УДК 517.5

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ON NUMERICAL STABILITY OF CONTINUED FRACTIONS

V. Hladun, V. Kravtsiv, M. Dmytryshyn, R. Rusyn. *On numerical stability of continued fractions*, Mat. Stud. **62** (2024), 168–183.

The paper considers the numerical stability of the backward recurrence algorithm (BR-algorithm) for computing approximants of the continued fraction with complex elements. The new method establishes sufficient conditions for the numerical stability of this algorithm and the error bounds of the calculation of the nth approximant of the continued fraction with complex elements. It follows from the obtained conditions that the numerical stability of the algorithm depends not only on the rounding errors of the elements and errors of machine operations but also on the value sets and the element sets of the continued fraction. The obtained results were used to study the numerical stability of the BR-algorithm for computing the approximants of the continued fraction expansion of the ratio of Horn's confluent functions H_7. Bidisc and bicardioid regions are established, which guarantee the numerical stability of the BR-algorithm.

The obtained result is applied to the study of the numerical stability of computing approximants of the continued fraction expansion of the ratio of Horn's confluent function H_7 with complex parameters. In addition, the analysis of the relative errors arising from the computation of approximants using the backward recurrence algorithm, the forward recurrence algorithm, and Lenz's algorithm is given.

The method for studying the numerical stability of the BR-algorithm proposed in the paper can be used to study the numerical stability of the branched continued fraction expansions and numerical branched continued fractions with elements in angular and parabolic domains.

1. Introduction. Continued fractions and their multidimensional generalizations (branched continued fractions) are effective tools for representing and approximating special functions that arise in physics, quantum mechanics, engineering, applied mathematics, and computer science (see, for example, [1–5]). An important property of these fractions is the property of stability to perturbations, which ensures their effective use for computing rational approximations of functions, ensuring numerical stability and accuracy of computation ([6–10]).

The numerical stability of the continued fraction (branched continued fraction), more precisely, the numerical stability of the algorithm for computing approximants of the continued fraction (branched continued fraction) lies in its ability to give the correct values of these approximants in the presence of rounding errors of its elements and errors of machine operations (see, [11, 12]). Note that this is an important characteristic of algorithms since small errors in computation can lead to significant deviations in the value of the computed approximants. The level of stability may vary depending on the specific calculation algorithm. Some algorithms may be more robust than others and minimize the impact of small changes on results. One of the ways to ensure the numerical stability of the algorithm is

2020 *Mathematics Subject Classification:* 32A17,33F05, 65G50.
Keywords: continued fraction; numerical approximation; roundoff error.
doi:10.30970/ms.62.2.168-183

to use methods of control and management of computation errors. For example, scaling, precision expansion, or rounding with certain rounding rules can be used in algorithms for computing approximants. For problems where computation accuracy is critical, it is possible to use algorithms for computing approximants with guaranteed numerical stability, which leads to minor deviations even with a large number of computation.

There are several algorithms for computing approximants. Among them, the most famous algorithms are the forward recurrent algorithm (FR-algorithm) and the backward recurrence algorithm (BR-algorithm). Computing of approximants by the FR-algorithm is in computing the ratio of the numerator to the denominator, which are computed by three-term recurrence relations. At the same time, if the sequence of numerators and denominators goes to infinity or zero, then when computing the sequence of approximants, it is possible to go beyond the limits of the bit grid. Also, the FR-algorithm is characterized by the accumulation of errors during the computation of approximants ([13]). The analysis of these errors is considered in [14]. G. Blanch, investigating the numerical stability of the BR-algorithm and FR-algorithm, concluded that the first one is more stable to the accumulation of errors than the second one ([15]). This problem was also studied by N. Macon, M. Baskervill [16] and W. B. Jones, W. J. Thron ([17]). In [17], the upper bounds are set for the relative errors of computing of approximants by the BR-algorithm. It is also established here that the numerical stability of this algorithm depends on the elements of the approximants. The problem of numerical stability of continued fraction expansions (branched continued fraction expansions) of special functions was also studied in the works [4, 18–20].

This paper continues the study of the numerical stability of the BR-algorithm for computing approximants of the continued fraction, which were considered (initiated) in works [16, 17]. The new method establishes sufficient conditions for the numerical stability of the BR-algorithm and the error bounds of the calculation of the nth approximant of the continued fraction with complex elements. At some values of the constant η, the obtained error bound is smaller than the error bound obtained in [3, Section 8.4], [17, Section 3]. It follows from the obtained conditions that the numerical stability of the algorithm depends not only on the rounding errors of the elements and errors of machine operations but also on the value sets and the element sets of the continued fraction. The obtained results were used to study the numerical stability of the BR-algorithm for computing the approximants of the continued fraction expansion of the ratio of Horn's confluent functions H_7. Bidisc and bicardioid regions are established, which guarantee the numerical stability of the BR-algorithm. Numerical experiments show that the relative errors of computing the approximants of the continued fraction by the BR-algorithm are limited and, unlike the FR-algorithm and Lenz's algorithm, do not exceed rounding units. The property of non-accumulation of relative errors allows the BR-algorithm to be used in the case when the order of approximate continued fraction is large, and the accuracy of computing is critically important.

The present paper is organized as follows. In Section 2, we provide a formula for the relative rounding error of the nth approximant of the continued fraction. In the next section, we establish sufficient conditions for the numerical stability of the BR-algorithm (Theorem 3.1). In Section 4, we consider the numerical stability of the BR-algorithm for computing the approximants of the continued fraction expansions of ratios of the Horn's confluent functions H_7. Section 5 provides an analysis of the relative errors of computing the approximants of this expansion by BR-algorithm, FR-algorithm, and Lenz's algorithm. Finally, we collect our conclusions in Section 6.

2. Formula of relative roundoff error. Consider a continued fraction with complex elements of the form

$$1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots}}. \tag{1}$$

Let n be a fixed natural number. For computing the nth approximant

$$f_n = 1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots+\cfrac{a_n}{1}}}.$$

of the continued fraction (1), we use the BR-algorithm, which consists in calculating

$$Q_k^{(n)} = 1 + G_{k+1}^{(n)}, \quad G_{k+1}^{(n)} = \frac{a_{k+1}}{Q_{k+1}^{(n)}}, \quad k \in \{n-1, n-2, \ldots, 0\}, \tag{2}$$

under the initial condition $Q_n^{(n)} = 1$. Then, $f_n = Q_0^{(n)}$.

We set

$$g_k^{(n)} = \frac{a_k}{Q_{k-1}^{(n)} Q_k^{(n)}}, \quad k \in \{1, \ldots, n\}. \tag{3}$$

For $k \in \{1, \ldots, n\}$, if a_k, $a_k \neq 0$, is approximated by $\widehat{a}_k = RN(a_k)$, where $RN(\cdot)$ is the rounding function, there is a relative rounding error

$$\varepsilon_k^{(a)} = \frac{\widehat{a}_k - a_k}{a_k}. \tag{4}$$

If $\widehat{a}_k = a_k = 0$, we assume that $\varepsilon_k^{(a)} = 0$.

Let $* \in \{+, -, \times, /\}$ be a binary arithmetic operation. Let $\circledast \in \{\oplus, \ominus, \otimes, \oslash\}$ denote the machine implementation of the arithmetic operation $*$ in floating-point arithmetic. Then

$$\varepsilon^{(*)} = \frac{x \circledast y - x * y}{x * y}$$

is the relative error of the machine operation $*$. The machine error can also be written as

$$\varepsilon^{(*)} = \frac{RN(x * y) - x * y}{x * y}.$$

In addition, the number $\widehat{f}_n = \widehat{Q}_0^{(n)}$, where

$$\widehat{Q}_k^{(n)} = 1 \oplus \widehat{G}_{k+1}^{(n)}, \quad \widehat{G}_{k+1}^{(n)} = \widehat{a}_{k+1} \oslash \widehat{Q}_{k+1}^{(n)}, \quad k \in \{n-1, n-2, \ldots, 0\},$$

under the initial condition $\widehat{Q}_n^{(n)} = 1$, is an approximate value of the approximant f_n.

Definition 1. The *algorithm* for calculating the nth approximant of the continued fraction (1) is called *stable* if, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|\widehat{f}_n - f_n|/|f_n| < \varepsilon$, if the following inequalities

$$\left|\frac{\widehat{a}_k - a_k}{a_k}\right| < \delta, \quad k \in \{1, \ldots, n\}, \quad \left|\frac{x \circledast y - x * y}{x * y}\right| < \delta, \quad * \in \{+, -, \times, /\}, \quad \circledast \in \{\oplus, \ominus, \otimes, \oslash\},$$

are valid.

Let $\varepsilon_{k,n}^{(Q)} = (\widehat{Q}_k^{(n)} - Q_k^{(n)})/Q_k^{(n)}$ be the relative errors of $Q_k^{(n)}$, $k \in \{1, \ldots, n\}$. Since $Q_n^{(n)} = \widehat{Q}_n^{(n)} = 1$, we have $\varepsilon_{n,n}^{(Q)} = 0$. We prove that

$$\varepsilon_{k,n}^{(Q)} = \left(1 - g_{k+1}^{(n)} + \frac{g_{k+1}^{(n)}(1 + \varepsilon_{k+1}^{(a)})(1 + \varepsilon_{k+1,n}^{(/)})}{1 + \varepsilon_{k+1,n}^{(Q)}}\right)(1 + \varepsilon_{k,n}^{(+)}) - 1, \quad k \in \{0, \ldots, n-1\}, \tag{5}$$

where

$$\varepsilon_{k,n}^{(+)} = \frac{1 \oplus \widehat{G}_{k+1}^{(n)} - (1 + \widehat{G}_{k+1}^{(n)})}{1 + \widehat{G}_{k+1}^{(n)}}, \quad k \in \{0, \ldots, n-1\}, \tag{6}$$

$$\varepsilon_{k,n}^{(/)} = \frac{\widehat{a}_k \oslash \widehat{Q}_{k+1}^{(n)} - \widehat{a}_k / \widehat{Q}_{k+1}^{(n)}}{\widehat{a}_k / \widehat{Q}_{k+1}^{(n)}}, \quad k \in \{1, \ldots, n\}, \tag{7}$$

are relative errors of machine operations of addition and division, respectively, $\varepsilon_k^{(a)}$ is the relative error of a_k, $k \in \{1, \ldots, n\}$, and $g_k^{(n)}$, $k \in \{1, \ldots, n\}$ are defined by (3).

For any $0 \leq k \leq n-1$ we have

$$\varepsilon_{k,n}^{(Q)} = \frac{\widehat{Q}_k^{(n)} - Q_k^{(n)}}{Q_k^{(n)}} = \frac{1 \oplus \widehat{a}_{k+1} \oslash \widehat{Q}_{k+1}^{(n)}}{Q_k^{(n)}} - 1 =$$
$$= \frac{1}{Q_k^{(n)}} \left(1 + \frac{a_{k+1}(1 + \varepsilon_{k+1}^{(a)})(1 + \varepsilon_{k+1,n}^{(/)})}{Q_{k+1}^{(n)}(1 + \varepsilon_{k+1,n}^{(Q)})} \right) (1 + \varepsilon_{k,n}^{(+)}) - 1 =$$
$$= \left(\frac{1}{Q_k^{(n)}} + \frac{a_{k+1}(1 + \varepsilon_{k+1}^{(a)})(1 + \varepsilon_{k+1,n}^{(/)})}{Q_k^{(n)} Q_{k+1}^{(n)}(1 + \varepsilon_{k+1,n}^{(Q)})} \right) (1 + \varepsilon_{k,n}^{(+)}) - 1.$$

From here we get (5) since

$$\frac{1}{Q_k^{(n)}} = \frac{1}{Q_k^{(n)}} \left(Q_k^{(n)} - \frac{a_{k+1}}{Q_{k+1}^{(n)}} \right) = 1 - g_{k+1}^{(n)}.$$

Using the recurrent relation (5), for any $0 \leq k \leq n-1$ we obtain

$$\varepsilon_{k,n}^{(Q)} = (1 - g_{k+1}^{(n)})(1 + \varepsilon_{k,n}^{(+)}) - 1 + \frac{g_{k+1}^{(n)}(1 + \varepsilon_{k+1}^{(a)})(1 + \varepsilon_{k,n}^{(+)})(1 + \varepsilon_{k+1,n}^{(/)})}{(1 - g_{k+2}^{(n)})(1 + \varepsilon_{k+1,n}^{(+)}) + \ddots + \frac{g_n^{(n)}(1 + \varepsilon_n^{(a)})(1 + \varepsilon_{n-1,n}^{(+)})(1 + \varepsilon_{n,n}^{(/)})}{1}}, \tag{8}$$

where $k \in \{0, \ldots, n-1\}$. It follows that

$$\varepsilon_n^{(f)} = \frac{\widehat{f}_n - f_n}{f_n} = (1 - g_1^{(n)})(1 + \varepsilon_{0,n}^{(+)}) - 1 + \frac{g_1^{(n)}(1 + \varepsilon_1^{(a)})(1 + \varepsilon_{0,n}^{(+)})(1 + \varepsilon_{1,n}^{(/)})}{(1 - g_2^{(n)})(1 + \varepsilon_{1,n}^{(+)}) + \ddots + \frac{g_n^{(n)}(1 + \varepsilon_n^{(a)})(1 + \varepsilon_{n-1,n}^{(+)})(1 + \varepsilon_{n,n}^{(/)})}{1}},$$

that is, the relative error of the nth approximant of the continued fraction (1).

3. Numerical stability of the backward recurrence algorithm. In this section, we establish sufficient conditions for the numerical stability of the BR-algorithm for computing the nth approximant of the continued fraction (1).

Theorem 1. *Let $n \in \mathbb{N}$. The BR-algorithm for computing the nth approximant of the continued fraction (1) is stable if there exist non-negative constants $\varepsilon^{(a)}$, $\varepsilon^{(+)}$, and $\varepsilon^{(/)}$ such that*

$$|\varepsilon_k^{(a)}| \le \varepsilon^{(a)} u, \quad k \in \{1, \ldots, n\}, \quad |\varepsilon_{k,n}^{(+)}| \le \varepsilon^{(+)} u, \quad k \in \{0, \ldots, n-1\},$$
$$|\varepsilon_{k,n}^{(/)}| \le \varepsilon^{(/)} u, \quad k \in \{1, \ldots, n\}, \tag{9}$$

where u denotes the unit roundoff, $\varepsilon_k^{(a)}$, $\varepsilon_{k,n}^{(+)}$, $\varepsilon_{k,n}^{(/)}$ are defined by (4), (6), and (7), respectively, and there exists a constant η, $0 < \eta < 1$, such that

$$|g_k^{(n)}| \le \eta, \quad k \in \{1, \ldots, n\}, \tag{10}$$

and

$$\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2} \le \frac{1}{4}. \tag{11}$$

In addition, the following

$$|\varepsilon_n^{(f)}| \le \frac{(1-\eta)(1+\varepsilon^{(+)}u)}{2} -$$
$$- \frac{\sqrt{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2 - 4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}}{2} \tag{12}$$

is valid for the relative error of the nth approximant of the continued fraction (1).

Proof. Using (8), we estimate the absolute value of the relative error of computing the quantities $Q_k^{(n)}$ defined in (2).

Let $\{t_n(\omega)\}_{n\ge0}$ and $\{T_n(\omega)\}_{n\ge0}$ be the sequence of linear fractional transformations

$$t_0(\omega) = 2-(1-\eta)(1+\varepsilon^{(+)}u)+\omega, \quad t_n(\omega) = -\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{2-(1-\eta)(1+\varepsilon^{(+)}u)+\omega}, \quad n \ge 1,$$

and $T_0(\omega) = t_0(\omega)$, $T_n(\omega) = T_{n-1}(t_n(\omega))$, $n \ge 1$.

Let us establish the conditions under which $T_n(\omega) > 0$, $n \ge 1$. To do this, let us set

$$\omega' = \frac{\omega}{2-(1-\eta)(1+\varepsilon^{(+)}u)}$$

and consider sequences $\{t'_n(\omega')\}_{n\ge0}$ and $\{T'_n(\omega')\}_{n\ge0}$ of linear fractional transformations

$$t'_0(\omega) = (2-(1-\eta)(1+\varepsilon^{(+)}u))(1+\omega'),$$
$$t'_n(\omega) = -\frac{\dfrac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2}}{1+\omega'}, \quad n \ge 1,$$

and $T'_0(\omega') = t'_0$, $T'_n(\omega') = T_{n-1}(t'_n(\omega'))$, $n \ge 1$.

With (11), we write

$$\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2} = p(1-p),$$

where

$$p=\frac{1}{2}\left(1-\sqrt{1-\frac{4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2}}\right),\quad 0<p<\frac{1}{2}.$$

Now, we consider a periodic continued fraction

$$1-\cfrac{p(1-p)}{1-\cfrac{p(1-p)}{1-\ddots}}. \tag{13}$$

Let $h_n=A_n/B_n$ be the nth approximant of the continued fraction (13), $n\geq 1$. According to [21, Theorem 3.2] the continued fraction (14) converges to the value $(1-\sqrt{1-4p(1-p)})/2$. In addition, it is easy to show that the approximants h_n, $n\geq 1$, form a monotonically decreasing sequence such that

$$1>h_n>\frac{1-\sqrt{1-4p(1-p)}}{2},\quad n\geq 1, \tag{14}$$

and $B_n=\sum_{i=0}^{n}p^i(1-p)^{n-i}$, $A_n=B_{n+1}$, $n\geq 1$.

Let n be an arbitrary natural number. Then

$$T'_n(\omega')=(2-(1-\eta)(1+\varepsilon^{(+)}u))\frac{A_n+\omega' A_{n-1}}{B_n+\omega' B_{n-1}}=(2-(1-\eta)(1+\varepsilon^{(+)}u))\frac{B_{n+1}+\omega' B_n}{B_n+\omega' B_{n-1}}.$$

Hence $T'_n(\omega')>0$, if

$$-\omega'<\frac{B_{n+1}}{B_n}\quad\text{or}\quad -\omega'>\frac{B_{n+1}}{B_n},\quad n\geq 1.$$

It follows from (14) that $T'_n(\omega')>0$, if

$$-\omega'<\frac{1-\sqrt{1-4p(1-p)}}{2}\quad\text{or}\quad -\omega'>1.$$

We choose $\omega=(1-\eta)(1+\varepsilon^{(+)}u)-1$. Then

$$\omega'=\frac{(1-\eta)(1+\varepsilon^{(+)}u)-1}{2-(1-\eta)(1+\varepsilon^{(+)}u)}\quad\text{and}\quad T'_n(\omega')>0,$$

if

$$\omega'<\frac{1}{2}\left(1-\sqrt{1-\frac{4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2}}\right).$$

Note that the last inequality holds for all $0<\eta<1$.

Using (5), by induction on k, we show that

$$|\varepsilon^{(Q)}_{k,n}|\leq(1-\eta)(1+\varepsilon^{(+)}u)-1-\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_{n-k}}{d_{n-k}}}},\quad k\in\{n-1,n-2,\dots,0\}, \tag{15}$$

where

$$c_r = -\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u), \quad r \in \{1,\dots,n-k\},$$
$$d_r = 2-(1-\eta)(1+\varepsilon^{(+)}u), \quad r \in \{1,\dots,n-k-1\}, \quad d_{n-k}=1.$$

Note that

$$2-(1-\eta)(1+\varepsilon^{(+)}u) + \cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_{n-k}}{d_{n-k}}}} = T'_{n-k-1}\left(\frac{(1-\eta)(1+\varepsilon^{(+)}u)-1}{2-(1-\eta)(1+\varepsilon^{(+)}u)}\right) > 0$$

if $0<\eta<1$ and hence the condition (11) holds.

For $k=n-1$ we have

$$\begin{aligned}
|\varepsilon^{(Q)}_{n-1,n}| &= |(1-g^{(n)}_n + g^{(n)}_n(1+\varepsilon^{(a)}_n)(1+\varepsilon^{(/)}_{n,n}))(1+\varepsilon^{(+)}_{n-1,n})-1| \le \\
&\le |\varepsilon^{(+)}_{n-1,n} + g^{(n)}_n((1+\varepsilon^{(a)}_n)(1+\varepsilon^{(/)}_{n,n})-1)(1+\varepsilon^{(+)}_{n-1,n})| \le \\
&\le |\varepsilon^{(+)}_{n-1,n}| + |g^{(n)}_n|(1+|\varepsilon^{(+)}_{n-1,n}|)(|\varepsilon^{(a)}_n| + |\varepsilon^{(/)}_n| + |\varepsilon^{(a)}_n||\varepsilon^{(/)}_n|) \le \\
&\le \varepsilon^{(+)}u + \eta(1+\varepsilon^{(+)}u)(\varepsilon^{(a)}u + \varepsilon^{(/)}u + \varepsilon^{(a)}\varepsilon^{(/)}u^2) = \\
&= (1-\eta)(1+\varepsilon^{(+)}u) - 1 + \eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u).
\end{aligned}$$

Let (15) be true for $k=r+1$, $0\le r\le n-2$. Then, for $k=r$,

$$\begin{aligned}
|\varepsilon^{(Q)}_{r,n}| &= \left|\left(1-g^{(n)}_{r+1} + \frac{g^{(n)}_{r+1}(1+\varepsilon^{(a)}_{r+1})(1+\varepsilon^{(/)}_{r+1,n})}{1+\varepsilon^{(Q)}_{r+1,n}}\right)(1+\varepsilon^{(+)}_{r,n})-1\right| = \\
&= \left|\varepsilon^{(+)}_{r,n} + g^{(n)}_{r+1}\left(\frac{(1+\varepsilon^{(a)}_{r+1})(1+\varepsilon^{(/)}_{r+1,n})}{1+\varepsilon^{(Q)}_{r+1,n}} - 1\right)(1+\varepsilon^{(+)}_{r,n})\right| \le \\
&\le |\varepsilon^{(+)}_{r,n}| + |g^{(n)}_{r+1}|(1+\varepsilon^{(+)}_{r,n})\frac{|\varepsilon^{(a)}_{r+1}| + |\varepsilon^{(/)}_{r+1,n}| + |\varepsilon^{(a)}_{r+1}||\varepsilon^{(/)}_{r+1,n}| + |\varepsilon^{(Q)}_{r+1,n}|}{1-|\varepsilon^{(Q)}_{r+1,n}|} \le \\
&\le \varepsilon^{(+)}u + \eta(1+\varepsilon^{(+)}u)\left(\frac{(1+\varepsilon^{(a)})(1+\varepsilon^{(/)})}{1-|\varepsilon^{(Q)}_{r+1,n}|} - 1\right) = \\
&= (1-\eta)(1+\varepsilon^{(+)}u) - 1 + \frac{\eta(1+\varepsilon^{(+)}u)(1+\varepsilon^{(a)})(1+\varepsilon^{(/)})}{1-|\varepsilon^{(Q)}_{r+1,n}|} \le \\
&\le (1-\eta)(1+\varepsilon^{(+)}u) - 1 + \frac{\eta(1+\varepsilon^{(+)}u)(1+\varepsilon^{(a)})(1+\varepsilon^{(/)})}{2-(1-\eta)(1+\varepsilon^{(+)}u) + \cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_{n-r-1}}{d_{n-r-1}}}}} = \\
&= (1-\eta)(1+\varepsilon^{(+)}u) - 1 - \cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_{n-r}}{d_{n-r}}}}.
\end{aligned}$$

Setting $k = 0$ in (15), for the relative error of the nth approximant of the continued fraction (1), we obtain the following

$$|\varepsilon_n^{(f)}| \le (1-\eta)(1+\varepsilon^{(+)}u) - 1 - \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}}. \tag{16}$$

Now, we find the value of the right-hand side of (16). It is sufficient to find

$$\lim_{n\to+\infty} \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}}.$$

The transformation

$$t(\omega) = -\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{2-(1-\eta)(1+\varepsilon^{(+)}u)+\omega}$$

is a linear fractional transformation with fixed points

$$\omega_1 = \frac{(1-\eta)(1+\varepsilon^{(a)}u)}{2} - 1 + \\ + \frac{\sqrt{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2 - 4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}}{2},$$

$$\omega_2 = \frac{(1-\eta)(1+\varepsilon^{(a)}u)}{2} - 1 - \\ - \frac{\sqrt{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2 - 4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}}{2},$$

herewith ω_1 is an attractive point.

Let

$$(2-(1-\eta)(1+\varepsilon^{(+)}u))^2 - 4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u) > 0.$$

Then $\omega_1 \ne \omega_2$. Since $(1-\eta)(1+\varepsilon^{(+)}u) - 1 \ne t^{-1}(\omega_2)$, then (see, [21])

$$\lim_{n\to+\infty} \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}} = \omega_1$$

and, therefore,

$$(1-\eta)(1+\varepsilon^{(+)}u) - 1 - \omega_1 = \\ = \frac{(1-\eta)(1+\varepsilon^{(+)}u)}{2} - \frac{\sqrt{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2 - 4\eta(1+\varepsilon^{(a)})(1+\varepsilon^{(+)})(1+\varepsilon^{(/)})}}{2}.$$

Using the formula for the difference between compositions of linear fractional transformations $T_{n+1}((1-\eta)(1+\varepsilon^{(+)}u)-1) - T_n((1-\eta)(1+\varepsilon^{(+)}u)-1)$ (see, [7]), we have

$$\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{T_{n+1}((1-\eta)(1+\varepsilon^{(+)}u)-1)} - \frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{T_n((1-\eta)(1+\varepsilon^{(+)}u)-1)} \ge 0,$$

that is, the sequence

$$(1-\eta)(1+\varepsilon^{(+)}u)-1-\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_n}{d_n}}},\quad n\ge 1,$$

is monotonically nondecreasing and the estimate (12) holds for the relative error of the nth approximant of the continued fraction (1).

Next, we show that from (12) it follows that the conditions of Definition 1 are fulfilled. Consider the function

$$\varphi(u)=\frac{(1-\eta)(1+\varepsilon^{(+)}u)}{2}-$$
$$-\frac{\sqrt{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2-4\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}}{2}. \tag{17}$$

We set $\varepsilon^{(a)}u=x_1$, $\varepsilon^{(+)}u=x_2$, and $\varepsilon^{(/)}=x_3$. Then

$$\varphi(u)=\psi(x_1,x_2,x_3)=$$
$$=\frac{(1-\eta)(1+x_2)-\sqrt{(2-(1-\eta)(1+x_2))^2-4\eta(1+x_1)(1+x_2)(1+x_3)}}{2}.$$

Since $\lim\limits_{u\to+0}x_i=0,\ i\in\{1,2,3\},\ \lim\limits_{u\to+0}\varphi(u)=0$, we have

$$\lim_{\substack{x_1\to+0\\x_2\to+0\\x_3\to+0}}\psi(x_1,x_2,x_3)=0.$$

In addition, for any $\varepsilon>0$ there exists $\delta>0$ such that for all x_1, x_2, x_3 such that

$$0<\sqrt{x_1^2+x_2^2+x_3^2}<\delta,$$

the inequality $\psi(x_1,x_2,x_3)<\varepsilon$ holds. If

$$x_1=\varepsilon^{(a)}u<\frac{\delta}{\sqrt3},\quad x_2=\varepsilon^{(+)}u<\frac{\delta}{\sqrt3},\quad x_3=\varepsilon^{(/)}u<\frac{\delta}{\sqrt3},$$

then provide

$$|\varepsilon_k^{(a)}|\le\varepsilon^{(a)}u<\frac{\delta}{\sqrt3},\quad k\in\{1,\dots,n\},\quad |\varepsilon_{k,n}^{(+)}|\le\varepsilon^{(+)}u<\frac{\delta}{\sqrt3},\quad k\in\{0,\dots,n-1\},$$

and

$$|\varepsilon_{k,n}^{(/)}|\le\varepsilon^{(/)}u<\frac{\delta}{\sqrt3},\quad k\in\{1,\dots,n\},$$

the inequality

$$|\varepsilon_n^{(f)}|<\varepsilon$$

holds, which proves the numerical stability of the BR-algorithm for computing the nth approximant of the continued fraction (1).

Finally, let

$$(2-(1-\eta)(1+\varepsilon^{(+)}u))^2-4\eta(1+\varepsilon^{(a)})(1+\varepsilon^{(+)})(1+\varepsilon^{(/)})=0.$$

Then $\omega_1 = \omega_2$ and

$$((1-\eta)(1+\varepsilon^{(+)}u)-1-\lim_{n\to+\infty}\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_n}{d_n}}}=\frac{1}{2}(1-\eta)(1+\varepsilon^{(+)}u),$$

where

$$\eta = 1-\frac{2\mu(u)}{1+\varepsilon^{(+)}u},$$

$$\mu(u)=\Big(((1+\varepsilon^{(a)}u)(1+\varepsilon^{(/)}u)-1)^2+(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)-1\Big)^{1/2}-$$
$$-(1+\varepsilon^{(a)}u)(1+\varepsilon^{(/)}u)+1. \tag{18}$$

Thus,

$$|\varepsilon_n^{(f)}|\le\mu(u)$$

is valid and, as in the previous case, we obtain the numerical stability of the BR-algorithm for computing the nth approximants of the continued fraction (1). □

Remark 1. It follows from inequalities (9) that $\varepsilon_k^{(a)}=O(u)$, $\varepsilon_{k,n}^{(+)}=O(u)$, $\varepsilon_{k,n}^{(/)}=O(u)$. If

$$\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2}<\frac{1}{4},\quad\text{then}\ \lim_{u\to+0}\frac{\varphi(u)}{u}=\frac{\varepsilon^{(+)}+\eta(1+\varepsilon^{(a)})(1+\varepsilon^{(/)})}{1-\eta},$$

where $\varphi(u)$ is defined by (17), and, therefore, $\varepsilon_n^{(f)}=O(u)$. And if

$$\frac{\eta(1+\varepsilon^{(a)}u)(1+\varepsilon^{(+)}u)(1+\varepsilon^{(/)}u)}{(2-(1-\eta)(1+\varepsilon^{(+)}u))^2}=\frac{1}{4},\quad\text{then}\ \lim_{u\to+0}\frac{\mu(u)}{\sqrt{u}}=\sqrt{\varepsilon^{(a)}+\varepsilon^{(+)}+\varepsilon^{(/)}},$$

where $\mu(u)$ is defined by (18), and, therefore, $\varepsilon_n^{(f)}=O(\sqrt{u})$.

4. Applications. In this section, we consider the numerical stability of the BR-algorithm for computing the approximants of the continued fraction expansion of ratio of Horn's confluent functions H_7.

Recall that Horn's confluent function H_7 is defined as follows

$$\mathrm{H}_7(\alpha;\gamma_1,\gamma_2;\mathbf{z})=\sum_{r,s=0}^{\infty}\frac{(\alpha)_{2r+s}}{(\gamma_1)_r(\gamma_2)_s}\frac{z_1^r z_2^s}{r!\,s!},\quad |z_1|<1/4,\ |z_2|<+\infty,$$

where $\alpha,\gamma_1,\gamma_2\in\mathbb{C}$, $\gamma_1,\gamma_2\notin\{0,-1,-2,\ldots\}$, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$, $(\alpha)_k$ is the Pochhammer symbol.

In [22] it was established, in particular, that

$$\frac{\mathrm{H}_7(\alpha;(\alpha+1)/2,\alpha;\mathbf{z})}{\mathrm{H}_7(\alpha+1;(\alpha+3)/2,\alpha;\mathbf{z})}=1+\cfrac{d_1z_1}{1+\cfrac{d_2z_2}{1+\cfrac{d_3z_2}{1+\cfrac{d_4z_1}{1+\cfrac{d_5z_2}{1+\cfrac{d_6z_2}{1+\ddots}}}}}}, \tag{19}$$

where

$$d_{3k+1} = -4, \quad d_{3k+2} = -\frac{1}{\alpha + 2k + 1}, \quad d_{3k+3} = \frac{1}{\alpha + 2k + 1}, \quad k \geq 0. \tag{20}$$

In addition, in [22, 23], some domains of convergence for the expansion (19) under certain conditions on the parameter α are investigated.

Let us investigate the numerical stability of the BR-algorithm for computing the nth approximant of continued fraction (19).

Theorem 2. *Let the relative roundoff errors of the elements of the continued fraction (19) and the machine operation errors satisfy the conditions (9). Let $n \in \mathbb{N}$. The BR-algorithm for computing the nth approximant of the continued fraction (19) is stable if $\mathbf{z} \in \Omega_{\alpha,\rho}$, where*

$$\Omega_{\alpha,\rho} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| \leq \frac{\rho(1-\rho)}{4}, \ |z_2| \leq |1+\alpha|\rho(1-\rho) \right\}, \quad 0 < \rho < \frac{1}{2}, \quad \mathrm{Re}(\alpha) > -1,$$

and for $\eta = \rho/(1-\rho)$, the inequality (11) holds.

Proof. For any $\mathbf{z} \in \Omega_{\alpha,\rho}$ and $k \geq 0$, we have

$$|d_{3k+1} z_1| \leq \rho(1-\rho), \quad |d_{3k+2} z_2| \leq \frac{|\alpha+1|\rho(1-\rho)}{|\alpha + 2k + 1|} \leq \rho(1-\rho),$$

and $|d_{3k+3} z_2| \leq \rho(1-\rho)$. Then it is easy to see that $V = \{\omega \in \mathbb{C} : \ |\omega| \leq \rho\}$ is the value set corresponding to the element set $E = \{\omega \in \mathbb{C} : \ |\omega| \leq \rho(1-\rho)\}$ of the continued fraction (19) (see, [3, Section 3.2]).

Let n be an arbitrary natural number. By setting

$$a_{3k+1} = d_{3k+1} z_1, \quad a_{3k+2} = d_{3k+2} z_2, \quad a_{3k+3} = d_{3k+3} z_2, \quad k \geq 0, \tag{21}$$

we use the settings (2) and (3).

It follows from $0 \in V$ and $\frac{a_k}{1+V} \subseteq V$ that ([17]) $G_k^{(n)} = \frac{a_k}{Q_k^{(n)}} \in V$, $k \in \{1, \ldots, n\}$. For any $k \in \{1, \ldots, n\}$ we rewrite $g_k^{(n)}$ as

$$g_k^{(n)} = \frac{a_k}{Q_{k-1}^{(n)} Q_k^{(n)}} = \frac{a_k}{\left(1 + \dfrac{a_k}{Q_k^{(n)}}\right) Q_k^{(n)}} = 1 - \frac{1}{1 + \dfrac{a_k}{Q_k^{(n)}}} = 1 - \frac{1}{1 + G_k^{(n)}}.$$

Then, $g_k^{(n)} \in 1 - \frac{1}{1+V}$, $k \in \{1, \ldots, n\}$. Now, since $0 < \rho < 1/2$, then $0 \notin 1 + V$ and the function

$$t(\omega) = 1 - \frac{1}{1+\omega}$$

maps the set V into the closed disk

$$1 - \frac{1}{1+V} = \left\{ w \in \mathbb{C}\colon \left| w + \frac{\rho^2}{1-\rho^2} \right| \leq \frac{\rho}{1-\rho^2} \right\}.$$

Thus, $|g_k^{(n)}| \leq \dfrac{\rho}{1-\rho}$, $k \in \{1, \ldots, n\}$.

Finally, we set $\eta = \rho/(1-\rho)$. It follows from $0 < \rho < 1/2$ that $0 < \eta < 1$. Thus, according to (11), we have the numerical stability of the BF-algorithm for computing the nth approximant of the continuous fraction (19). □

Theorem 3. *Let the relative roundoff errors of the elements of the continued fraction (19) and the machine operation errors satisfy the conditions (9). Let $n \in \mathbb{N}$. The BR-algorithm for computing the nth approximant of the continued fraction (19) is stable if the parameter α is such that $\operatorname{Re}\alpha > -1$ and*

$$|d_{3k+2}| - \operatorname{Re} d_{3k+2} \le r q_{3k+1}(1 - q_{3k+2}), \quad k \ge 0, \tag{22}$$
$$|d_{3k+3}| - \operatorname{Re} d_{3k+3} \le r q_{3k+2}(1 - q_{3k+3}), \quad k \ge 0, \tag{23}$$

where d_{3k+2} and d_{3k+3} are defined by (20), r is a positive constant, $\{q_k\}$ is a sequence of real numbers such that $\delta \le q_k \le 1-\delta$, $0 < \delta \le 1/2$, and

$$8 \le s q_{3k}(1 - q_{3k+1}), \quad k \ge 0, \tag{24}$$

where s is a positive constant, in addition, $\mathbf{z} \in \Omega^{r,s}_{\alpha,\rho}$, where

$$\Omega^{r,s}_{\alpha,\rho} = \Big\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| \le \frac{1+\cos(\arg(z_1))}{s}, \ |z_2| \le \frac{1+\cos(\arg(z_2))}{r}, \ \arg(z_1) = \arg(z_1) \Big\}, \tag{25}$$

and for

$$\eta = \max\Big\{ \frac{8}{s\delta^2}, \frac{2}{r|1+\alpha|\delta^2} \Big\},$$

the inequality (11) holds.

Proof. It follows from (25) and (24) that $|d_{3k+1}| - \operatorname{Re} d_{3k+1} \le s q_{3k}(1 - q_{3k+1})$ for $k \ge 0$, where a_{3k+1}, $k \ge 0$, are defined by (20).

We set $z_1 = |z_1| e^{i\varphi}$, $\quad z_2 = |z_2| e^{i\varphi}$, and choose $q_k = \frac{p_k}{\cos(\varphi/2)}$, $k \ge 0$. Then for $k \ge 0$

$$|d_{3k+1} z_1| - \operatorname{Re}(d_{3k+1} z_1 e^{-i\varphi}) \le \frac{s|z_1| p_{3k}(\cos(\varphi/2) - p_{3k+1})}{\cos^2(\varphi/2)} \le 2 p_{3k}(\cos(\varphi/2) - p_{3k+1}).$$

Similarly,

$$|d_{3k+2} z_2| - \operatorname{Re}(d_{3k+2} z_2 e^{-i\varphi}) \le p_{3k+1}(\cos\psi - p_{3k+2}), \quad k \ge 0,$$
$$|d_{3k+3} z_2| - \operatorname{Re}(d_{3k+3} z_2 e^{-i\varphi}) \le p_{3k+2}(\cos\psi - p_{3k+3}), \quad k \ge 0.$$

Thus, $V_k = \{\omega \in \mathbb{C} \colon \operatorname{Re}(\omega e^{-i\varphi/2}) \ge -p_k\}$, $k \ge 0$, is the sequence of value sets corresponding the sequence of element sets

$$E_k = \{\omega \in \mathbb{C} : \ |\omega| - \operatorname{Re}(\omega e^{-i\varphi}) \le 2 p_{k-1}(\cos(\varphi/2) - p_k)\}, \quad k \ge 1,$$

of the continued fraction (19) (see, [3, Section 3.2]).

Setting as (2), (3), and (21), it follows from

$$0 \in V_k \quad \text{and} \quad \frac{d_{k+1}}{1 + V_{k+1}} \subseteq V_k, \quad k \ge 0,$$

that

$$G^{(n)}_{k+1} = \frac{a_{k+1}}{Q^{(n)}_{k+1}} \in V_k \quad \text{and} \quad Q^{(n)}_k = 1 + G^{(n)}_{k+1} \in 1 + V_k, \quad k \ge 0.$$

Next, since

$$0 \notin 1 + V_k = \{\omega \in \mathbb{C}\colon \operatorname{Re}((\omega - 1)e^{-i\varphi/2}) \geq -p_{k-1}\},$$

we have

$$\min_{\omega \in 1+V_k} |\omega| = \cos(\varphi/2) - p_k = \cos(\varphi/2)(1 - q_k).$$

Thus,

$$|g_{3k+1}^{(n)}| = \frac{|d_{3k+1}z_1|}{|Q_{3k}^{(n)}||Q_{3k+1}^{(n)}|} \leq \frac{4(1+\cos\varphi)}{s\cos^2(\varphi/2)(1-q_{3k})(1-q_{3k+1})} = \frac{8}{s\delta^2},$$

$$|g_{3k+2}^{(n)}| = \frac{|d_{3k+2}z_2|}{|Q_{3k+1}^{(n)}||Q_{3k+2}^{(n)}|} \leq \frac{1+\cos\varphi}{r|\alpha+2k+1|\cos^2(\varphi/2)(1-q_{3k})(1-q_{3k+1})} \leq \frac{2}{r|\alpha+1|\delta^2},$$

and similarly $|g_{3k+3}^{(n)}| \leq \frac{2}{r|\alpha+1|\delta^2}$.

Let

$$\eta = \max\left\{\frac{8}{s\delta^2}, \frac{2}{r|1+\alpha|\delta^2}\right\}.$$

Then, according to Theorem 1, the BR-algorithm is stable for computing the nth approximant of the continued fraction (19). □

Note that results similar to Theorems 2 and 3 can be obtained in the same way for expansions

$$\frac{\mathrm{H}_7(\alpha; \alpha/2, \alpha; \mathbf{z})}{\mathrm{H}_7(\alpha+1; \alpha/2, \alpha+1; \mathbf{z})} = 1 + \cfrac{c_1 z_2}{1 + \cfrac{c_2 z_2}{1 + \cfrac{c_3 z_1}{1 + \cfrac{c_4 z_2}{1 + \cfrac{c_5 z_2}{1 + \cfrac{c_6 z_1}{1 + \ddots}}}}}},$$

where $c_{3k+1} = -1/(\alpha+2k)$, $c_{3k+2} = 1/(\alpha+2k)$, $c_{3k+3} = -4$, $k \geq 0$, and

$$\frac{\mathrm{H}_7(\alpha; (\alpha+1)/2, \alpha-1; \mathbf{z})}{\mathrm{H}_7(\alpha; (\alpha+1)/2, \alpha; \mathbf{z})} = 1 + \cfrac{b_1 z_2}{1 + \cfrac{b_2 z_1}{1 + \cfrac{b_3 z_2}{1 + \cfrac{b_4 z_2}{1 + \cfrac{b_5 z_1}{1 + \cfrac{b_6 z_2}{1 + \ddots}}}}}},$$

where $b_{3k+1} = 1/(\alpha+2k-1)$, $b_{3k+2} = -4$, $b_{3k+3} = -1/(\alpha+2k+1)$, $k \geq 0$.

5. Numerical Experiments. In this section, we analyze the relative errors of computing the approximants of the continued fraction (19) by BR-algorithm, FR-algorithm [15] and the Lentz algorithm [24]. We use floating-point arithmetic (IEEE 754-2019 standard) with $p = 15$ precision and rounding mode as nearest.

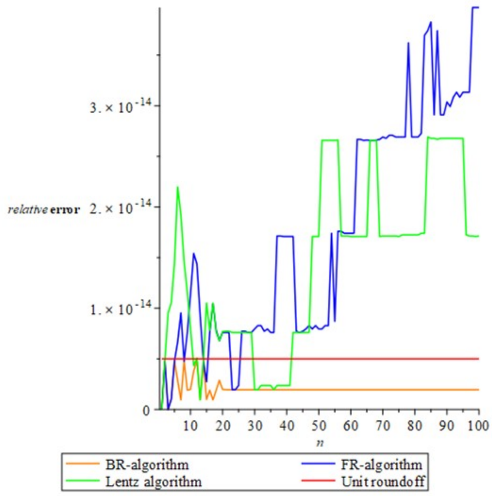
Figures 1a and 1b show graphs computing of relative errors at the point

$$z = (-0.0624i, -1.248i) \in \Omega_{2+4i,\rho}$$

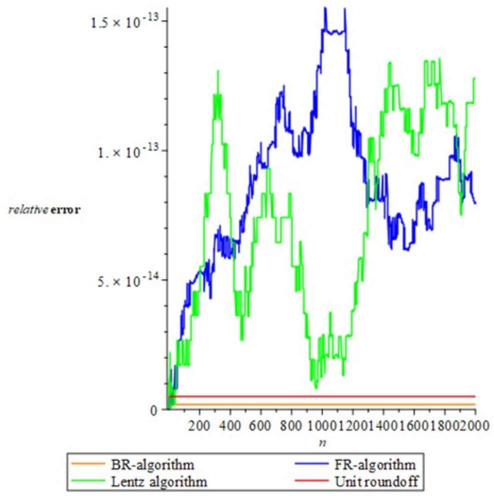
for the first 100 and 2000 approximants of continued fraction (19), respectively. Similarly, graphs of computing of relative errors at the point

$$z = (0.025i, i) \in \Omega_{5+8i,\rho}^{r,s}$$

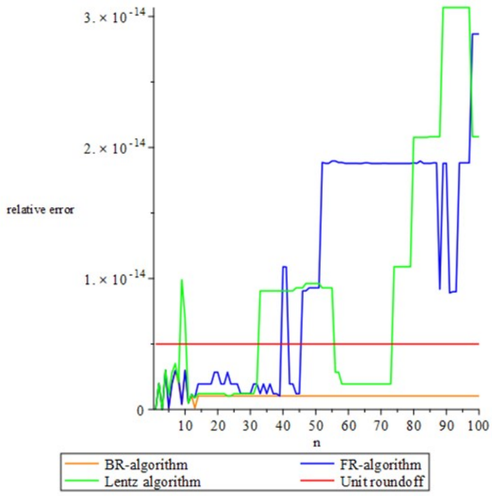
are shown in Figures 1c and 1d.



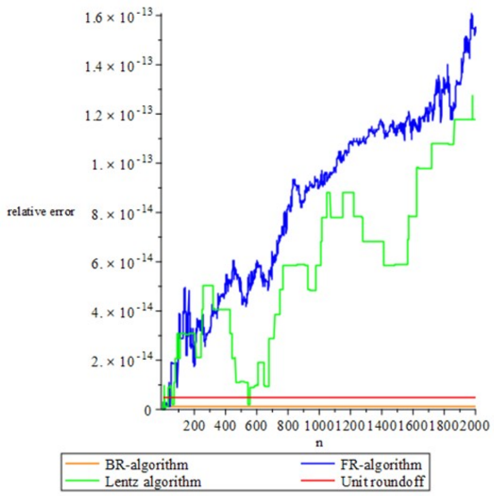
(a) $1 \le n \le 100$, $\mathbf{z} = (-0.0624i, -1.248i)$



(b) $1 \le n \le 2000$, $\mathbf{z} = (-0.0624i, -1.248i)$



(c) $1 \le n \le 100$, $\mathbf{z} = (0.025i, i)$



(d) $1 \le n \le 2000$, $\mathbf{z} = (0.025i, i)$

Fig. 1: Relative errors in the computing of nth approximants of the continued fraction (19).

The numerical experiments presented here and similar to them show that the BR-algorithm is stable to the error accumulation, and the maximum value of the relative error of the computing of approximants does not exceed the rounding unit, which ensures high accuracy of computing. Instead, the computing of the relative errors by the FR-algorithm and Lenz's algorithm tends to accumulate and exceed the rounding unit value.

Computing was performed in Maple software 2022.2 for Windows.

6. Conclusions. In the future, the method for studying the numerical stability of the BR-algorithm proposed here can be used to study the numerical stability of the branched continued fraction expansions and numerical branched continued fractions with elements in angular and parabolic domains, considered in [25–29] and [30–33], respectively.

REFERENCES

1. T. Antonova, R. Dmytryshyn, V. Goran, *On the analytic continuation of Lauricella-Saran hypergeometric function* $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$, Mathematics, **11** (2023), 4487. http://dx.doi.org/10.3390/math11214487
2. T. Antonova, R. Dmytryshyn, S. Sharyn, *Branched continued fraction representations of ratios of Horn's confluent function* H_6, Constr. Math. Anal., **6** (2023), 22–37. http://dx.doi.org/10.33205/cma.1243021
3. A. Cuyt, V.B. Petersen, B. Verdonk, H. Waadeland, W.B. Jones, Handbook of Continued Fractions for Special Functions, Springer, Dordrecht, 2008.
4. N. Hoyenko, V. Hladun, O. Manzij, *On the infinite remains of the Nörlund branched continued fraction for Appell hypergeometric functions*, Carpathian Math. Publ., **6** (2014), 11–25. (in Ukrainian) http://dx.doi.org/10.15330/cmp.6.1.11-25
5. R. Dmytryshyn, I.-A. Lutsiv, M. Dmytryshyn, *On the analytic extension of the Horn's hypergeometric function* H_4, Carpathian Math. Publ., **16** (2024), 32–39. http://dx.doi.org/10.15330/cmp.16.1.32-39
6. D.I. Bodnar, V.R. Hladun, *On the stability of branched continued fractions with complex elements under perturbations*, Mat. Stud., **25** (2006), 207–212. (in Ukrainian)
7. D. Bodnar, V. Hladun, *Sufficient conditions of stability of branched continued fractions with positive elements*, Mat. Metody Fiz.-Mekh. Polya, **45** (2002), 22–27. (in Ukrainian)
8. V.R. Hladun, D.I. Bodnar, R.S. Rusyn, *Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements*, Carpathian Math. Publ., **16** (2024), 16–31. http://dx.doi.org/10.15330/cmp.16.1.16-31
9. V. Hladun, D. Bodnar, *Some domains of relative stability under perturbations of branched continued fractions with complex elements*, Bukovinian Math. J., **288** (2018), 18–27. (in Ukrainian)
10. V.R. Hladun, *Some sets of relative stability under perturbations of branched continued fractions with complex elements and a variable number of branches*, J. Math. Sci., **215** (2016), 11–25. https://doi.org/10.1007/s10958-016-2818-x
11. N.J. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Philadelphia, 2002.
12. J.M. Muller, N. Brisebarre, F. De Dinechin, C.P. Jeannerod, V. Lefevre, G. Melquiond, S. Torres, Handbook of floating-point arithmetic, Birkhäuser, Cham, 2018.
13. W. Gautschi, *Computational aspects of three-term recurrence relations*, SIAM Rev., **9** (1967), 24–82.
14. A. Cuyt, P. Van der Cruyssen, *Rounding error analysis for forward continued fraction algorithms*, Comput. Math. Appl., **11** (1985), 541–564. http://dx.doi.org/10.1016/0898-1221(85)90037-9
15. G. Blanch, *Numerical evaluation of continued fractions*, SIAM Review, **6** (1964), 383–421. http://dx.doi.org/10.1137/1006092
16. N. Macon, M. Baskervill, *On the generation of errors in the digital evaluation of continued fractions*, J. Assoc. Comput. Math., **3** (1956), 199–202. http://dx.doi.org/10.1145/320831.320838
17. W.B. Jones, W.J. Thron, *Numerical stability in evaluating continued fractions*, Math. Comp., **28** (1974), 795–810. http://dx.doi.org/10.2307/2005701
18. F. Backeljauw, S. Becuwe, A. Cuyt, *Validated evaluation of special mathematical functions*, In: Autexier, S., Campbell, J., Rubio, J., Sorge, V., Suzuki, M., Wiedijk, F. (Eds.) Intelligent Computer Mathematics. CICM 2008. Lecture Notes in Computer Science, 5144. Springer, Berlin, Heidelberg. https://dx.doi.org/10.1007/978-3-540-85110-3_17
19. D.I. Bodnar, O.S. Manzii, *Expansion of the ratio of Appel hypergeometric functions* F_3 *into a branching continued fraction and its limit behavior*, J. Math. Sci., **107** (2001), 3550–3554. http://dx.doi.org/10.1023/A:1011977720316
20. R. Dmytryshyn, C. Cesarano, I.-A. Lutsiv, M. Dmytryshyn, *Numerical stability of the branched continued fraction expansion of Horn's hypergeometric function* H_4, Mat. Stud., **61** (2024), 51–60. https://dx.doi.org/10.30970/ms.61.1.51-60
21. W.B. Jones, W.J. Thron, Continued Fractions: Analytic Theory and Applications, Addison-Wesley Pub. Co., Reading, 1980.
22. T. Antonova, R. Dmytryshyn, P. Kril, S. Sharyn, *Representation of some ratios of Horn's hypergeometric functions* H_7 *by continued fractions*, Axioms, **12** (2023), 738. http://dx.doi.org/10.3390/axioms12080738

23. V. Hladun, R. Rusyn, M. Dmytryshyn, *On the analytic extension of three ratios of Horn's confluent hypergeometric function* H_7, Res. Math., **32** (2024), 60–70. https://dx.doi.org/10.15421/242405
24. W.J. Lentz, A method of computing spherical Bessel functions of complex argument with tables, United States Army Electronics Command, Fort Monmouth, New Jersey, 1973.
25. T. Antonova, C. Cesarano, R. Dmytryshyn, S. Sharyn, *An approximation to Appell's hypergeometric function* F_2 *by branched continued fraction*, Dolomites Res. Notes Approx., **17** (2024), 22–31. http://dx.doi.org/10.14658/PUPJ-DRNA-2024-1-3
26. R. Dmytryshyn, V. Goran, *On the analytic extension of Lauricella–Saran's hypergeometric function* F_K *to symmetric domains*, Symmetry, **16** (2024), 220. http://dx.doi.org/10.3390/sym16020220
27. R. Dmytryshyn, I.-A. Lutsiv, O. Bodnar, *On the domains of convergence of the branched continued fraction expansion of ratio* $H_4(a, d+1; c, d; \mathbf{z})/H_4(a, d+2; c, d+1; \mathbf{z})$, Res. Math., **31** (2023), 19–26. http://dx.doi.org/10.15421/242311
28. V.R. Hladun, N.P. Hoyenko, O.S. Manzij, L. Ventyk, *On convergence of function* $F_4(1, 2; 2, 2; z_1, z_2)$ *expansion into a branched continued fraction*, Math. Model. Comput., **9** (2022), 767–778. http://dx.doi.org/10.23939/mmc2022.03.767
29. O. Manziy, V. Hladun, L. Ventyk, *The algorithms of constructing the continued fractions for any rations of the hypergeometric Gaussian functions*, Math. Model. Comput., **4** (2017), 48–58. http://dx.doi.org/10.23939/mmc2017.01.048
30. T. Antonova, R. Dmytryshyn, I.-A. Lutsiv, S. Sharyn, *On some branched continued fraction expansions for Horn's hypergeometric function* $H_4(a, b; c, d; z_1, z_2)$ *ratios*, Axioms, **12** (2023), 299. http://dx.doi.org/10.3390/axioms12030299
31. I.B. Bilanyk, D.I. Bodnar, *Two-dimensional generalization of the Thron-Jones theorem on the parabolic domains of convergence of continued fractions*, Ukr. Math. J., **74** (2023), 1317–1333. https://dx.doi.org/10.1007/s11253-023-02138-1
32. D.I. Bodnar, O.S. Bodnar, I.B. Bilanyk, *A truncation error bound for branched continued fractions of the special form on subsets of angular domains*, Carpathian Math. Publ., **15** (2023), 437–448. https://dx.doi.org/10.15330/cmp.15.2.437-448
33. O.S. Bodnar, R.I. Dmytryshyn, S.V. Sharyn, *On the convergence of multidimensional* S*-fractions with independent variables*, Carpathian Math. Publ., **12** (2020), 353–359. http://dx.doi.org/10.15330/cmp.12.2.353-359

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Received 25.04.2024
Revised 25.10.2024