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ON AN ITERATIVE MOSER-KURCHATOV METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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This paper is devoted to analysis of an iterative method for solving nonlinear equations. The method, inspired by the Kurchatov-type methods, is specifically designed to avoid the need for derivative calculations or inverses of linear operators. By employing a sequence of approximating operators and divided differences, the method achieves semilocal convergence. Numerical experiments demonstrate the method's efficiency and robustness, highlighting its potential advantages over traditional methods like Newton's method, especially in scenarios where derivative calculations are impractical and computationally expensive. The results indicate that the method is a viable and efficient alternative for solving nonlinear equations, especially in large-scale problems or scenarios, where derivative information is not readily available. The robustness and efficiency of the method make it a valuable tool in various scientific and engineering applications.

1. Introduction. Solving nonlinear equations is a fundamental problem in numerical analysis with widespread applications in science and engineering [1–13, 21–23]. The general problem involves finding a solution $x_* \in X$ such that

$$F(x) = 0,$$

where $F : D \subseteq X \to Y$ is a nonlinear operator between Banach spaces X and Y, and D is an open and convex subset of X.

Newton's method is widely used for its quadratic convergence rate in solving such equations. However, it requires the computation and inversion of the Frechet derivative $F'(x)$ at each iteration. It may not be not effective, when the derivative is difficult to obtain or when inverting the linear operator is computationally intensive [14–17]. We are interested in methods that are free of derivatives and inverses. By approximating the inverse operator instead of computing it directly, we eliminate the need to calculate inverse operators. This leads to a Newton-type method and does not require the computation of inverse operators.

A representative example of these methods is the iterative Moser-Secant method [2], which combines ideas from Newton-like methods and the Secant method to avoid the computation of Jacobians and their inverses. This method approximates the inverse of the divided difference operator without explicitly computing inverses, making it computationally attractive, when derivatives are unavailable or difficult to compute.

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Building upon the Moser-Secant method [2], we propose an enhanced iterative method that improves its properties, leading to better performance in practice. In this paper, we analyze the Moser-Kurchatov method [12, 13, 18–20], an iterative technique that combines elements of the secant and quasi-Newton methods to solve nonlinear equations without derivative computations. The method is defined by the following iterative scheme

$$\begin{cases} x_{m+1} = x_m - A_m F(x_m), \\ y_{m+1} = x_m + p_m(x_{m+1} - x_m), \quad p_m \in [0,1], \\ K_{m+1} = [2y_{m+1} - x_{m+1}, x_{m+1}; F], \\ A_{m+1} = 2A_m - A_m K_{m+1} A_m, \end{cases} \tag{1}$$

where A_m are approximations of the inverse of F', p_m are relaxation parameters, and $[v, w; F]$ denotes the divided difference operator of F at points v and w.

Our objective is to analyze the semilocal convergence of the method (1). We establish conditions under which the iterative sequence $\{x_m\}$ converges to a solution x_* of the equation $F(x) = 0$. The convergence analysis is based on certain assumptions about the operator F and utilizes properties of the divided differences and approximating operators.

The main contributions of this paper are:

– Establishing semilocal convergence results for the Moser-Kurchatov iterative method under specific conditions on F and the initial approximation x_0.

– Providing explicit bounds on the convergence and error estimates, demonstrating the efficiency of the method without requiring derivative computations.

– Presenting numerical experiments that illustrate the practical performance of proposed iterative method on several benchmark nonlinear systems.

The structure of the paper is organized as follows. In Section 2, we provide a detailed analysis of the convergence of the method (1), including necessary conditions and supporting lemmas that lead to the main convergence theorem. In Section 3, we present numerical experiments that demonstrate the effectiveness of the method on various nonlinear problems. Finally, in Section 4, we conclude with a summary of our findings and discuss potential directions for future research.

By researching the Moser-Kurchatov method, we aim to contribute to the understanding of derivative-free iterative methods for solving nonlinear equations, offering insights into their convergence properties and practical performance.

2. Analysis of convergence. The semilocal convergence of the method (1) relies on certain conditions.

(C_1) There exists $\lambda \geq 0$ such that $\|[v_1, v_2; F] - [v_3, v_4; F]\| \leq \lambda(\|v_1 - v_3\| + \|v_2 - v_4\|)$ for each $v_1, v_2, v_3, v_4 \in D$ with $v_1 \neq v_2$ and $v_3 \neq v_4$. It is worth noting that if $v_1 = v_2, v_3 = v_4$ and the Frechet derivative exists in D, then $F'(v_1) = [v_1, v_2; F]$.

(C_2) There exist $x_0 \in D$ and $A_0 \in \mathcal{L}(X, X)$ such that $\|F(x_0)\| \leq \mu$, $\|A_0\| \leq b$, $2\lambda b_\nu < 1$, and $\|I - [2y_0 - x_0, x_0; F]\, A_0\| = \delta < 1$ provided that $\nu = \delta\mu$, and for $x_0, x_0 \in D$ it follows that $2y_0 - x_0 \in D$.

(C_3) There exist $x_{-1} \in D$ and $p_{-1} \in (0,1)$ such that $\frac{(1+\delta+4\lambda b\nu)\nu}{1-2\lambda b\nu} < \alpha = \|x_{-1} - x_0\|$ and $y_0 = x_{-1} + p_{-1}(x_0 - x_{-1}) \in D$.

We assume without loss of generality that $F(x_0) \neq 0$. Otherwise, $x_m = x_0$ for each $n \in \{0, 1, 2, \ldots\}$. In this case, $F(x_0) = 0$ and $\lim\limits_{m\to\infty} x_m = x_0$.

It is convenient to introduce the operators

$$K_m = [2y_m - x_m, x_m; F] \ \text{ and } S_m = [x_{m-1}, x_m; F]\,.$$

The iterate x_1 exists by the first substep of the method (1) and by the condition (C_2)

$$\|x_1 - x_0\| = \|A_0 F(x_0)\| \le \beta\mu = \nu.$$

If $x_1 \in D$, and from the existence of $[x_0, x_1; F]$, the definition of the divided difference and the method (1), we can write in turn

$$F(x_1) = F(x_0) - [x_0, x_1; F](x_0 - x_1) = (I - [x_0, x_1; F]A_0)F(x_0).$$

Consequently,

$$\|F(x_1)\| \le \|I - S_1 A_0\| \|F(x_0)\|. \tag{2}$$

An upper bound is needed for the first norm at the right hand side of (2). Let us consider the operator $K_0 = [2y_0 - x_0, x_0; F]$ provided that $y_0 \in D$ and $2y_0 - x_0 \in D$. Notice that

$$I - S_1 A_0 = (I - K_0 A_0) + (K_0 - S_1)A_0.$$

But by (C_1), we get in turn

$$\begin{aligned} \|K_0 - S_1\| = \|\,[2y_0 - x_0, x_0; F] - [x_0, x_1; F]\,\| \le \lambda(\|y_0 - x_0\| + \|x_0 - x_1\|) \le \\ \le \lambda(2\|y_0 - x_0\| + \|x_0 - x_1\|) \le \lambda\,(2(1 - p_{-1})\|x_0 - x_{-1}\| + \|x_0 - x_{-1}\|) \le \\ \le \lambda(2\|x_0 - x_{-1}\| + \|x_1 - x_0\|) \end{aligned}$$

leading to

$$\|I - S_1 A_0\| \le \|I - K_0 A_0\| + \|K_0 - S_1\| \|A_0\| \le \delta + \lambda(2b\alpha + b\nu) = \delta + \lambda(2a_0 + b_0) := c_0.$$

So, we have

$$\|F(x_1)\| \le c_0 \|F(x_0)\|.$$

Moreover, by $A_1 = 2A_0 - A_0[2y_1 - x_1, x_1; F]A_0$ and the triangle inequality $\|A_1\| \le \|A_0\|(1 + \|I - K_0 A_0\|)$. We can also write

$$I - K_1 A_0 = I - S_1 A_0 + (S_1 - K_1)A_0.$$

Then, for the second term at the right hand side and (C_1) we have in turn the estimates

$$\begin{aligned} \|S_1 - K_1\| = \|[x_0, x_1; F] - [2y_1 - x_1, x_1; F]\| \le \lambda(\|2y_1 - x_1 - x_0\|) \le \\ \le \lambda(2\|y_1 - x_0\| + \|x_1 - x_0\|) \le \lambda(2p_0\|x_1 - x_0\| + \|x_1 - x_0\|) \le \lambda(3\|x_1 - x_0\|), \end{aligned}$$

so

$$\|I - K_1 A_0\| \le \|I - S_1 A_0\| + \|S_1 - K_1\| \|A_0\| \le c_0 + 3\lambda b_0 := d_0$$

and

$$\|A_1\| \le (1 + d_0)\|A_0\|.$$

Next, as in [2] we define some nonnegative real sequences $\{a_n\}, (b_n), (c_n), (d_n)$ for $n \in \{1, 2, \ldots\}$ that play a role in the convergence of the method (1) as

$$\begin{aligned} a_n = (1 + d_{n-1})b_{n-1}, \quad b_n = (1 + d_{n-1})b_{n-1}, \\ c_n = d_{n-1}^2 + \lambda(2a_n + b_n), \quad d_n = c_n + 3\lambda b_n. \end{aligned} \tag{3}$$

A convergence criterion is needed for these sequences.

Lemma 1. *Suppose that*

$$c_1 < c_0 \quad \text{and} \quad c_0(1+d_0)^2 < 1. \tag{4}$$

Then, the following items hold:
(a) $c_0 < 1$ *and* $c_0(1+d_0)^2 < 1$,
(b) The sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, *and* $\{d_n\}$ *are strictly decreasing.*

Proof. (a) The second condition in (4) and $(1+b_0) > 1$ give $c_0(1+d_0) < 1$, so the second item in (a) holds. Similarly, the first item in (a) holds.

The item (b) is shown by induction. We first check if $a_1 \le a_0$. By the definition of these sequences and the choice of α in the condition (C_3) we must show

$$a_1 = (1+d_0)b_0 = (1+c_0+3\lambda b_0)b_0 \le a_0$$

or

$$(1+c_0+3\lambda b_n)b_n < b_0\alpha$$

or

$$(1+\delta+\lambda(2a_0+b_0))\nu + 3\lambda b\nu < b\alpha$$

or

$$(1+\delta+3\lambda b\nu+\lambda b\nu+2\lambda b\alpha)\nu < \alpha,$$

which is true by the choice of α.

Then, by the second condition in (4) one has $b_1 < b_0$. Moreover, by the first condition in (4), we have $d_1 < d_0$.

Next, suppose these sequences decrease up to an integer m. Then, we obtain in turn that

$$\begin{gathered}
a_{m+1} = (1+d_m)b_m < (1+d_{m-1})b_{m-1} = a_m, \\
b_{m+1} = (1+d_m)^2 b_m c_m < (1+d_{m-1})^2 b_{m-1} c_{m-1} = b_m, \\
c_{m+1} = d_m^2 + \lambda(2a_{m+1}+b_{m+1}) < d_{m-1}^2 + \lambda(2a_m+b_m) = c_m, \\
d_{m+1} = c_{m+1} + 2\lambda b_{m+1} < c_m + 3\lambda b_m = d_m.
\end{gathered}$$

The induction is completed. Thus, the item (b) is true. □

Lemma 2. *Let* $\{x_m\}$ *be the sequence generated by the method* (1). *Suppose that* x_m, y_m, $2y_m - x_m \in D$ *for each* $m \in \{1,2,\ldots\}$. *Then, the following items hold:*
(a) $\|K_m - S_{m+1}\| \le \lambda(2\|x_m - x_{m-1}\| + \|x_{m+1} - x_m\|)$,
(b) $\|I - K_m A_m\| \le \|I - K_m A_{m-1}\|^2$,
(c) $\|S_{m+1} - K_{m+1}\| \le 3\lambda\|x_{m+1} - x_m\|$,
(d) $\|I - S_{m+1}A_m\| \le \|I - K_m A_m\| + \|K_m - S_{m+1}\|\|A_m\|$.

Proof. (a) Using the condition (C_1) and the definition of the operators K_m and S_{m+1}, we have in turn

$$\begin{aligned}
\|K_m - S_{m+1}\| = \|[x_m - x_m, x_m; F] - [x_m, x_{m+1}; F]\| \le \lambda(2\|y_m - x_m\| + \|x_{m+1} - x_m\|) \le \\
\le \lambda(2(1-p_{m-1})\|x_m - x_{m-1}\| + \|x_{m+1} - x_m\|) \le \lambda(2\|x_m - x_{m-1}\| + \|x_{m+1} - x_m\|).
\end{aligned}$$

Thus, (a) holds.

(b) Notice that $I - K_m A_m = I - 2K_m A_{m-1} + K_m A_{m-1} K_m A_{m-1} = I - K_m A_{m-1})^2$. By taking norms we show (b).

(c) As in (a) we have in turn

$$\|S_{m+1} - K_{m+1}\| = \|[x_m, x_{m+1}; F] - [y_{m+1} - x_{m+1}, x_{m+1}; F]\| \le \lambda \|2y_{m+1} - x_{m+1} - x_m\|.$$

But $2y_{m+1} - x_{m+1} - x_m = 2(y_{m+1} - x_m) + (x_m - x_{m+1})$, so

$$\|2y_{m+1} - x_{m+1} - x_m\| \le 2\|y_{m+1} - x_m\| + \|x_m - x_{m+1}\| \le$$
$$\le 2p_m\|x_{m+1} - x_m\| + \|x_{m+1} - x_m\| \le 2\|x_{m+1} - x_m\| + \|x_{m+1} - x_m\| = 3\|x_{m+1} - x_m\|,$$

which shows (c).

(d) We can write $I - S_{m+1}A_m = (I - K_mA_m) + (K_m - S_{m+1})A_m$ from which item (d) follows. □

Lemma 3. *Under the hypotheses of the Lemma 1 further suppose that* $x_m, y_m, 2y_m - x_m \in D$ *for each* $m \in \{1, 2, \ldots\}$. *Then, the following items hold:*
(I_m) $\|x_{m+1} - x_m\| \le (1 + d_{m-1})c_{m-1}\|A_{m-1}\|\|F(x_{m-1})\|$,
(II_m) $\|x_m - x_0\| \le \frac{1}{1-q}\|A_0\|\|F(x_0)\|$ *provided that* $q = (1 + d_0)c_0 < 1$,
(III_m) $\|A_m\|\|x_m - x_{m-1}\| \le \|A_m\|\|A_{m-1}\|\|F(x_{m-1})\| \le a_m$,
(IV_m) $\|A_m\|\|x_{m+1} - x_m\| \le \|A_m\|^2\|F(x_m)\| \le b_m$,
(V_m) $\|K_m - S_{m+1}\| \le \lambda(2a_m + b_m)$,
(VI_m) $\|I - S_{m+1}A_m\| = c_m$,
(VII_m) $\|F(x_{m+1})\| \le c_m\|F(x_m)\|$,
($VIII_m$) $\|S_{m+1} - K_{m+1}\| \le \lambda b_m$,
(IX_m) $\|I - K_{m+1}A_m\| = d_m$,
(X_m) $\|A_{m+1}\| \le (1 + d_{m-1})\|A_{m-1}\|$.

Proof. Mathematical induction is used on the integer m to show all the items starting from $m = 1$. Items (I_1) , (II_1) , (III_1) , and (IV_1) are shown in Lemma 2.3 [2]. By hypothesis $x_2 \in D$. Then, by (1) we can write in turn that

$$F(x_2) = F(x_1) - S_2(x_1 - x_2) = F(x_1) - S_2A_1F(x_1) = (I - S_2A_1)F(x_1),$$

so

$$\|F(x_2)\| \le \|I - S_2A_1\|\|F(x_1)\|.$$

By Lemma 2, we have

$$\|I - S_2A_1\| = \|I - K_1A_1\| + \|K_1 - S_2\|\|A_1\| \le \|I - K_1A_0\|^2 + \|K_1 - S_2\|\|A_1\|,$$

and

$$\|K_1 - S_2\|\|A_1\| \le \lambda(2\|y_1 - x_1\| + \|x_2 - x_1\|)\|A_1\| \le \lambda(2(1 - p_0)\|x_1 - x_0\| + \|x_2 - x_1\|)\|A_1\| \le$$
$$\le \lambda\left[2(1 + d_0)b_0 + a_0(1 + d_0)^2c_0b_0\right] = \lambda(2a_1 + b_1).$$

Hence, we get $\|I - S_2A_1\| \le \|I - K_1A_0\|^2 + \|K_1 - S_2\|\|A_1\| \le d_0^2 + \lambda(2a_1 + b_1) = c_2$. So, $\|F(x_2)\| \le c_1\|F(x_1)\|$ and the items (V_1) , (VI_1) , and (VII_1) hold.

Notice that $A_2 = 2A_1 - A_1K_2A_1 = A_1(I + (I - K_2A_1))$, leading to

$$\|A_2\| \le (1 + \|I - K_2A_1\|)\|A_1\|.$$

We need an upper bound of the right-hand side of the preceding estimate

$$\|S_2 - K_2\|\|A_1\| = \|[x_1, x_2; F] - [2y_2 - x_2, x_2; F]\|\|A_1\| \le 3\lambda p_1\|x_2 - x_1\|\|A_1\| \le$$
$$\le 3\lambda\|A_1\|\|x_2 - x_1\| \le 3\lambda b_1.$$

Furthermore,

$$\|I - K_2A_1\| = \|I - S_2A_1\| + \|S_2 - K_2\|\|A_1\| \le c_1 + 3\lambda b_1 = d_1,$$
$$\|A_2\| \le (1 + d_1)\|A_1\|,$$

showing (VIII_1), (IX_1), and (X_1). The induction is completed for $m = 1$.

Suppose that $(I_k) - (X_k)$ hold for $k \in \{1, 2, \ldots, n-1\}$. Then, the preceding calculations can be repeated with k replacing $m = 1$. □

Next, we present the main semilocal convergence of the method (1) based on the preceding lemmas.

Theorem 1. *Suppose that conditions* (C_1)–(C_3) *hold,* $c_1 < c_0$, $c_0(1+d_0)^2 < 1$, *and* $U_0 = U(x_0, 3r + 2\nu) \subseteq D$ *for* $r = \nu/(1-q)$. *Then the sequence,* $\{n_m\}$ *generated by* (1), *is well defined in* U_0, *remains in* U_0 *for each* $n \in \{0, 1, 2, \ldots\}$, *and converges to a solution* $x_\star \in U_0$ *of the equation* $F(x) = 0$.

Proof. Using the triangle inequality, we get for $k \in \{1, 2, \ldots\}$ in turn that

$$\begin{gathered}\|x_{m+k} - x_m\| \le \|x_{m+k} - x_{m+k-1}\| + \|x_{m+k-1} - x_{m+k-2}\| + \cdots + \|x_{m+1} - x_m\| \le \\ \le (1 + d_{m+k-2})c_{m+k-2}(1 + d_{m+k-3})c_{m+k-3} \cdots (1 + d_{m+1})c_{m+1}(1 + d_m)c_m\|A_m\|\|F(x_m)\| + \\ +(1 + d_{m+k-3})C_{m+k-3} \cdots (1 + d_{m+1})c_{m+1}(1 + d_m)c_m\|A_m\|\|F(x_m)\| + \ldots \\ \ldots + (1 + d_{m+1})a_m\|A_m\|\|F(x_m)\| + \|A_m\|\|F(x_m)\| = \gamma_m\|A_m\|\|F(x_m)\|,\end{gathered}$$

where $\gamma_m = \prod_{i=m}^{m+k-2}(1 + d_i)c_i + \prod_{i=m}^{m+k-3}(1 + d_i)c_i + \cdots + (1 + d_m)c_m + 1$. Hence,

$$\|x_{m+k} - x_m\| < \left(q^{k-1} + q^{k-2} + \cdots + 1\right) q^m\|A_0\|\|F(x_0)\| \le \frac{q^m(1 - q^k)}{1 - q}\|A_0\|\|F(x_0)\|.$$

Next, we show $x_m, y_m, 2y_m - x_m \in D$ for each $m \in \{1, 2, \ldots\}$. We have in turn that

$$\begin{gathered}\|x_k - x_0\| \le \frac{1 - q^k}{1 - q}\|A_0\|\|F(x_0)\| < \frac{\nu}{1 - q} \le r, \\ \|y_m - x_0\| \le \|y_m - x_m\| + \|x_m - x_0\| \le (1 - p_m)\|x_{m+1} - x_m\| + \|x_m - x_0\| \le \\ \le r + q^m\|A_0\|\|F(x_0)\| < r + \nu,\end{gathered}$$

and

$$\|2y_m - x_m - x_0\| \le 2\|y_m - x_0\| + \|x_m - x_0\| \le 2(r + \nu) + r = 3r + 2\nu,$$

showing that $x_m \in U(x_0, r)$, $y_m \in U(x_0, r + \nu)$, and $2y_m - x_m \in U(x_0, 3r + 2\nu)$.

Moreover, the sequence $\{x_m\}$ is complete in the Banach space X and as such it converges to some $x_\star \in U(x_0, r)$.

Finally, by the continuity of the operator F, the condition (C_1) , $a_0 < 1$ and the estimate

$$\|F(x_m)\| \le \prod_{i=0}^{m-1} c_i\|F(x_0)\| \le c_0^m\|F(x_0)\|$$

we obtain $F(x_\star) = 0$ by letting $m \to \infty$. □

Remark 1.
(a) The R-order $\frac{1+\sqrt{5}}{2}$ of convergence for the method (1) is as given in the Lemma 2.5, Lemma 2.6 and Theorem 2.7 in [2].
(b) The sequence $\{A_m\}$ converges to $A_\star$, which is the right inverse of $F'(x_\star) = [x_\star, x_\star; F]$ (see section 2.3 in [2]).
(c) As in the Remark 2.8 [2], a possible choice of $\{p_n\}$ is to select it as an increasing sequence of real numbers converging to 1. So, we consider

$$\begin{gathered}a_0 = (1 - p_{-1})b\alpha, \quad b_0 = 6\nu, \quad c_0 = \delta + \lambda(2a_0 + b_0), \quad d_0 = c_0 + 3\lambda b_0, \\ a_m = (1 - p_{m-1})(1 + d_{m-1})b_{m-1}, \quad b_m = (1 + d_{m-1})^2 b_{m-1}c_{m-1}, \\ c_m = d_{m-1}^2 + \lambda(2a_m + b_m), \quad d_m = c_m + 3\lambda b_m.\end{gathered}$$

The sequence $\{a_m\}$ converges faster to zero, making $\{c_m\}$ and $\{d_m\}$ behave similarly. Thus, the sequence $\{\|x_{m+1}-x_m\|\}$ decreases faster, forcing a higher speed of convergence for $\{x_m\}$.

3. Numerical Results. In this section, we present the numerical results of applying the Moser-Secant method

$$\begin{cases} x_{m+1} = x_m - A_m F(x_m), \\ y_{m+1} = x_m + p_m(x_{m+1} - x_m), \\ K_{m+1} = [y_{m+1}, x_{m+1}; F], \\ A_{m+1} = 2A_m - A_m K_{m+1} A_m, \end{cases} \tag{5}$$

and the Moser-Steffensen-Kurchatov method

$$\begin{cases} x_{m+1} = x_m - A_m F(x_m), \\ y_{m+1} = x_m + (1 - p_m)(x_{m+1}), \\ K_{m+1} = [2y_{m+1} - x_{m+1}, x_{m+1}; F], \\ A_{m+1} = 2A_m - A_m K_{m+1} A_m \end{cases} \tag{6}$$

as a baseline for comparison. These methods combine secant and quasi-Newton techniques to approximate solutions efficiently.

3.1. Example 1: Academic example. In this example, we consider a basic nonlinear system, which is defined by the following set of equations

$$\begin{cases} (2x - x^2) + (y - \frac{y^2}{2}) = 0, \\ x + y = 0. \end{cases}$$

The solution to this system is known to be $x_* = (0, 0)$. We started with an initial guess of $\mathbf{x}_0 = (0.1, -0.3)$, and an initial approximation of $A_0 = (F'(x_0))^{-1}$ was used for the iterative method. The stopping criteria was set with a tolerance of $tol_u = 10^{-8}$ and a parameter $p_m = 0.15$. The results from the iterations and their convergence are detailed in the Table 1.

m	$\|\mathbf{F}(\mathbf{x_m})\|$ (5)	$\|\mathbf{F}(\mathbf{x_m})\|$ (1)	$\|\mathbf{F}(\mathbf{x_m})\|$ (6)
...	...	...	...
6	6.083968e-03	2.01025e-03	1.391648e-07
7	5.437738e-04	6.468644e-05	1.813980e-11
8	8.621826e-06	3.235434e-07	7.264144e-18
9	8.183577e-09	6.178086e-11	-
10	9.737637e-14	4.870678e-17	-
11	1.029824e-21	-	-

Table 1: Residual's norms for iterations of Example 1 using method (5), method (1) and method (6).

3.2. Example 2: Freudenstein and Roth Function. In this example, we consider the Freudenstein and Roth function, a classic nonlinear system often used to evaluate the performance of numerical methods. The system is defined by the following set of equations

$$\begin{cases} -13 + x_1 + ((5 - x_2)x_2 - 2)x_2 = 0, \\ -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2 = 0. \end{cases}$$

The known solution for this system is $x_* = (5, 4)$. To approach this solution, we began with an initial guess of $\mathbf{x}_0 = (0.5, 3.4)$ and used an initial estimate of $A_0 = (F'(x_0))^{-1}$. The parameters were set with a tolerance of $tol_u = 10^{-8}$ and $p_m = 0.9$. The iterative results and their convergence behavior are shown in the Table 2.

m	$\|\mathbf{F}(\mathbf{x_m})\|$ (5)	$\|\mathbf{F}(\mathbf{x_m})\|$ (1)	$\|\mathbf{F}(\mathbf{x_m})\|$ (6)
...	...	...	...
5	7.53000e+00	4.49947e+00	1.159506e-04
6	6.43237e+00	2.30754e+00	4.357130e-09
7	4.51643e+00	6.56425e-01	8.881784e-15
8	6.31642e+00	6.12237e-02	-
9	6.79762e-01	6.49258e-04	-
10	6.39402e-02	1.08621e-07	-
11	8.85630e-04	3.16273e-14	-
12	8.85630e-07	8.88178e-16	-
13	3.16273e-14	-	-
14	8.81784e-16	-	-

Table 2: Residual's norms for iterations of Example 2 using method (5), method (1) and method (6).

3.3. Example 3: Coupled oscillator system.

$$\begin{cases} (3x_1 - x^2) + \left(x_2 - \frac{x_2^2}{3}\right) + \left(x_3 - \frac{x_3^3}{4}\right) = 0, \\ x_1 + x_2 + x_3 = 1, \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

This system models the interaction between multiple oscillators, with each equation balancing various contributions from the variables x_1, x_2, and x_3. To solve this system, we used an initial guess of $x_0 = (-0.4, 1.0, 1.6)$ and an initial approximation of the inverse Jacobian matrix, $A_0 = (F'(x_0))^{-1}$. The parameters for the iterative method were set to $tol_u = 10^{-8}$ and $p_m = 0.7$. The results of the numerical solution are provided in Table 3.

m	$\|\mathbf{F}(\mathbf{x_m})\|$ (5)	$\|\mathbf{F}(\mathbf{x_m})\|$ (1)	$\|\mathbf{F}(\mathbf{x_m})\|$ (6)
...	...	...	...
15	3.712782e-04	1.476267e-04	3.446419e-07
16	8.940192e-05	1.940265e-05	5.587108e-10
17	1.892783e-05	7.066557e-07	4.082210e-14
18	1.440488e-06	2.593621e-09	-
19	1.594533e-08	2.445511e-13	-
20	5.942390e-12	1.570092e-16	-
21	1.110223e-16	-	-

Table 3: Residual's norms for iterations of Example 3 using method (5), method (1) and method (6).

3.4. Example 4: Benchmarking example. In this section, we solve the following system of nonlinear equations

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0, \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0, \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0. \end{cases}$$

This system is serves as a benchmark for testing numerical algorithms that solve nonlinear systems of equations. It incorporates polynomial, trigonometric, and exponential functions, making it a challenging and comprehensive test case.

We applied two methods to solve the system using a $p_m = 0.94$, $tol_u = 10^{-8}$ and an initial guess of $[-1.8, 0.1, 0.9]$. The known solution vector is $x = [0.5, 0.0, -0.523...]$.

m	$\lVert\mathbf{F}(\mathbf{x_m})\rVert$ (5)	$\lVert\mathbf{F}(\mathbf{x_m})\rVert$ (1)	$\lVert\mathbf{F}(\mathbf{x_m})\rVert$ (6)
...	...	...	...
8	1.354285e-01	3.197802e-03	5.520144e-05
9	9.972021e-02	6.896170e-05	1.189791e-07
10	5.500205e-02	4.398313e-08	3.548948e-12
11	1.749767e-02	1.312284e-13	-
12	1.934708e-03	-	-
13	2.704585e-05	-	-
14	6.881804e-09	-	-
15	4.373771e-15	-	-

Table 4: Residual's norms for iterations of Example 4 using method (5), method (1) and method (6).

The final solution after 15 iterations for all methods is presented in Table 4.

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REFERENCES

1. S. Amat, J.A. Ezquerro, M.A. Hernandez-Veron, *Approximation of inverse operators by a new family of high-order iterative methods*, Numerical Linear Algebra with Applications, **21** (2014), 629–644. https://doi.org/10.1002/nla.1917
2. S. Amat, S. Busquier, M.A. Hernandez-Verson, D. Levin, D.F. Yanez, *On an iterative Moser-Secant's method with applications in optimization problems*, Preprint (2024). (Private message)
3. I.K. Argyros, *On the Secant method for solving nonsmooth equations*, Journal of Mathematical Analysis and Applications, **322** (2006), 146–157. https://doi.org/10.1016/j.jmaa.2005.09.006
4. I. Argyros, S. Shakhno, Y. Shunkin, *Improved convergence analysis of Gauss-Newton-Secant method for solving nonlinear Least Squares Problems*, Mathematics, **7** (2019), 99. https://doi.org/10.3390/math7010099
5. I.K. Argyros, S. Shakhno, *Extended Two-Step-Kurchatov method for solving Banach space valued nondifferentiable equations*, International Journal of Applied and Computational Mathematics, **6** (2020), 32. https://doi.org/10.1007/s40819-020-0784-y
6. I.K. Argyros, *The theory and applications of iteration methods*, 2nd ed., Engineering Series, CRC Press, Taylor & Francis Group, Boca Raton, Florida, USA, 2022. https://doi.org/10.1201/9780203719169

7. A.S. Berahas, O. Shoham, L.N. Vicente, *Full-low evaluation methods for derivative-free optimization*, Optimization Methods and Software, **38** (2023), 386–411. https://doi.org/10.48550/arXiv.2107.11908
8. A. Diaconu, *Sur la maniere d'obtenir et sur la convergence de certaines methodes iteratives*, Seminar on Functional Analysis and Numerical Methods, Univ. "Babe?-Bolyai", Cluj-Napoca, Preprint 87-1 (1987), 25–74.
9. M.J. Ebadi, A. Fahs, H. Fahs, R. Dehghani, *Competitive secant (BFGS) methods based on modified secant relations for unconstrained optimization*, Optimization, **72** (2023), 1691–1706. https://doi.org/10.1080/02331934.2022.2048381
10. M. Grau-Sanchez, M. Noguera, S. Amat, *On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods*, Journal of Computational and Applied Mathematics, **237** (2013), 363–372. https://doi.org/10.1016/j.cam.2012.06.005
11. O.H. Hald, *On a Newton-Moser type method*, Numerische Mathematik, **23** (1975), 411–425. https://doi.org/10.1007/BF01437039
12. N. Hamel, N. Benrabia, M. Ghiat, H. Guebbai, *A new hybrid conjugate gradient algorithm based on the Newton direction to solve unconstrained optimization problems*, Journal of Applied Mathematics and Computing, **69** (2023), 2531–2548. https://doi.org/10.1007/s12190-022-01821-z
13. M.A. Hernandez-Veron, A.A. Magzenuin, E. Munez, E.G. Villalba, *Solving non-differentiable Hammerstein integral equations via first-order divided differences*, Numerical Algorithms, **97** (2024), 567–594. https://doi.org/10.1007/s11075-023-01715-6
14. I. Moret, *A note on Newton type iterative methods*, Computing, **33** (1984), 65–73. https://doi.org/10.1007/BF02243076
15. M.A. Nawkhas, N.A. Sulaiman, *Using geometric arithmetic mean to solve non-linear fractional programming problems*, International Journal of Applied and Computational Mathematics, **8** (2022), 217. https://doi.org/10.1007/s40819-022-01424-z
16. Y. Nesterov, *Superfast second-order methods for unconstrained convex optimization*, Journal of Optimization Theory and Applications, **191** (2021), 1–30. https://doi.org/10.1007/s10957-021-01930-y
17. H. Petzeltova, *Remark on a Newton-Moser type method*, Commentationes Mathematicae Universitatis Carolinae, **21** (1980), 719–725.
18. S.M. Shakhno, *Nonlinear majorants for investigation of methods of linear interpolation for the solution of nonlinear equations*, ECCOMAS 2004—European Congress on Computational Methods in Applied Sciences and Engineering, 2004.
19. S.M. Shakhno, *Gauss–Newton–Kurchatov Method for the solution of nonlinear Least-Squares Problems*, Journal of Mathematical Sciences, **247** (2020), 58–72. https://doi.org/10.1007/s10958-020-04789-y
20. S. Ulm, *Ob obobshchennykh razdelennykh raznostyakh*, Eesti NSV Teaduste Akadeemia Toimetised Fuusika-Matemaatika, **16** (1967), 146–156.
21. T. Wang, Q. Huang, *A new Newton method for convex optimization problems with singular Hessian matrices*, AIMS Mathematics, **8** (2023), 21161–21175. https://doi.org/10.3934/math.20231078
22. G.L. Yuan, Y.-J. Zhou, M.-X. Zhang, *A hybrid conjugate gradient algorithm for nonconvex functions and its applications in image restoration problems*, Journal of the Operations Research Society of China, **11** (2023), 759–781. https://doi.org/10.1007/s40305-022-00424-6
23. E.J. Zehnder, *A remark about Newton's method*, Communications on Pure and Applied Mathematics, **27** (1974), 361–366. https://doi.org/10.1002/cpa.3160270305

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