lemma 3. *(I) The operator $\mathcal{H}$ isometrically maps the space X_0 to X_1.*

(II) The operator $\overset{\circ}{\mathcal{H}}$ isometrically maps the space Y_0 to Y_1.

(III) The topological embeddings $Y_0 \subset G_-$ and $Y_1 \subset G_-$ hold.

(IV) The space G_+ is topologically and everywhere densely embedded in the spaces X_0 and X_1.

Proof. (I) It follows from the definitions that $\mathcal{H}\colon X_0 \to X_1$ is a bijection and $\|\mathcal{H}x\|_{X_1} = \|x\|_{X_0}$, $x \in X_0$. Therefore, the operator $\mathcal{H}$ isometrically maps the space X_0 to X_1.

(II) According to Theorem 2, the operator $\overset{\circ}{\mathcal{H}}\colon G_- \to G_-$ is a bijection, and by definition $\|\overset{\circ}{\mathcal{H}}y\|_{Y_1} = \|y\|_{Y_0}$, $y \in Y_0$. Therefore, $\overset{\circ}{\mathcal{H}}$ isometrically maps Y_0 to Y_1.

(III) Let $c = \big(\sum_{n\in\mathbb{Z}}(1+n^2)^{-1}\big)^{1/2}$. Since $\|y\|_- \le c\|y\|_\infty = c\|y\|_{Y_0}$, $y \in Y_0$, we have

$$\|y\|_- \le c_1\|(\overset{\circ}{\mathcal{H}})^{-1}y\|_- \le cc_1\|(\overset{\circ}{\mathcal{H}})^{-1}y\|_{Y_0} = cc_1\|y\|_{Y_1}, \quad y \in Y_1,$$

where $c_1 = \|\overset{\circ}{\mathcal{H}}\|_{G_-\to G_-}$. Thus, the embeddings $Y_0 \subset G_-$ and $Y_1 \subset G_-$ are topological.
(IV) Since $G_+ \subset \ell_1$ and $F_d(x) = 0$ for all $x \in G_+$, it follows that $G_+ \subset X_0$. Hence, $\mathcal{H}G_+ \subset X_1$. According to Theorem 2, $\mathcal{H}G_+ = G_+$, and therefore, $G_+ \subset X_1$.

Using the Cauchy-Schwarz inequality, we obtain $\|x\|_{X_0} = \|x\|_1 \le c\|x\|_+$, $x \in G_+$, where $c = \left(\sum_{n\in\mathbb{Z}}(1+n^2)^{-1}\right)^{1/2}$. From this, it follows that $\|x\|_{X_1} = \|\mathcal{H}^{-1}x\|_{X_0} \le c\|\mathcal{H}^{-1}x\|_+ \le cc_2\|x\|_+$, $x \in G_+$, where $c_2 = \|\mathcal{H}^{-1}\|_{G_+\to G_+}$. Hence, the embeddings $G_+ \subset X_0$ and $G_+ \subset X_1$ are topological.

Let $x \in X_0$. For an arbitrary $n \in \mathbb{N}$, we define

$$x_n(j) := \begin{cases} x(j), & \text{if } |j| \le n; \\ 0, & \text{if } |j| > n, \end{cases} \qquad u_n := x_n + F_d(x - x_n)e_0.$$

It is easy to see that $u_n \in G_+$ and $\|x - u_n\|_1 \to 0$ as $n \to \infty$. Thus, G_+ is everywhere dense in X_0. Consequently, according to (I), the set $\mathcal{H}G_+$ is everywhere dense in X_1. Since $\mathcal{H}G_+ = G_+$, it follows that G_+ is everywhere dense in X_1. □

Lemma 4. *If $F_j \in X_j'$, then there exists $y_j \in Y_j$ such that $F_j(x) = \langle x, y_j\rangle$, $x \in G_+$ $(j = 0, 1)$.*

Proof. Let $F_0 \in X_0'$. Since X_0 is a subspace of ℓ_1, by the Hahn-Banach theorem, F_0 can be extended to a continuous functional on ℓ_1. Therefore, there exists $u \in \ell_\infty$ such that $F_0(x) = \langle x, u\rangle$, $x \in G_+$. Put $y_0 = u - u(0)d$. Clearly, $y_0 \in Y_0$. Since $\langle x, d\rangle = 0$ for all $x \in G_+$, we have $F_0(x) = \langle x, u\rangle = \langle x, y_0\rangle - u(0)\langle x, d\rangle = \langle x, y_0\rangle, \quad x \in G_+$.

Let $F_1 \in X_1'$. Since the operator $\mathcal{H}\colon X_0 \to X_1$ is an isometry, the functional

$$F(x) := F_1(\mathcal{H}x), \quad x \in X_0,$$

is continuous on X_0. Thus, there exists $u \in Y_0$ such that $F(x) = \langle x, u\rangle$, $x \in G_+$.

Let $y_1 = \overset{\circ}{\mathcal{H}}u$. Then $y_1 \in Y_1$. Taking into account (9), we obtain $F_1(x) = F(\mathcal{H}^{-1}x) = \langle \mathcal{H}^{-1}x, u\rangle = \langle x, \overset{\circ}{\mathcal{H}}u\rangle = \langle x, y_1\rangle$, $x \in G_+$. □

Let X denote the *intersection of the Banach spaces* X_0 and X_1, and let Y denote the *sum of the Banach spaces* Y_0 and Y_1, i.e. (see the Appendix) $X = X_0 \cap X_1$, $Y = Y_0 + Y_1$,

$$\|x\|_X := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}, \quad x \in X,$$
$$\|y\|_Y = \inf\{\|y_0\|_{Y_0} + \|y_1\|_{Y_1} : y_0 \in Y_0,\ y_1 \in Y_1,\ y = y_0 + y_1\}, \quad y \in Y.$$

It follows from (11) that $\|x\|_{X_1} = \|\mathcal{H}^{-1}x\|_1$, $x \in X_1$, and from (3) we get $\mathcal{H} = -S\mathcal{H}^{-1}$. Since the operator $S\colon \ell_1 \to \ell_1$ is an isometry, we have $\|\mathcal{H}x\|_1 = \|S\mathcal{H}^{-1}x\|_1 = \|\mathcal{H}^{-1}x\|_1$, and thus,

$$\|x\|_X = \max\{\|x\|_1, \|\mathcal{H}^{-1}x\|_1\} = \max\{\|x\|_1, \|\mathcal{H}x\|_1\}, \quad x \in X. \tag{12}$$

Theorem 3. *The space X' is isomorphic to the space Y.*

Proof. Let $F \in X'$. We will show that there exists $y \in Y$ such that $F(x) = F_y(x)$, $x \in G_+$. Lemma 3 yields that $G_+ \subset X_0 \cap X_1$ and G_+ is everywhere dense in both X_0 and X_1. By Theorem 6 (see the Appendix), $X' = (X_0 \cap X_1)' = X_0' + X_1'$. Thus, there exist functionals $F_j \in X_j'$ $(j = 0, 1)$ such that $F(x) = F_0(x) + F_1(x)$, $x \in X_0 \cap X_1$. Lemma 4 implies that there exist $y_j \in Y_j$ such that $F_j(x) = \langle x, y_j\rangle$, $x \in G_+$ $(j = 0, 1)$. Therefore,

$$F(x) = \langle x, y\rangle = F_y(x), \quad x \in G_+,$$

where $y = (y_0 + y_1) \in Y$.

Let us show that for an arbitrary $y \in Y$ the functional $F_y \in G_+'$ can be uniquely extended to a functional $\overline{F}_y \in X'$. Indeed, if $y \in Y$, then $y = y_0 + y_1$, where $y_0 \in Y_0$ and $y_1 \in Y_1$.

Therefore, $F_y(x) = \langle x, y_0\rangle + \langle x, y_1\rangle$, $x \in G_+$. Using (10) and (11), we obtain $y_1 = \overset{\circ}{\mathcal{H}}u$, where $u \in Y_0$, moreover

$$\|y_1\|_{Y_1} = \|\overset{\circ}{\mathcal{H}}u\|_{Y_1} = \|u\|_{Y_0}.$$

Thus (see (5)), $F_y(x) = \langle x, y_0\rangle + \langle x, \overset{\circ}{\mathcal{H}}u\rangle = \langle x, y_0\rangle + \langle \mathcal{H}^{-1}x, u\rangle$, $x \in G_+$. Consequently, taking into account (12), for an arbitrary $x \in G_+$, we have

$$|F_y(x)| \le |\langle x, y_0\rangle| + |\langle \mathcal{H}^{-1}x, u\rangle| \le \|x\|_1\|y_0\|_\infty + \|\mathcal{H}^{-1}x\|_1\|u\|_\infty \le$$
$$\le \max\{\|x\|_1, \|\mathcal{H}^{-1}x\|_1\} \cdot (\|y_0\|_{Y_0} + \|u\|_{Y_0}) = \|x\|_X(\|y_0\|_{Y_0} + \|y_1\|_{Y_1}).$$

Since G_+ is everywhere dense in X, the functional F_y can be uniquely extended to $\overline{F}_y \in X'$ with

$$\|\overline{F}_y\| \le \inf\{\|y_0\|_{Y_0} + \|y_1\|_{Y_1} : y = y_0 + y_1, \;\; y_0 \in Y_0, \; y_1 \in Y_1\} = \|y\|_Y.$$

Now, consider the mapping $Y \ni y \mapsto \Gamma y := \overline{F}_y \in X'$. It follows from the above that Γ is a continuous surjection. Let us check that $\ker\Gamma = \{0\}$. Indeed, if $y \in \ker\Gamma$, then $F_y(x) = 0$, $x \in G_+$. Thus, by statement (III) of Proposition 2, we conclude $y = 0$. Therefore, the operator Γ is a continuous bijection. Consequently, by the Banach inverse theorem, Γ is a linear homeomorphism. Thus, $X' \sim Y$. □

4. The space $\mathrm{BMO}(\mathbb{Z})$**.** The spaces $\mathrm{BMO}(\mathbb{R}^n)$ of functions of bounded mean oscillation were introduced by John and Nirenberg in [3]. Similarly, one can introduce the spaces $\mathrm{BMO}(X)$ in the case when X is a measure space (see [4]). In this section, we describe the space $\mathrm{BMO}(\mathbb{Z})$ in terms of the discrete Hilbert transform. The main result is Theorem 5, which is an analogue of Fefferman's theorem (see [5]).

Let $\mathscr{I}$ be the set of all bounded intervals in $\mathbb{R}$ of positive length. For an arbitrary $f \in L_{1,\mathrm{loc}}(\mathbb{R})$ and an arbitrary $\mathcal{I} \in \mathscr{I}$, we put

$$f_{\mathcal{I}} := \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} f(t)\,dt, \quad f^*_{\mathcal{I}} := \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - f_{\mathcal{I}}|\,dt, \quad \|f\|_* := \sup_{\mathcal{I}\in\mathscr{I}} f^*_{\mathcal{I}}.$$

If $\|f\|_* < \infty$, then we say that f has *bounded mean oscillation*, $f \in \mathrm{BMO}(\mathbb{R})$. The value $\|f\|_*$ is the norm in $\mathrm{BMO}(\mathbb{R})$. Since constant functions have zero BMO-norm, we identify $f \in \mathrm{BMO}(\mathbb{R})$ with $f+\mathrm{const}$ and consider $\mathrm{BMO}(\mathbb{R})$ as a subset of the quotient space $L_{1,\mathrm{loc}}/C$, where C is the one-dimensional subspace of constant functions.

Let $\tilde{\mathscr{I}}$ be the set of all non-empty bounded intervals $\tilde{\mathcal{I}}$ in $\mathbb{Z}$. For an arbitrary sequence $\varphi\colon \mathbb{Z} \to \mathbb{C}$ and an artitrary $\tilde{\mathcal{I}} \in \tilde{\mathscr{I}}$, we define

$$\varphi_{\tilde{\mathcal{I}}} := \frac{1}{|\tilde{\mathcal{I}}|}\sum_{k\in\tilde{\mathcal{I}}} \varphi(k), \quad \varphi^*_{\tilde{\mathcal{I}}} := \frac{1}{|\tilde{\mathcal{I}}|}\sum_{k\in\tilde{\mathcal{I}}} |\varphi(k) - \varphi_{\tilde{\mathcal{I}}}| \quad (|\tilde{\mathcal{I}}| = \operatorname{card}\tilde{\mathcal{I}}), \quad \|\varphi\|_* := \sup_{\tilde{\mathcal{I}}\in\tilde{\mathscr{I}}} \varphi^*_{\tilde{\mathcal{I}}}.$$

If $\|\varphi\|_* < \infty$, then we say that φ has *bounded mean oscillation*, $\varphi \in \mathrm{BMO}(\mathbb{Z})$. The value $\|\varphi\|_*$ is the norm in $\mathrm{BMO}(\mathbb{Z})$. Since constant sequences have zero BMO-norm, we identify $\varphi \in \mathrm{BMO}(\mathbb{R})$ with $\varphi+\mathrm{const}$ and consider $\mathrm{BMO}(\mathbb{Z})$ as a subset of the quotient space $\ell_{1,\mathrm{loc}}/C$, where C is the one-dimensional subspace of constant sequences.

Remark 2. To avoid complicating the notation, we use the same symbols for similar objects in the definitions of the spaces $\mathrm{BMO}(\mathbb{R})$ and $\mathrm{BMO}(\mathbb{Z})$, in particular for norms. This should not lead to misunderstandings.

Remark 3. The formula $P_0\varphi := \varphi - \varphi(0)d$ $(\varphi \in \mathbb{C}^{\mathbb{Z}})$ defines a projector in the space $\mathbb{C}^{\mathbb{Z}}$. In particular, it projects the space ℓ_∞ onto Y_0 and $\|P_0\|_{\ell_\infty\to\ell_\infty} \le 2$. Moreover, if $\varphi \in \mathrm{BMO}(\mathbb{Z})$, then $P_0\varphi \in \mathrm{BMO}(\mathbb{Z})$ and $\|P_0\varphi\|_* \le 2\|\varphi\|_*$.

Let χ_k denote the characteristic function of the interval $\mathcal{I}_k := [k, k+1)$ for $k \in \mathbb{Z}$, and consider the linear operators

$$U\colon \mathbb{C}^{\mathbb{Z}} \to L_{1,\mathrm{loc}}(\mathbb{R}), \quad V\colon L_{1,\mathrm{loc}}(\mathbb{R}) \to \mathbb{C}^{\mathbb{Z}},$$

defined by the formulas

$$U\varphi := \sum_{n\in\mathbb{Z}} \varphi(n)\chi_n \quad (\varphi \in \mathbb{C}^{\mathbb{Z}}), \quad (Vf)(n) := f_{\mathcal{I}_n} \quad (n \in \mathbb{Z},\ f \in L_{1,\mathrm{loc}}(\mathbb{R})).$$

Proposition 3. *The operator U continuously maps* $\mathrm{BMO}(\mathbb{Z})$ *into* $\mathrm{BMO}(\mathbb{R})$, *and the operator V continuously maps* $\mathrm{BMO}(\mathbb{R})$ *into* $\mathrm{BMO}(\mathbb{Z})$, *satisfying*

$$\|Vf\|_* \le \|f\|_*, \quad f \in \mathrm{BMO}(\mathbb{R}), \tag{13}$$
$$\|U\varphi\|_* \le 6\|\varphi\|_*, \quad \varphi \in \mathrm{BMO}(\mathbb{Z}), \tag{14}$$
$$VU\varphi = \varphi, \qquad \varphi \in \mathrm{BMO}(\mathbb{Z}). \tag{15}$$

Proof. First, let us make a few remarks.
(a) Let $f \in \mathrm{BMO}(\mathbb{R})$, $\varphi = Vf$, and $[n,m] =: \mathcal{I} \in \mathscr{I}$, where $n, m \in \mathbb{Z}\,(n < m)$ and $\tilde{\mathcal{I}} := [n,m) \cap \mathbb{Z}$. Clearly, $f_{\mathcal{I}} = \varphi_{\tilde{\mathcal{I}}}$ and

$$|\varphi(k) - \varphi_{\tilde{\mathcal{I}}}| = \left|\int_{\mathcal{I}_k} (f(t) - f_{\mathcal{I}})\,dt\right| \le \int_{\mathcal{I}_k} |f(t) - f_{\mathcal{I}}|\,dt, \quad k \in \tilde{\mathcal{I}},$$

therefore,

$$\varphi^*_{\tilde{\mathcal{I}}} = \frac{1}{|\tilde{\mathcal{I}}|}\sum\nolimits_{k\in\tilde{\mathcal{I}}} |\varphi(k) - \varphi_{\tilde{\mathcal{I}}}| \le \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - f_{\mathcal{I}}|\,dt = f^*_{\mathcal{I}}.$$

(b) For an arbitrary $f \in L_{1,\mathrm{loc}}(\mathbb{R})$, $\alpha \in \mathbb{C}$ and $\mathcal{I} \in \mathscr{I}$

$$|f_{\mathcal{I}} - \alpha| = \left|\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} (f(t) - \alpha)\,dt\right| \le \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - \alpha|\,dt.$$

Thus,

$$f^*_{\mathcal{I}} = \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - f_{\mathcal{I}}|\,dt \le \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - \alpha|\,dt + \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |\alpha - f_{\mathcal{I}}|\,dt \le$$
$$\le |f_{\mathcal{I}} - \alpha| + \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - \alpha|\,dt \le \frac{2}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - \alpha|\,dt. \tag{16}$$

(c) If $k \in \mathbb{Z}$ and $\tilde{\mathcal{I}} = \{k, k+1\}$, then

$$\varphi_{\tilde{\mathcal{I}}} = \frac{\varphi(k) + \varphi(k+1)}{2} \quad \text{and} \quad \varphi^*_{\tilde{\mathcal{I}}} = \frac{|\varphi(k) - \varphi(k+1)|}{2}. \tag{17}$$

Let $f \in \mathrm{BMO}(\mathbb{R})$ and $\varphi = Vf$. From (a), it follows that $\varphi \in \mathrm{BMO}(\mathbb{Z})$ and $\|\varphi\|_* \le \|f\|_*$, thus, (13) holds.

Now we prove (14). Let $\varphi \in \mathrm{BMO}(\mathbb{Z})$, $f = U\varphi$ and $\mathcal{I} \in \mathscr{I}$. First, consider the case when $|\mathcal{I}| \le 1$. Then there exists $k \in \mathbb{Z}$ such that $\mathcal{I} \subset \mathcal{I}_k \cup \mathcal{I}_{k+1}$. Taking into account (16) and (17), we obtain

$$f^*_{\mathcal{I}} \le \frac{2}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - \varphi(k)|\,dt = \frac{2}{|\mathcal{I}|}\int_{\mathcal{I}\cap\mathcal{I}_{k+1}} |\varphi(k+1) - \varphi(k)|\,dt \le 2|\varphi(k+1) - \varphi(k)| = 4\varphi^*_{\tilde{\mathcal{I}}}.$$

Thus,

$$f^*_{\mathcal{I}} \le 4\varphi^*_{\tilde{\mathcal{I}}} \le 4\|\varphi\|_*, \quad \text{when} \quad |\mathcal{I}| \le 1. \tag{18}$$

Let $|\mathcal{I}| > 1$ and $\mathcal{I}_1 = [n, m]$ be the smallest interval in $\mathscr{I}$ that contains $\mathcal{I}$ with $n, m \in \mathbb{Z}$. Then $|\mathcal{I}_1| \le 3|\mathcal{I}|$. Take into account (16), we have

$$f^*_{\mathcal{I}} \le \frac{2}{|\mathcal{I}|}\int_{\mathcal{I}} |f(t) - f_{\mathcal{I}_1}|\,dt \le \frac{2}{|\mathcal{I}|}\int_{\mathcal{I}_1} |f(t) - f_{\mathcal{I}_1}|\,dt \le \frac{6}{|\mathcal{I}_1|}\int_{\mathcal{I}_1} |f(t) - f_{\mathcal{I}_1}|\,dt = 6f^*_{\mathcal{I}_1}.$$

According to (a) $f^*_{\mathcal{I}_1} = \varphi^*_{\tilde{\mathcal{I}}_1} \le \|\varphi\|_*$, where $\tilde{\mathcal{I}}_1 := [n, m) \cap \mathbb{Z}$. Therefore,

$$f^*_{\mathcal{I}} \le 6f^*_{\mathcal{I}_1} \le 6\|\varphi\|_*, \quad \mathcal{I} \in \mathscr{I}, \tag{19}$$

thus, $\|f\|_* \le 6\|\varphi\|_*$.

The verification of the equality (15) is straightforward. □

The Hilbert transform in the spaces $L_p(\mathbb{R}),\ 1 < p < \infty$, is defined by the formula

$$(Hf)(x) := \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{|x-t|\ge\varepsilon} \frac{f(t)}{x-t}\,dt, \quad x \in \mathbb{R}. \tag{20}$$

In these spaces, H is a linear homeomorphism (see [5]). However, H does not map the space $L_1(\mathbb{R})$ into itself, and the formula (20) does not allow us to correctly define its action on functions from $L_\infty(\mathbb{R})$. Using a one-dimensional perturbation of the operator H, we can obtain a regularized operator $\overset{\circ}{H}$, which is defined on functions from $L_{1,\mathrm{loc}}(\mathbb{R})$ for which

$$\int_{-\infty}^{\infty} |f(t)|(1+|t|)^{-1}\,dt < \infty.$$

This regularization is given by the formula

$$(\overset{\circ}{H}f)(x) := \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{|x-t|\ge\varepsilon} f(t)\left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)dt, \quad x \in \mathbb{R}.$$

Fefferman proved the theorem (see [5]) that describes $\mathrm{BMO}(\mathbb{R})$ in terms of $\overset{\circ}{H}$.

Theorem 4 (Fefferman)**.** *The following equality holds*

$$\mathrm{BMO}(\mathbb{R}) = \{c + f_1 + \overset{\circ}{H}f_2 \colon f_1, f_2 \in L_\infty(\mathbb{R}), c \in \mathbb{C}\}/\,\mathrm{const},$$

where the formula

$$\|f\| := \inf\{\|f_1\|_\infty + \|f_2\|_\infty \colon f = c + f_1 + \overset{\circ}{H}f_2, \quad c \in \mathbb{C},\ f_1, f_2 \in L_\infty(\mathbb{R})\}$$

defines a norm in $\mathrm{BMO}(\mathbb{R})$ *that is equivalent to the norm* $\|\cdot\|_*$.

The discrete analogue of this result is as follows.

Theorem 5. *The following equality holds*

$$\mathrm{BMO}(\mathbb{Z}) = \{c + \varphi_1 + \overset{\circ}{\mathcal{H}}\varphi_2 \colon \varphi_1, \varphi_2 \in \ell_\infty,\ c \in \mathbb{C}\}/\,\mathrm{const},$$

where the formula

$$\|\varphi\|_\triangle := \inf\{\|\varphi_1\|_\infty + \|\varphi_2\|_\infty \colon \varphi = c + \varphi_1 + \overset{\circ}{\mathcal{H}}\varphi_2,\ \varphi_1, \varphi_2 \in \ell_\infty,\ c \in \mathbb{C}\}$$

defines a norm in $\mathrm{BMO}(\mathbb{Z})$ *that is equivalent to the norm* $\|\cdot\|_*$.

The proof of Theorem 5 is based on Fefferman's theorem and the statement that is proved below.

Proposition 4. *The operator* $V\mathring{H} - \mathring{\mathcal{H}}V$ *continuously maps* $L_\infty(\mathbb{R})$ *to* ℓ_∞.

Proof. Let $f \in L_\infty(\mathbb{R})$ and $\varphi := (V\mathring{H} - \mathring{\mathcal{H}}V)f$. Fix an arbitrary $n \in \mathbb{Z}$ and estimate $|\varphi(n)|$. Let $\mathcal{I} := \mathcal{I}_{n-1} \cup \mathcal{I}_n \cup \mathcal{I}_{n+1}$ and define

$$f_1 := \chi_{\mathcal{I}} f, \quad f_2 := (1-\chi_{\mathcal{I}})f, \quad \varphi_j := (V\mathring{H} - \mathring{\mathcal{H}}V)f_j \quad (j \in \{1,2\}),$$

where $\chi_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$. Since $|\varphi(n)| \le |\varphi_1(n)| + |\varphi_2(n)|$, it suffices to estimate the values $|\varphi_j(n)|$.

Note that $\|f_j\|_\infty \le \|f\|_\infty$ $(j \in \{1,2\})$ and $f_1 \in L_2(\mathbb{R}), Vf_1 \in \ell_2$, and

$$\|f_1\|_2 \le \sqrt{3}\|f\|_\infty, \quad \|Vf_1\|_2 \le \sqrt{3}\|f\|_\infty. \tag{21}$$

From the definitions of the operators $\mathring{H}$ and $\mathring{\mathcal{H}}$ we obtain

$$(V\mathring{H}f_1)(n) = \int_{\mathcal{I}_n} (Hf_1)(x)\,dx + \frac{1}{\pi}\int_{\mathcal{I}_n} \frac{tf_1(t)\,dt}{1+t^2}, \quad (\mathring{\mathcal{H}}Vf_1)(n) = (\mathcal{H}Vf_1)(n) - (\mathcal{H}Vf_1)(0).$$

Thus, taking into account (21) and the fact that the operators $H\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ and $\mathcal{H}\colon \ell_2 \to \ell_2$ are unitary, we have

$$|(V\mathring{H}f_1)(n)| \le \int_{\mathcal{I}_n} |(Hf_1)(x)|\,dx + \frac{1}{\pi}\int_{\mathcal{I}_n} \frac{|tf_1(t)|\,dt}{1+t^2} \le \|Hf_1\|_2 + \frac{1}{6}\|f\|_\infty = (\sqrt{3}+1/6)\|f\|_\infty$$

and $|(\mathring{\mathcal{H}}Vf_1)(n)| \le |(\mathcal{H}Vf_1)(n)| + |(\mathcal{H}Vf_1)(0)| \le 2\|\mathcal{H}Vf_1\|_2 = 2\|Vf_1\|_2 \le 2\sqrt{3}\|f\|_\infty$. Therefore,

$$|\varphi_1(n)| \le |(V\mathring{H}f_1)(n)| + |(\mathring{\mathcal{H}}Vf_1)(n)| \le 6\|f\|_\infty. \tag{22}$$

Next, let us estimate $|\varphi_2(n)|$. For this, consider the functions

$$\Phi(x,t) := \frac{1}{\pi}\left(\frac{1}{x-t} + \frac{t}{t^2+1}\right), \quad \Phi_0(x,t) := \frac{1}{\pi}\left(\frac{1}{[x]-[t]+1/2} + \frac{1}{[t]-1/2}\right),$$

where $[x]$ is the integer part of $x \in \mathbb{R}$. It is easy to verify that

$$\varphi_2(n) = \int_{\mathcal{I}_n}\int_{\mathbb{R}\setminus\mathcal{I}} (\Phi(x,t) - \Phi_0(x,t))f(t)\,dt\,dx. \tag{23}$$

We will show that if $x, t \in \mathbb{R}$ and $|x-t| \ge 1$, then

$$|\Phi(x,t) - \Phi_0(x,t)| \le \frac{1}{\pi}\left(\frac{6}{|x-t|^2} + \frac{8}{1+t^2}\right). \tag{24}$$

Let $t \in \mathbb{R}$ and set $u = [t] - 1/2,\ v = t - u$. Since $|u| \ge 1/2$ and $v \in [0, 3/2]$, we have

$$\left|\frac{1}{[t]-1/2} - \frac{t}{1+t^2}\right| = \frac{|1+(u+v)^2 - u(u+v)|}{(1+t^2)|u|} \le$$
$$\le \frac{1}{1+t^2}\left(v + \frac{1+v^2}{|u|}\right) \le \frac{8}{1+t^2}. \tag{25}$$

Let $x, t \in \mathbb{R}$ and $|x-t| \ge 1$. Put $u = x - t$ and $v = [x] - [t] + 1/2$. Obviously, $|v| \ge 1/2$, $|u - v| \le 3/2$. Therefore, $|u| \le |v| + |u-v| \le |v| + 3/2 \le 4|v|$, and thus,

$$\left|\frac{1}{x-t} - \frac{1}{[x]-[t]+1/2}\right| \le \frac{|u-v|}{|u|\,|v|} \le \frac{6}{|u|^2} \le \frac{6}{|t-x|^2}.$$

From this, taking into account (25), we obtain (24). Using (23) and (24), we have

$$|\varphi_2(n)| \le \frac{\|f\|_\infty}{\pi} \int_{\mathcal{I}_n} \int_{\mathbb{R}\setminus\mathcal{I}} \left(\frac{6}{|x-t|^2} + \frac{8}{1+t^2} \right) dt\, dx \le \frac{\|f\|_\infty (12+8\pi)}{\pi} \le 12\|f\|_\infty.$$

Thus, considering (22), we obtain that $|\varphi(n)| \le 18\|f\|_\infty$. From the arbitrariness of $n \in \mathbb{Z}$, it follows that

$$\|V\overset{\circ}{H} - \overset{\circ}{\mathcal{H}}V\|_{L_\infty(\mathbb{R})\to\ell_\infty} \le 18. \tag{26}$$

□

From Proposition 4, we get the following corollary.

Corollary 1. *The operator* $\overset{\circ}{\mathcal{H}}$ *continuously maps* ℓ_∞ *to* BMO($\mathbb{Z}$).

Proof. Let $\varphi \in \ell_\infty$ and $f = U\varphi$. It is obvious that $\|f\|_\infty \le \|\varphi\|_\infty$. In view of (15), we have $\varphi = VU\varphi = Vf$. Thus,

$$\overset{\circ}{\mathcal{H}}\varphi = V\overset{\circ}{H}f - (V\overset{\circ}{H} - \overset{\circ}{\mathcal{H}}V)f. \tag{27}$$

From Fefferman's theorem, it follows that the operator $\overset{\circ}{H}$ continuously maps the space $L_\infty(\mathbb{R})$ into BMO($\mathbb{R}$). Therefore, taking into account (13), we obtain

$$\|V\overset{\circ}{H}f\|_* \le \|\overset{\circ}{H}f\|_* \le B\|f\|_\infty \le B\|\varphi\|_\infty, \tag{28}$$

where $B = \|\overset{\circ}{H}\|_{L_\infty \to \mathrm{BMO}(\mathbb{R})}$. Note that $L_\infty(\mathbb{R}) \subset \mathrm{BMO}(\mathbb{R})$ and $\|g\|_* \le \|g\|_\infty$, $g \in L_\infty(\mathbb{R})$. Therefore, (27), (28) and (26) imply that

$$\|\overset{\circ}{\mathcal{H}}\varphi\|_* \le \|V\overset{\circ}{H}f\|_* + \|(V\overset{\circ}{H} - \overset{\circ}{\mathcal{H}}V)f\|_* \le B\|\varphi\|_\infty + 18\|f\|_\infty \le (B+18)\|\varphi\|_\infty.$$

Thus, the operator $\overset{\circ}{\mathcal{H}}$ continuously maps ℓ_∞ into BMO($\mathbb{Z}$). □

Proof of Theorem 5. Let $\varphi \in \mathrm{BMO}(\mathbb{Z})$ and $f = U\varphi$. From Proposition 3, it follows that $f \in \mathrm{BMO}(\mathbb{R})$. Therefore, according to Fefferman's theorem, f can be expressed as

$$f = f_1 + \overset{\circ}{H}f_2 + c, \tag{29}$$

where c is a constant, and $f_1, f_2 \in L_\infty(\mathbb{R})$. The functions f_1 and f_2 can be chosen such that

$$\|f_j\|_\infty \le A\|f\|_*, \quad j \in \{1,2\}, \tag{30}$$

where A is an absolute constant. Since (see (15)) $Vf = VU\varphi = \varphi$, it follows from (29) that

$$\varphi = Vf_1 + V\overset{\circ}{H}f_2 + c. \tag{31}$$

Clearly, the operator V continuously maps $L_\infty(\mathbb{R})$ into ℓ_∞ and

$$\|Vf\|_\infty \le \|f\|_\infty, \quad f \in L_\infty(\mathbb{R}). \tag{32}$$

Put $\varphi_1 := Vf_1 + (V\overset{\circ}{H} - \overset{\circ}{\mathcal{H}}V)f_2$, $\varphi_2 := Vf_2$. Then $\varphi = \varphi_1 + \overset{\circ}{\mathcal{H}}\varphi_2 + c$. Taking into account (32) and (26), we get

$$\|\varphi_1\|_\infty \le \|Vf_1\|_\infty + \|(V\overset{\circ}{H} - \overset{\circ}{\mathcal{H}}V)f_2\|_\infty \le \|f_1\|_\infty + 18\|f_2\|_\infty.$$

Thus, taking into account (30) and (14), we have $\|\varphi_1\|_\infty \le 19A\|f\|_* = 19A\|U\varphi\|_* \le 114A\|\varphi\|_*$. Additionally, considering (32), (30) and (14), we obtain

$$\|\varphi_2\|_\infty = \|Vf_2\|_\infty \le \|f_2\|_\infty \le A\|f\|_* = A\|U\varphi\|_* \le 6A\|\varphi\|_*.$$

Thus, we proved that $\mathrm{BMO}(\mathbb{Z}) \subset \{\varphi_1 + \mathring{\mathcal{H}}\varphi_2 + c \mid \varphi_1, \varphi_2 \in \ell_\infty\}/\,\mathrm{const}$ and $\|\varphi\|_\triangle \le A_1\|\varphi\|_*$, $\varphi \in \mathrm{BMO}(\mathbb{Z})$, where A_1 is an absolute constant. It remains to show that there exists $A_2 > 0$ such that

$$\|\varphi\|_* \le A_2\|\varphi\|_\triangle,\ \varphi \in \mathrm{BMO}(\mathbb{Z}).$$

Let $\varphi \in \mathrm{BMO}(\mathbb{Z})$. Then there exist $\varphi_1, \varphi_2 \in \ell_\infty$ and $c \in \mathbb{C}$ such that

$$\varphi = \varphi_1 + \mathring{\mathcal{H}}\varphi_2 + c, \quad \|\varphi\|_\triangle \le \|\varphi_1\|_\infty + \|\varphi_2\|_\infty \le 2\|\varphi\|_\triangle.$$

Note that $\|\varphi\|_* = \|\varphi_1 + \mathring{\mathcal{H}}\varphi_2 + c\|_* \le \|\varphi_1\|_* + \|\mathring{\mathcal{H}}\varphi_2\|_* \le \tilde{A}_2(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty)$, where $\tilde{A}_2 = 1 + \|\mathring{\mathcal{H}}\|_{\ell_\infty \to \mathrm{BMO}(\mathbb{Z})}$. Therefore, $\|\varphi\|_* \le 2\tilde{A}_2\|\varphi\|_\triangle$, $\varphi \in \mathrm{BMO}(\mathbb{Z})$, and thus, the norm $\|\cdot\|_\triangle$ is equivalent to the norm $\|\cdot\|_*$. □

5. Proof of Theorem 1. First, we prove that $Y \sim \mathrm{BMO}(\mathbb{Z})$. Consider the linear operator

$$Y \ni y \mapsto Wy := \{y + c\colon c \in \mathbb{C}\}.$$

Let us show that W is a homeomorphism between the space Y and $\mathrm{BMO}(\mathbb{Z})$. Let $\varphi \in \mathrm{BMO}(\mathbb{Z})$. Then there exist the functions $\varphi_0, \varphi_1 \in \ell_\infty$ such that $\varphi = \varphi_0 + \mathring{\mathcal{H}}\varphi_1 + c$, $c \in \mathbb{C}$, and $\|\varphi\|_\triangle \le \|\varphi_0\|_\infty + \|\varphi_1\|_\infty \le 2\|\varphi\|_\triangle$.

Note that the projector P_0 (see Remark 3) projects ℓ_∞ onto Y_0 and

$$\mathring{\mathcal{H}}x = \mathring{\mathcal{H}}P_0x = P_0\mathring{\mathcal{H}}x, \quad x \in \ell_\infty.$$

Thus, $\varphi = P_0\varphi_0 + \mathring{\mathcal{H}}P_0\varphi_1 + c$, $c \in \mathbb{C}$. Since the elements $P_0\varphi_j$ belong to the space Y_0, the element $y = P_0\varphi_0 + \mathring{\mathcal{H}}P_0\varphi_1$ belongs to the space Y, it means that $\varphi = Wy$. From the arbitrariness of the element φ, it follows that W maps Y onto $\mathrm{BMO}(\mathbb{Z})$. Additionally,

$$\|y\|_Y \le \|P_0\varphi_0\|_{Y_0} + \|\mathring{\mathcal{H}}P_0\varphi_0\|_{Y_1} = \|P_0\varphi_0\|_{Y_0} + \|P_0\varphi_1\|_{Y_0}.$$

According to Remark 3 $\|P_0\|_{\ell_\infty\to\ell_\infty} \le 2$. Therefore, $\|y\|_Y \le 2(\|\varphi_0\|_\infty + \|\varphi_1\|_\infty) \le 4\|\varphi\|_\triangle$, which means $\|Wy\|_\triangle \ge \frac{1}{4}\|y\|_Y$. According to Theorem 5 the norm $\|\cdot\|_\triangle$ is equivalent to the norm $\|\cdot\|_*$. Therefore, the operator W is bounded below. This implies that the operator $W\colon Y \to \mathrm{BMO}(\mathbb{Z})$ is a linear bijection, and the operator W^{-1} continuously maps $\mathrm{BMO}(\mathbb{Z})$ to Y. Hence, by the Banach inverse operator theorem, W is a homeomorphism of the space Y onto $\mathrm{BMO}(\mathbb{Z})$, meaning $Y \sim \mathrm{BMO}(\mathbb{Z})$.

To complete the proof, it is sufficient to show that $\mathcal{L}_1 \sim X$. Indeed, in that case, $\mathcal{L}_1' \sim X'$ and in view of Theorem 3 we have

$$\mathcal{L}_1' \sim X' \sim Y \sim \mathrm{BMO}(\mathbb{Z}).$$

Thus, we prove that $\mathcal{L}_1 \sim X$. Let the operator $J\colon \mathcal{L}_1 \to \mathbb{C}^{\mathbb{Z}}$ be defined by formula (1). From the results of [1], it follows that $J\mathcal{L}_1 = \{x \in \ell_1\colon \mathcal{H}x \in \ell_1\}$. And by Lemma 2,

$$\{x \in \ell_1\colon \mathcal{H}x \in \ell_1\} = \{x \in \ell_1\colon \mathcal{H}x \in \ell_1,\ F_d(x) = F_d(\mathcal{H}x) = 0\} = X_0 \cap X_1,$$

i.e., $J\mathcal{L}_1 = X$. In [7], it is proved that for all $f \in \mathcal{L}_1$ the following inequality holds

$$4^{-1}(\|Jf\|_1 + \|\mathcal{H}Jf\|_1) \le \|f\|_{\mathcal{L}_1} \le \|Jf\|_1 + \|\mathcal{H}Jf\|_1. \tag{33}$$

Take into account (12), we have

$$\|Jf\|_X = \max\{\|Jf\|_1,\ \|\mathcal{H}Jf\|_1\} \le \|Jf\|_1 + \|\mathcal{H}Jf\|_1,$$
$$\|Jf\|_1 + \|\mathcal{H}Jf\|_1 \le 2\max\{\|Jf\|_1,\ \|\mathcal{H}Jf\|_1\} = 2\|Jf\|_X.$$

Therefore, from (33) it follows that $4^{-1}\|Jf\|_X \le \|f\|_{\mathcal{L}_1} \le 2\|Jf\|_X$. Thus, J is an isomorphism between the spaces $\mathcal{L}_1$ and X, i.e., $\mathcal{L}_1 \sim X$.

Appendix. Some definitions and facts of the theory of Banach space. Banach spaces A and B are called *isomorphic* (abbreviated as $A \sim B$) if there exists a linear homeomorphism from space A to B. If additionally this homeomorphism is an isometry, then spaces A and B are called *isometrically isomorphic* (abbreviated as $A \simeq B$).

Let A and B be Banach spaces that are algebraically and topologically embedded in some Hausdorff linear topological space. The Banach space $A \cap B$, consisting of elements common to A and B, with the norm

$$\|x\|_{A\cap B} = \max(\|x\|_A, \|x\|_B), \quad x \in A \cap B,$$

is called the *intersection of Banach spaces* A and B.

The Banach space $A+B$, consisting of elements of the form $x = u+v$, where $u \in A, v \in B$, and endowed with the norm

$$\|x\|_{A+B} = \inf\{\|u\|_A + \|v\|_B\},$$

where the infimum is taken over all elements $u \in A, v \in B$ such that $x = u + v$, is called *the sum of Banach spaces* A and B.

In the theory of Banach spaces, the following theorem is well-known (see, for example, [8]).

Theorem 6. *Let A and B be Banach spaces that are algebraically and topologically embedded in some Hausdorff linear topological space, and if the intersection $A\cap B$ is dense in spaces A and B, then the dual spac $(A\cap B)'$ of the intersection of A and B is isometrically isomorphic to the sum $A' + B'$.*

Acknowledgements. The authors express their gratitude to Prof. Rostyslav Hryniv for careful reading of the article and valuable remarks and suggestions.

REFERENCES

1. B.Ya. Levin, Lectures on entire functions, in collaboration with G.Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translations of Mathematical Monographs, V.150, AMS, 1996.
2. R. Banuelos, M. Kwasnicki, *On the l_p-norm of the discrete Hilbert transform*, Duke Math. J., **168**, (2019), №3, 471–504. https://doi.org/10.1215/00127094-2018-0047.
3. F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415–426. http://dx.doi.org/10.1002/cpa.3160140317.
4. G. Dafni, R. Gibara, A. Lavigne, *BMO and the John-Nirenberg Inequality on Measure Spaces*, Anal. Geom. Metr. Spaces 2020; 8: 335-362. https://doi.org/10.1515/agms-2020-0115.
5. J.B. Garnett, Bounded analytic functions, Academic Press, 1981.
6. E.C. Titchmarsh, *Reciprocal formulae involving series and integrals*, Math. Z., **25** (1926), №1, 321–347. MR 1544814.
7. N. Sushchyk, D. Lukivska, *Some inequalities for entire functions*, Mat. Stud. **62** (2024), №1, 109–112. https://doi.org/10.30970/ms.62.1.109-112.
8. S.G. Krein, Ju.I. Petunin, E.M. Semenov, Interpolation of linear operators, Transl. Math. Monographs, V.54, AMS, 1982.

Ivan Franko National University of Lviv
Lviv, Ukraine
yamykytyuk@yahoo.com
n.sushchyk@gmail.com
d.lukivska@gmail.com

Received 10.08.2024