Yu. M. Gal', M. M. Sheremeta

ON SOME PROPERTIES OF THE MAXIMAL TERM OF SERIES IN SYSTEMS OF FUNCTIONS

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For an entire transcendental function f and a sequence (λ_n) of positive numbers increasing to $+\infty$ a series $A(z)=\sum_{n=1}^{\infty}a_nf(\lambda_nz)$ in the system $\{f(\lambda_nz)\}$ is said to be regularly convergent in $\mathbb{C}$ if $\mathfrak{M}(r,A)=\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty$ for all $r\in(0,+\infty)$, where $M_f(r)=\max\{|f(z)|\colon |z|=r\}$. Conditions are found on (λ_n) and f, under which $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$, where $\mu(r,A)=\max\{|a_n|M_f(r\lambda_n)\colon n\ge 1\}$ is the maximal term of the series. A formula for finding the lower generalized order

$$\lambda_{\alpha,\beta}[A]=\varliminf_{r\to+\infty}\frac{\alpha(\ln\mathfrak{M}(r,A))}{\beta(r)}$$

is obtained, where the functions α and β are positive, continuous and increasing to $+\infty$.

The open problems are formulated.

1. Introduction. Let
$$f(z)=\sum_{k=0}^{\infty}f_kz^k \tag{1}$$
be an entire transcendental function, $M_f(r)=\max\{|f(z)|\colon |z|=r\}$ and $\Lambda=(\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$. Suppose that the series
$$A(z)=\sum_{n=1}^{\infty}a_nf(\lambda_nz) \tag{2}$$
in the system $f(\lambda_nz)$ regularly convergent in $\mathbb{C}$, that is
$$\mathfrak{M}(r,A):=\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty \text{ for all } r\in[0,+\infty).$$
It is clear that many functional series arising in various sections of the analysis can be written as series by a system of functions $\{f(\lambda_nz)\}$. In particular, for example, in articles [1–4] B.V. Vinnitskii investigated under the most general conditions on a function f itself and on the sequence (λ_n), both the basicity of this system of functions and the properties of series on this system. In [5, 6] there were obtained the conditions under which, for series of the form (2), as well as integrals of the form $\int_0^{+\infty}a(t)f(tx)\nu(dt)$ that are a generalization of such series, the Borel-type asymptotic relation holds outside some set of finite Lebesgue measure, where f is a positive functions on $(0,+\infty)$ such that the function $\ln f(x)$ is convex on $(0,+\infty)$. In [7], the Borel-type relation was considered for more general positive integrals of the form

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$\int_0^{+\infty} a(t)f(tx+\beta(t)\tau(x))\nu(dt)$, which are, in particular, generalizations of series of the Taylor-Dirichlet type. In the end, modern e-search systems will allow the reader to easily find both other articles about the series on this general system of functions, and on the specific systems of functions, such as the Mittag-Leffler functions, the Bessel functions, and many others. This article continues the study of the properties series of form (2), which was started by the first author in articles [8–10].

Let $\mu(r,A)=\max\{|a_n|M_f(r\lambda_n)\colon n\geq 1\}$ be the maximal term of series (1) and $\nu(r,A)=\max\{n\geq 1\colon |a_n|M_f(r\lambda_n)=\mu(r,A)\}$ be its central index.

Since function (1) is transcendental, the function $\ln M_f(r)$ is logarithmically convex and, thus,

$$\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}\nearrow +\infty,\quad r\to+\infty,$$

(at points where the derivative does not exist, under $\frac{d\ln M_f(r)}{d\ln r}$ means right-hand side derivative). The function $\Gamma_f(r)$ plays an important role in the study of the properties of series in system of functions. For example, in [9] it is proved that the functions $\ln\mu(r,A)$, $\lambda_{\nu(r,A)}$ and $\nu(r,A)$ are non-decreasing and

$$\ln\mu(r,A)-\ln\mu(r_0,A)=\int_{r_0}^{r}\frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t}dt,\quad 0\leq r_0\leq r<+\infty.$$

Here we will study the conditions for the equivalence of the logarithms of the functions $\mathfrak{M}(r,A)$ and $\mu(r,A)$, a behavior of $\mu(r,A)$ and $\nu(r,A)$ in the case when $\Gamma_f(r)\uparrow+\infty$ and apply the results obtained to finding formulas for lower generalized orders.

2. Equivalence of $\ln\mathfrak{M}(r,A)$ and $\ln\mu(r,A)$. Let h be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$ and $\mathbf{S}_h(f,\Lambda)$ be a class of the function A such that $|a_n|M_f(\lambda_n h(\lambda_n))\to 0$ as $n\to+\infty$, whence $|a_n|\leq 1/M_f(\lambda_n h(\lambda_n))$ for $n\geq n^0$. For simplicity, we will assume that $n^0=1$.

Let $n_0(r)=\min\{n\colon h(\lambda_n)\geq qr\}$, $q>e$. Then

$$\mathfrak{M}(r,A)\leq\sum_{n=1}^{n_0(r)-1}|a_n|M_f(r\lambda_n)+\sum_{n=n_0(r)}^{\infty}M_f(r\lambda_n)/M_f(\lambda_n h(\lambda_n))\leq$$

$$\leq(n_0(r)-1)\mu(r,A)+\sum_{n=n_0(r)}^{\infty}M_f(\lambda_n h(\lambda_n)/q)/M_f(\lambda_n h(\lambda_n)).$$

Since $\int_{\lambda_n h(\lambda_n)/q}^{\lambda_n h(\lambda_n)}\Gamma_f(x)d\ln x\geq\Gamma_f(\lambda_n h(\lambda_n)/q)\ln q$, we have

$$\frac{M_f(\lambda_n h(\lambda_n)/q)}{M_f(\lambda_n h(\lambda_n))}=\exp\Big\{-\int_{\lambda_n h(\lambda_n)/q}^{\lambda_n h(\lambda_n)}\Gamma_f(x)d\ln x\Big\}\leq\exp\{-\Gamma_f(\lambda_n h(\lambda_n)/q)\ln q\}$$

and

$$\mathfrak{M}(r,A)\leq(n_0(r)-1)\mu(r,A)+\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(h(\lambda_n))\ln q\right\},\tag{3}$$

provided $\lambda_n\geq q$. Suppose that $\ln n\leq p\Gamma_f(h(\lambda_n))$ for all $n\geq 1$. Then for $q>e^p$ we obtain

$$\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(h(\lambda_n))\ln q\right\}\leq\sum_{n=1}^{\infty}\exp\Big\{-\frac{\ln q}{p}\ln n\Big\}=K_1(q)<+\infty.$$

Therefore, (3) implies $\mathfrak{M}(r,A)\leq(n_0(r)-1)\mu(r,A)+K_1(q)$ and, thus,

$$\frac{\ln\mu(r,A)}{\ln\mathfrak{M}(r,A)}\leq 1\leq\frac{\ln(n_0(r)-1)}{\ln\mathfrak{M}(r,A)}+\frac{\ln\mu(r,A)}{\ln\mathfrak{M}(r,A)}+o(1),\quad r\to+\infty.\tag{4}$$

Now, by E we denote a class of entire functions (1) such that $r=O(\ln M_f(r))$ and $\ln M_f(r)=O(\Gamma_f(r))$ as $r\to+\infty$. Then for $f\in E$, $\ln M_f(r)=o(\ln\mathfrak{M}(r,A))$ as $r\to+\infty$.

Indeed, $\mathfrak{M}(r,A)\ge|a_n|M_f(r\lambda_n)$ implies $\ln\mathfrak{M}(r,A)\ge\ln|a_n|+\ln M_f(r\lambda_n)$. On the other hand, since $\Gamma_f(r)/\ln M_f(r)\ge\eta>0$ for all r, we get

$$\frac{\ln M_f(r\lambda_n)}{\ln M_f(r)}=\exp\left\{\int_r^{r\lambda_n}\frac{d\ln\ln M_f(r)}{d\ln r}d\ln r\right\}=\exp\left\{\int_r^{r\lambda_n}\frac{\Gamma_f(r)}{\ln M_f(r)}d\ln r\right\}\ge$$
$$\ge\exp\left\{\int_r^{r\lambda_n}\eta d\ln r\right\}=\exp\{\eta\ln\lambda_n\}=\lambda_n^{\eta}$$

for all n. In view of the arbitrariness of λ_n we get $\ln M_f(r))=o(\ln M_f(r\lambda_n))$, i.e. $\ln M_f(r))=$ $=o(\ln\mathfrak{M}(r,A))$ as $r\to+\infty$.

Finally, suppose that $\ln n=O(h(\lambda_n))$ as $n\to\infty$. Then $\ln n=O(\Gamma_f(h(\lambda_n)))$ as $n\to\infty$ and, since $h(\lambda_{n_0(r)-1})\le qr$, we get $\ln(n_0(r)-1)=O(r)=O(\ln M_f(r))=o(\ln\mathfrak{M}(r,A))$ as $r\to+\infty$. Therefore, from (4) it follows that $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$.

Thus, the following theorem is true.

Theorem 1. *Let h be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. If $\ln n=O(h(\lambda_n))$ as $n\to\infty$ then $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$ for every functions $A\in\mathbf{S}_h(f,\Lambda)$ and $f\in E$.*

Condition $\ln n=O(h(\lambda_n))$ as $n\to\infty$ in Theorem 1 cannot be relaxed; this is shown by the following statement.

Proposition 1. *Let h be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. For any positive continuous function α on $[0,+\infty)$, slowly increasing to $+\infty$, there exist functions $f\in E$ and $A\in\mathbf{S}_h(f,\Lambda)$ such that $\ln n=O(\alpha(\lambda_n)h(\lambda_n))$ as $n\to\infty$ and the relation $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$ is not satisfied.*

Indeed, the function $f(z)=e^z$ belongs to E, and then $A(z)=\sum_{n=1}^{\infty}a_ne^{\lambda_nz}$ is an entire Dirichlet series. For such function A this proposition is prowed in [11].

Theorem 1 is supplemented by the following assertion.

Proposition 2. *Let h be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. If $\ln r=O(\Gamma_f(r))$ as $r\to+\infty$ and $\ln n=O(\ln h(\lambda_n))$ as $n\to\infty$ then $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$ for every function $A\in\mathbf{S}_h(f,\Lambda)$.*

Indeed, since $\ln n=O(\ln h(\lambda_n))=O(\Gamma_f(h(\lambda_n)))$ as $n\to\infty$, from (3) we obtain again (4). Also $\ln h(\lambda_{n_0(r)-1})\le\ln(qr)$ and, thus, $\ln(n_0(r)-1)=O(\ln h(\lambda_{n_0(r)-1}))=O(\ln r)=$ $o(\ln M_f(r))=o(\ln\mathfrak{M}(r,A))$ as $r\to+\infty$. Therefore, from (4) it follows that $\ln\mathfrak{M}(r,A)\sim$ $\ln\mu(r,A)$ as $r\to+\infty$.

3. Maximal term and central index of some class of series in the system of functions. Now, we suppose that $\Gamma_f(r)\uparrow+\infty$ as $r\to+\infty$. Then for all $n\ge1$

$$\int_{\lambda_n}^{\lambda_{n+1}}\Gamma_f(rx)d\ln x\uparrow+\infty,\quad r\to+\infty,$$

and as above

$$\frac{M_f(r\lambda_{n+1})}{M_f(r\lambda_n)}=\exp\left\{\int_{r\lambda_n}^{r\lambda_{n+1}}\Gamma_f(x)d\ln x\right\}=\exp\left\{\int_{\lambda_n}^{\lambda_{n+1}}\Gamma_f(rx)d\ln x\right\}\uparrow+\infty,\quad r\to+\infty.$$

If all $a_n\ne0$ then from hence it follows that the equation $M_f(r\lambda_{n+1})/M_f(r\lambda_n)=|a_n|/|a_{n+1}|$ has unique solution $r=\varkappa_n$, i.e.

$$|a_n|/|a_{n+1}|=M_f(\varkappa_n\lambda_{n+1})/M_f(\varkappa_n\lambda_n).$$

Let $\varkappa_n\nearrow+\infty$ and $j<n$. Then

$$|a_j|M_f(\varkappa_n\lambda_j)=|a_n|\frac{|a_j|}{|a_{j+1}|}\frac{|a_{j+1}|}{|a_{j+2}|}\dots\frac{|a_{n-1}|}{|a_n|}M_f(\varkappa_n\lambda_j)\le$$
$$\le|a_n|\frac{M_f(\varkappa_n\lambda_{j+1})}{M_f(\varkappa_n\lambda_j)}\frac{M_f(\varkappa_n\lambda_{j+2})}{M_f(\varkappa_n\lambda_{j+1})}\dots\frac{M_f(\varkappa_n\lambda_n)}{M_f(\varkappa_n\lambda_{n-1})}M_f(\varkappa_n\lambda_j)=|a_n|M_f(\varkappa_n\lambda_n).$$

If $j>n$ then

$$|a_j|M_f(\varkappa_n\lambda_j)=|a_n|\frac{|a_j|}{|a_{j-1}|}\frac{|a_{j-1}|}{|a_{j-2}|}\dots\frac{|a_{n+1}|}{|a_n|}M_f(\varkappa_n\lambda_j)=$$
$$=|a_n|\frac{M_f(\varkappa_{j-1}\lambda_{j-1})}{M_f(\varkappa_{j-1}\lambda_j)}\frac{M_f(\varkappa_{j-2}\lambda_{j-2})}{M_f(\varkappa_{j-2}\lambda_{j-1})}\dots\frac{M_f(\varkappa_n\lambda_n)}{M_f(\varkappa_n\lambda_{n+1})}M_f(\varkappa_n\lambda_j)\le|a_n|M_f(\varkappa_n\lambda_n).$$

Thus, $\mu(\varkappa_n,A)=|a_n|M_f(\varkappa_n\lambda_n)$ for all $n\ge1$.

Now, let $\varkappa_n\uparrow+\infty$ and $\varkappa_{n-1}\le r<\varkappa_n$. Then for $j<n$ as above

$$\frac{|a_j|M_f(r\lambda_j)}{|a_n|M_f(r\lambda_n)}=\frac{M_f(\varkappa_j\lambda_{j+1})}{M_f(\varkappa_j\lambda_j)}\dots\frac{M_f(\varkappa_{n-1}\lambda_n)}{M_f(\varkappa_{n-1}\lambda_{n-1})}\frac{M_f(r\lambda_j)}{M_f(r\lambda_n)}\le\frac{M_f(\varkappa_{n-1}\lambda_n)}{M_f(\varkappa_{n-1}\lambda_j)}\frac{M_f(r\lambda_j)}{M_f(r\lambda_n)}\le1$$

and for $j>n$

$$\frac{|a_j|M_f(r\lambda_j)}{|a_n|M_f(r\lambda_n)}=\frac{M_f(\varkappa_{j-1}\lambda_{j-1})}{M_f(\varkappa_{j-1}\lambda_j)}\dots\frac{M_f(\varkappa_n\lambda_n)}{M_f(\varkappa_n\lambda_{n+1})}\frac{|M_f(r\lambda_j)}{M_f(r\lambda_n)}\le\frac{M_f(\varkappa_n\lambda_n)}{M_f(\varkappa_n\lambda_j)}\frac{|M_f(r\lambda_j)}{M_f(r\lambda_n)}\le1$$

and, thus, $\mu(r,A)=|a_n|M_f(r\lambda_n)$ and $\nu(r,A)=n$. Therefore, the following theorem is correct.

Theorem 2. *If $\varkappa_n\nearrow+\infty$ as $n\to+\infty$ then $\mu(\varkappa_n,A)=|a_n|M_f(\varkappa_n\lambda_n)$. If $\varkappa_n\uparrow+\infty$ as $n\to+\infty$ then $\mu(r,A)=|a_n|M_f(r\lambda_n)$ and $\nu(r,A)=n$ for all $r\in[\varkappa_{n-1},\,\varkappa_n)$ and all $n\ge1$.*

4. Lover generalized order. As in [12] by L we denote a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$, i.e. α is a slowly increasing function. Clearly, $L_{si}\subset L^0$.

For $\alpha\in L$ and $\beta\in L$ quantity

$$\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[\mathfrak{M}]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln\mathfrak{M}(r,A))}{\beta(r)}$$

is called [9] generalized (α,β)-order of the entire function A.

Suppose that $\alpha(e^x)\in L^0$, $\beta(x)\in L^0$ and $\frac{\ln r}{\ln\alpha^{-1}(c\beta(r))}\to0$ as $r\to+\infty$ for each $c\in(0,+\infty)$. If $\ln n=o(\Gamma_f(\lambda_n))$ as $n\to\infty$, $\ln M_f(r)=O(\Gamma_f(r))$ and $\Gamma_f(r)=O(r)$ as $r\to+\infty$ then [9]

$$\varrho_{\alpha,\beta}[A]=\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\big(M_f^{-1}\left(1/|a_n|\right)/\lambda_n\big)}.$$

Here we define a lower generalized (α,β)-order

$$\lambda_{\alpha,\beta}[A]=\lambda_{\alpha,\beta}[\mathfrak{M}]=\varliminf_{r\to+\infty}\frac{\alpha(\ln\mathfrak{M}(r,A))}{\beta(r)}$$

and prove the same formula for $\lambda_{\alpha,\beta}[A]$.

We need the following lemma [9].

Lemma 1. *If $\ln n=O(\Gamma_f(\lambda_n))$ as $n\to\infty$ then $\mu(r,A)\le\mathfrak{M}(r,A)\le K\mu(qr,A)$, ($K=const>0$) for $q>1$ and all $r\ge1$. If $\ln n=o(\Gamma_f(\lambda_n))$ as $n\to\infty$ then $\mu(r,A)\le\mathfrak{M}(r,A)\le K(\varepsilon)\mu((1+\varepsilon)r,A)$, ($K(\varepsilon)>0$) for every $\varepsilon>0$ and all $r\ge1$.*

Lemma 1 implies the following statement.

Proposition 1. *Let $\alpha(\ln x) \in L_{si}$. If either $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L_{si}$ or $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L^0$ then*

$$\lambda_{\alpha,\beta}[A] = \lambda_{\alpha,\beta}[\mu] := \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)}.$$

Proof. If $\ln n = O(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $\beta \in L_{si}$ then by Lemma 1

$$\underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \le \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mathfrak{M}(r, A)}{\beta(r)} \le \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(qr, A) + \ln K)}{\beta(qr)} \overline{\lim}_{r\to+\infty} \frac{\beta(qr)}{\beta(r)} =$$
$$= \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \overline{\lim}_{r\to+\infty} \frac{\beta(qr)}{\beta(r)} = \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)}.$$

If $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then similarly we obtain

$$\underline{\lim}_{r\to+\infty} \alpha(\ln \mathfrak{M}(r, A)/\beta(r) \le \underline{\lim}_{r\to+\infty} \alpha(\ln \mu(r, A))/\beta(r) \overline{\lim}_{r\to+\infty} \beta((1 + \varepsilon)r)/\beta(r).$$

It is known [6] that if $\beta \in L^0$ then $\overline{\lim}_{r\to+\infty} \beta((1 + \varepsilon)r)/\beta(r) \searrow 1$ as $\varepsilon \searrow 0$. □

Using Theorem 2 and Proposition 3 we prove the following theorem.

Theorem 3. *Let $\alpha(e^x) \in L^0$, $\beta(x) \in L^0$, $\frac{\ln r}{\ln \alpha^{-1}(c\beta(r))} \to 0$ as $r \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $\Gamma_f(r) \asymp r$ as $r \to +\infty$. If $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$, $\varkappa_n \nearrow +\infty$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then*

$$\lambda_{\alpha,\beta}[A] = \sigma_{\alpha,\beta}[A] := \underline{\lim}_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(M_f^{-1}\left(1/|a_n|\right)/\lambda_n\right)}.$$

Proof. At first we remark that $0 < c_1 r \le \Gamma_f(r) \le c_2 r < +\infty$ for all r.

Suppose that $\sigma_{\alpha,\beta}[A] > 0$. Then $|a_n| \ge 1\big/M_f\big(\lambda_n\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)\big)$ for every $\sigma \in (0, \sigma_{\alpha,\beta}[A])$ and all $n \ge n_0(\sigma)$. We choose $r = r_n = \beta^{-1}\left(\alpha(\lambda_n)/\sigma\right) + 1$. Then for $r_n \le r \le r_{n+1}$ we have

$$\ln \mu(r, A) \ge \ln \mu(r_n, A) \ge \ln |a_n| + \ln M_f(r_n\lambda_n) \ge$$
$$\ge \ln M_f\left(\lambda_n\left(\beta^{-1}\left(\alpha(\lambda_n)/\sigma + 1\right)\right)\right) - \ln M_f\left(\lambda_n\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)\right) =$$
$$= \int_{\lambda_n\beta^{-1}(\alpha(\lambda_n)/\sigma)}^{\lambda_n\left(\beta^{-1}(\alpha(\lambda_n)/\sigma)+1\right)} \Gamma_f(x)d\ln x \ge \Gamma_f\left(\lambda_n\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)\right)\ln\left(1 + \frac{1}{\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)}\right) \ge$$
$$\ge c_1\lambda_n\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)\ln\left(1 + \frac{1}{\beta^{-1}\left(\alpha(\lambda_n)/\sigma\right)}\right) = (1 + o(1))c_1\lambda_n, \quad n \to \infty.$$

Therefore, since $\alpha \in L_{si}$ and $\beta \in L^0$,

$$\lambda_{\alpha,\beta}[\mu] = \underline{\lim}_{r\to+\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(r)} \ge \underline{\lim}_{n\to\infty} \frac{\alpha(\ln \mu(r_n, A))}{\beta(r_{n+1})} \ge \underline{\lim}_{n\to\infty} \frac{\sigma\alpha(c_1\lambda_n)}{\alpha(\lambda_{n+1})} = \sigma,$$

because $\alpha \in L_{si}$ and $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ as $n \to \infty$. In view of the arbitrariness of σ we obtain the inequality $\sigma_{\alpha,\beta}[A] \le \lambda_{\alpha,\beta}[\mu]$ that is obvious if $\sigma_{\alpha,\beta}[A] = 0$.

Now suppose that $\sigma_{\alpha,\beta}[A] < +\infty$. Then for every $\sigma > \sigma_{\alpha,\beta}[A])$ there exists a sequence $(n_j) \to \infty$ such that $|a_{n_j}| \le 1/M_f\left(\lambda_{n_j}\beta^{-1}\left(\alpha(\lambda_{n_j})/\sigma\right)\right)$. Since $\varkappa_n \nearrow +\infty$ as $n \to +\infty$, by Theorem 2 we have $\mu(\varkappa_{n_j}, A) = |a_{n_j}|M_f(\varkappa_{n_j}\lambda_{n_j})$. Let $m \in \{n_j\}$. Then

$$\mu(\varkappa_m, A) \le M_f(\varkappa_m\lambda_m)\big/M_f\big(\lambda_m\beta^{-1}\left(\alpha(\lambda_m)/\sigma\right)\big) \le \mu^*(\varkappa_m),$$

where $\mu^*(r) = \max\{M_f(r\lambda_n)\big/M_f\big(\lambda_n\beta^{-1}\big(\alpha(\lambda_n)/\sigma\big)\big) : n \ge 1\}$. Therefore,

$$\lambda_{\alpha,\beta}[\mu]=\varliminf_{r\to+\infty}\frac{\alpha(\ln\mu(r,A))}{\beta(r)}\le\varliminf_{m\to+\infty}\frac{\alpha(\ln\mu(\varkappa_m,A))}{\beta(\varkappa_m)}\le\varliminf_{m\to+\infty}\frac{\alpha(\ln\mu^*(\varkappa_m))}{\beta(\varkappa_m)}$$

In [9] (see the proof of Theorem 4) it is proved that if $\alpha(e^x)\in L^0$, $\frac{\ln r}{\ln\alpha^{-1}(c\beta(r))}\to 0$ as $r\to+\infty$ for each $c\in(0,+\infty)$ and $\Gamma_f(r)\le c_2r$ then

$$\varlimsup_{r\to+\infty}\alpha(\ln\mu^*(r))/\beta(r)\le\sigma.$$

Therefore, in view of the arbitrariness of σ we obtain the inequality $\lambda_{\alpha,\beta}[\mu]\le\sigma_{\alpha,\beta}[A]$ that is obvious if $\sigma_{\alpha,\beta}[A]=+\infty$. Thus, $\lambda_{\alpha,\beta}[\mu]=\sigma_{\alpha,\beta}[A]$ and by Proposition 2 $\lambda_{\alpha,\beta}[A]=\sigma_{\alpha,\beta}[A]$. □

Remark 1. In Theorem 3, conditions $\beta(x)\in L^0$ and $\ln n=o(\Gamma_f(\lambda_n))$ as $n\to\infty$ can be replaced by conditions $\beta(x)\in L_{si}$ and $\ln n=O(\Gamma_f(\lambda_n))$ as $n\to\infty$.

The functions $\alpha(x)=\ln^+x$ and $\beta(x)=x^+$ satisfy the conditions of Theorem 4. Therefore, the following statement is correct.

Corollary 1. *Let the function f and the sequence (λ_n) satisfy the conditions of Theorem 3. If $\varkappa_n\nearrow+\infty$ as $n\to\infty$ then*

$$\varliminf_{r\to+\infty}\frac{\ln\ln\mathfrak{M}(r,A))}{r}=\varliminf_{n\to\infty}\frac{\lambda_n\ln\lambda_n}{M_f^{-1}(1/|a_n|)}.$$

The functions $\alpha(x)=\beta(x)=\ln^+x$ not satisfy the conditions of Theorem 3. In this case we put

$$\lambda[A]=\varliminf_{r\to+\infty}\frac{\ln\ln\mathfrak{M}(r,A))}{\ln r}$$

and prove the following statement.

Proposition 2. *Let $r=O(\Gamma_f(r))$ as $r\to+\infty$ and $\varlimsup_{n\to\infty}\frac{\ln\ln M_f(r)}{\ln r}\le 1$. If $\ln\lambda_{n+1}\sim\ln\lambda_n$, $\varkappa_n\nearrow+\infty$ and $\ln n=O(\Gamma_f(\lambda_n))$ as $n\to\infty$ then*

$$\lambda[A]=\sigma[A]+1,\quad \sigma[A]:=\varliminf_{n\to\infty}\frac{\ln\lambda_n}{\ln(M_f^{-1}(1/|a_n|)/\lambda_n)}.$$

Proof. Suppose that $\sigma[A]>0$. Then $|a_n|\ge 1/M_f(\lambda_n^{1+1/\sigma})$ for every $\sigma\in(0,\sigma[A])$ and all $n\ge n_0(\sigma)$. We choose $r=r_n=2\lambda_n^{1/\sigma}$. Then for $r_n\le r\le r_{n+1}$ we have

$$\ln\mu(r_n,A)\ge\ln M_f(2\lambda_n^{1+1/\sigma})-\ln M_f(\lambda_n^{1+1/\sigma})=\int_{\lambda_n^{1+1/\sigma}}^{2\lambda_n^{1+1/\sigma}}\frac{\Gamma_f(x)}{x}dx\ge c_1\lambda_n^{1+1/\sigma}.$$

Since $\ln\lambda_{n+1}\sim\ln\lambda_n$ as $n\to\infty$,

$$\lambda[\mu]:=\varliminf_{r\to+\infty}\frac{\ln\ln\mu(r,A))}{\ln r}\ge\varliminf_{n\to\infty}\frac{\ln\ln\mu(r_n`,A))}{\ln r_{n+1}}\ge\varliminf_{n\to\infty}\frac{\ln(c_1\lambda_n^{1+1/\sigma})}{\ln(2\lambda_{n+1}^{1/\sigma})}=1+\sigma.$$

In view of the arbitrariness of $\sigma\in(0,\sigma[A])$ we obtain the inequality $\lambda[\mu]\ge\sigma[A]+1$.

The inequality $\Gamma_f(r)\ge c_1r$ implies

$$\ln\mu(r,A)\ge\ln M_f(r)\ge(1+o(1)c_1r$$

as $r_0\le r\to+\infty$, i.e. $\lambda[\mu]\ge 1$ and the inequality $\lambda[\mu]\ge\sigma[A]+1$ holds at $\sigma[A]=0$.

On the other hand, if $\sigma[A]<+\infty$ then $|a_{n_j}|\le 1/M_f(\lambda_{n_j}^{1+1/\sigma})$ for every $\sigma>\sigma[A])$ and some sequence $(n_j)\to\infty$. Therefore, for $m\in\{n_j\}$ as above we get $\mu(\varkappa_m,A)\le\mu^*(\varkappa_m)$, where $\mu^*(r)=\max\{M_f(r\lambda_n)/M_f(\lambda_n^{1+1/\sigma})\colon n\ge 1\}$. In [9] (see the proof of Theorem 5) it is proved that

$$\overline{\lim_{r\to+\infty}}\frac{\ln\ln\mu^*(r)}{\ln r}\le(1+\sigma)\overline{\lim_{r\to+\infty}}\frac{\ln\ln M_f(r)}{\ln r}.$$

Therefore,

$$\lambda[\mu]=\underline{\lim_{r\to+\infty}}\frac{\ln\ln\mu(r,A)}{\ln r}\le\underline{\lim_{m\to+\infty}}\frac{\ln\ln\mu(\varkappa_m,A)}{\ln\varkappa_m}\le\underline{\lim_{m\to+\infty}}\frac{\ln\ln\mu^*(\varkappa_m)}{\ln\varkappa_m}\le$$
$$\le\overline{\lim_{m\to+\infty}}\frac{\ln\ln\mu^*(\varkappa_m)}{\ln\varkappa_m}\le\overline{\lim_{r\to+\infty}}\frac{\ln\ln\mu^*(r)}{\ln r}\le(1+\sigma)\overline{\lim_{r\to+\infty}}\frac{\ln\ln M_f(r)}{\ln r}\le 1+\sigma.$$

In view of the arbitrariness of σ we obtain the inequality $\lambda[\mu]\le 1+\sigma[A]$ that is obvious if $\sigma[A]=+\infty$. □

5. Some open problems. If $f(z)=e^z$ then $A(z)=F(z)$, where

$$F(z)=\sum_{n=1}^{\infty}a_ne^{\lambda_nz},\ z=\sigma+it,$$

is an entire Dirichlet series. We put $\mathfrak{M}(\sigma)=\sum_{n=1}^{\infty}|a_n|e^{\lambda_n\sigma}$ and $\mu(\sigma)=\max\{|a_n|e^{\lambda_n\sigma}\colon n\ge 1\}$. Then [11] $\ln\mathfrak{M}(\sigma)\sim\ln\mu(\sigma)$ as $\sigma\to+\infty$, provided $|a_n|\le\exp\{-\lambda_n(h(\lambda_n)\}$ for $n\ge n_0$ and $\ln n=O(h(\lambda_n))$ as $n\to\infty$, where h is a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. Therefore, we can consider that Theorem 3 is a generalization of this result. In [14, 15] it is studied conditions under which $\varphi(\ln\mathfrak{M}(\sigma))\sim\varphi(\ln\mu(\sigma))$ as $\sigma\to+\infty$, where φ is a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. The following question arises.

Question 1. *For a given function φ under what conditions the relation*

$$\varphi(\ln\mathfrak{M}(r,A))\sim\varphi(\ln\mu(r,A))$$

holds as $r\to+\infty$ for the entire functions A of form (2)*?*

Using the Wiman-Valiron method, in [16] it is proved that the asymptotitc estimate $\ln\mathfrak{M}(\sigma)\sim\ln\mu(\sigma)$ holds as $0<\sigma\to+\infty$ outside some exceptional sets of finite measure for each Dirichlet series with a given sequence of exponents (λ_n) if and only if

$$\sum_{n=1}^{\infty}\frac{1}{n\lambda_n}<+\infty.$$

The conjecture about the correctness of this statement was posed in [17]. The following conjecture seems to be correct.

Conjecture 1. *If*

$$\sum_{n=1}^{\infty}\frac{1}{n\Gamma_f(\lambda_n)}<+\infty$$

then $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\to+\infty$ outside some exceptional set E such that

$$\int_E\frac{\Gamma_f(x)}{x}dx<+\infty$$

for the every entire functions A of form (2).

Question 2. *The condition $\Gamma_f(r)\asymp r$ as $r\to+\infty$ in Theorem 3 appeared due to the applied method. Can it be weakened?*

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Drogobych Ivan Franko Pedagogical State University
Drogobych, Ukraine
yuriyhal@gmail.com

Ivan Franko National University of Lviv
Lviv, Ukraine
m.m.sheremeta@gmail.com

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