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A. O. Kuryliak, M. R. Kuryliak, O. M. Trusevych

ARBITRARY RANDOM VARIABLES AND WIMAN'S INEQUALITY FOR ANALYTIC FUNCTIONS IN THE UNIT DISC

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We consider the class $\mathcal{A}(\varphi,\beta)$ of random analytic functions in the unit disk $\mathbb{C}=\{z\colon |z|<1\}$ of the form $f(z,\omega)=f(z,\omega_1,\omega_2)=\sum_{n=0}^{+\infty}R_n(\omega_1)\xi_n(\omega_2)a_nz^n$, where $a_n\in\mathbb{C}\colon \varlimsup\limits_{n\to+\infty}\sqrt[n]{|a_n|}=1$, $\big(R_n(\omega)\big)$ is the Rademacher sequence, $\big(\xi_n(\omega)\big)$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta>0$ and a function $\varphi(N,\beta)\colon\mathbb{N}\times\mathbb{R}_+\to[1;+\infty)$ non-decreasing by N and β for which

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^\beta\Big)\Big)^{1/\beta}\asymp\varphi(N,\beta),\quad N\to+\infty,\quad \alpha=\varlimsup_{N\to+\infty}\frac{\ln\varphi(N,\beta)}{\ln N}<+\infty,$$

$$(\exists\gamma>0)(\exists n_0\in\mathbb{N})\colon\ \sup\{\mathbf{E}|\xi_n|^{-\gamma}\colon\ n\ge n_0\}<+\infty.$$

By $\mathcal{A}_1(\varphi,\beta)$ we denote the class of random analytic functions in $\mathbb{D}$ of the form $f(z,\omega)=\sum_{n=0}^{+\infty}\xi_n(\omega)a_nz^n$, where a sequence $\big(\xi_n(\omega)\big)\in\Delta_\varphi$ and, in particular, may be not sub-gaussian and not independent. In the paper, there are proved the following statements:
Let $\delta>0$. 1) Theorem 3: For $f\in\mathcal{A}(\varphi,\beta)$ there exist $r_0(\omega)>0$, a set $E(\delta)\subset(0;1)$ of finite logarithmic measure such that for all $r\in(r_0(\omega);1)\backslash E$ we have with probability $p\in(0;1)$

$$M_f(r,\omega)\le\frac{\mu_f(r)}{(1-p)^{1/\beta}}\varphi(N(r),\beta)\Big((1-r)^{-2}\cdot\ln\frac{\mu_f(r)\varphi(N(r),\beta)}{(1-p)(1-r)}\Big)^{1/4+\delta}.$$

2) Theorem 4: For a function $f\in\mathcal{A}_1(\varphi,\beta)$ there exist $r_0(\omega)>0$, a set $E(\delta)\subset(0;1)$ of finite logarithmic measure such that for all $r\in(r_0(\omega);1)\backslash E$ we get with probability $p\in(0;1)$

$$M_f(r,\omega)\le\frac{\mu_f(r)}{(1-p)^{1/\beta}}\varphi(N(r),\beta)\Big((1-r)^{-2}\cdot\ln\frac{\mu_f(r)\varphi(N(r),\beta)}{(1-p)(1-r)}\Big)^{1/2+\delta}.$$

1. Introduction. Let us consider the class $\mathcal{A}$ of an analytic function f in the disc $\mathbb{D}:=\{z\colon |z|<1\}$ of the form

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n. \tag{1}$$

Let $M_f(r)=\max\{|f(z)|\colon |z|=r\}$, $\mu_f(r)=\max\{|a_n|r^n\colon n\ge 0\}$, $r>0$, be the maximum modulus and the maximal term of series (1), respectively.

The analogues of Wiman's inequality for analytic functions in the unit disc $\mathbb{D}$ one can find in [1, 2]. From results proved in [2] follows such statement.

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Theorem 1 ([2])**.** *Let $f \in \mathcal{A}$ be an analytic function of form* (1)*. Then for eveery $\delta > 0$ there exists a set $E_f(\delta) \subset (0;1)$ of finite logarithmic measure (f.l.m.), i.e. $\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty$, such that for all $r \in (0;1)\backslash E_f(\delta)$ we have*

$$M_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}. \tag{2}$$

For an analytic function $g(z) = \sum_{n=1}^{+\infty} \exp\{n^{\varepsilon}\} z^n,\ \varepsilon \in (0;1)$, we have ([2])

$$M_g(r) \geq C\frac{\mu_g(r)}{1-r} \ln^{1/2} \frac{\mu_g(r)}{1-r} \quad (r \in [r_0, 1)),\ C > 0.$$

Therefore, inequality (2) is sharp in the class of analytic functions in the unit disc $\mathbb{D}$. But this inequality can be improved in some subclasses of random analytic functions ([3, 4, 5]).

Denote by $K(f, Y)$ the class of random analytic functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} a_n Y_n(\omega) z^n, \tag{3}$$

where $\{Y_n(\omega)\}$ is a sequence of random variables on the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, and the sequence (a_n) satisfies the condition $\varlimsup\limits_{n\to+\infty} \sqrt[n]{|a_n|} = 1$.

Let $Y = (Y_n(\omega))$ be multiplicative system (MS) uniformly bounded by the number 1. That is, for all $n \in \mathbb{N}$ we have $|Y_n(\omega)| \leq 1$ almost surely (a.s.) and

$$\forall (j_1, j_2, \ldots, j_k) \in \mathbb{N}^k,\ 1 \leq j_1 < j_2 < \cdots < j_k\colon\ \mathbf{E}(Y_{j_1} Y_{j_2} \cdots Y_{j_k}) = 0,$$

where $\mathbf{E}\eta = \int_{\Omega} \eta(\omega) P(d\omega)$ is the expectation of a random variable η.

In 1997 P.V. Filevych proved the following theorem.

Theorem 2 ([3])**.** *Let $f(z, \omega)$ be random analytic function of the form* (3)*, $Y_n \in MS$ and $|Y_n(\omega)| \leq 1$ for almost all $\omega \in [0;1]$. Then a.s. in $K(f, Y)$ for any $\delta > 0$ there exists a set $E = E(f, \omega, \delta) \subset (0;1)$ f.l.m. such that for all $r \in (0;1)\backslash E$ we get*

$$M_f(r, \omega) \leq \mu_f(r)\Big((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{1-r}\Big)^{1/4+\delta}. \tag{4}$$

The constant $1/4$ in the previous inequality cannot be replaced by a smaller number. This is indicated by another statement from [3].

Let $\big(R_n(\omega)\big)$ be the *Rademacher sequence*, i.e. a sequence of independent random variables defined on Steinhaus probability space $(\Omega, \mathcal{A}, P)$, such that for any $n \in \mathbb{Z}_+$ we have $\mathbb{P}\{\omega\colon R_n(\omega) = -1\} = \mathbb{P}\{\omega\colon R_n(\omega) = 1\} = \frac{1}{2}$.

Remark that for random entire function of the form $f(z, \omega) = \sum_{n=0}^{+\infty} R_n(\omega) a_n z^n$ the above-mentioned theorems from [3] are valid.

Suppose that (Z_n) is a sequence of real independent centered sub-gaussian random variables, that is for any $n \in \mathbb{Z}_+$ we have $\mathbf{E}Z_n = 0$ and there exist a constant $C_1 > 0$ such that for any $t \in [0; +\infty)$ we have $P\{\omega\colon |Z_n(\omega)| \geq t\} \leq 2\exp\big(-t^2/C_1\big)$. For such random variables we have (see [9]):

1) there exists $D > 0$ such that $\mathbf{E}(e^{\lambda_0 Z_k}) \leq e^{D\lambda_0^2}$ for any $k \in \mathbb{N}$ and all $\lambda_0 \in \mathbb{R}$;
2) for any $k \in \mathbb{N}$: $\mathbf{E}Z_k = 0$ and $\sup\{\mathbf{E}(Z_k^2)\colon k \in \mathbb{N}\} = \sup\{\mathbf{D}Z_k\colon k \in \mathbb{N}\} \leq 2D$, where $\mathbf{D}Z_k$ is the variance of random variable Z_k.

Consider the class of random functions of the form

$$K(f,\mathcal{Z})=\left\{f(z,\omega)=\sum_{n=0}^{+\infty}a_nZ_n(\omega)z^n\colon \omega\in[0;1]\right\},$$

where $Z=(Z_n)$ is a sequence of real centered independent sub-gaussian random variables such that $(\exists\gamma>0)(\exists n_0\in\mathbb{N})\colon \sup\{\mathbf{E}|Z_n|^{-\gamma}\colon n\ge n_0\}<+\infty$. Analogues of inequality (2) for random analytic functions from class $K(f,\mathcal{Z})$ was considered in [5].

Remark, that in all above-mentioned statements about random analytic functions the obtained inequalities valid with probability 1 and only for sequences of random variables which are independent or MS and sub-gaussian in general (see also [3, 4, 5, 7, 8]).

In this regard Prof. O. B. Skaskiv formulated the following **problem:** *to obtain estimates of maximum modulus of random analytic functions: a) with probability $p\in(0;1)$; b) in case of sequence $(Z_n(\omega))$: 1) is not sub-gaussian; 2) may not be independent.*

In this paper we give answer to all this questions. Similar question was considered for random entire functions in [6].

2. Notations. Here $\varphi(N)\asymp\psi(N),\ N\to+\infty$, means the equivalence of functions up to constant factors. Precisely, $\varphi(N)\asymp\psi(N)$ means that there exist positive constants c,C such that the inequality $cf(N)\le g(N)\le Cf(N)$ holds for for all sufficiently large N.

Consider the random analytic functions of the form

$$f(z,\omega)=f(z,\omega_1,\omega_2)=\sum_{n=0}^{+\infty}R_n(\omega_1)\xi_n(\omega_2)a_nz^n, \tag{5}$$

where $a_n\in\mathbb{C}\colon \lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}=1$, $\big(R_n(\omega)\big)$ is the Rademacher sequence, $\big(\xi_n(\omega)\big)$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exists a constant $\beta>0$ and a function $\varphi(N,\beta)\colon\mathbb{N}\times\mathbb{R}_+\to[1;+\infty)$ non-decreasing by N and β such that

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^\beta\Big)\Big)^{1/\beta}\asymp\varphi(N,\beta),\quad N\to+\infty,\quad \alpha=\varlimsup_{N\to+\infty}\frac{\ln\varphi(N,\beta)}{\ln N}<+\infty, \tag{6}$$

$$(\exists\gamma>0)(\exists n_0\in\mathbb{N})\colon\ \sup\{\mathbf{E}|\xi_n|^{-\gamma}\colon\ n\ge n_0\}<+\infty. \tag{7}$$

Such class of random analytic functions we denote by $\mathcal{A}(\varphi,\beta)$. Remark that by conditions (6)–(7) radius of convergence of series (5) $R(\omega)=1$ almost surely ([5]).

Remark, that for any sequence $\big(\xi_n(\omega)\big)$ function $\psi(N,\beta)=\big(\mathbf{E}\big(\max\limits_{0\le n\le N}|\xi_n|^\beta\big)\big)^{1/\beta}$ is non-decreasing by N and β.

Also class of random analytic functions of the form $f(z,\omega)=\sum_{n=0}^{+\infty}\xi_n(\omega)a_nz^n$ we denote by $\mathcal{A}_1(\varphi,\beta)$.

In this paper we will use the following notations.

$$W_N(r,\omega)=\sum_{n=N(r)}^{+\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n,\ N(r)=\Big[\frac{1}{1-r}\ln\frac{\mu_f(r)}{1-r}\Big]^m,\ m=\Big[\alpha+\frac{2}{\beta}\Big]+4,$$

where $[x]$ means integer part of x.

3. Main results. We obtain the asymptotic estimates for maximum modulus of functions $f\in\mathcal{A}(\varphi,\beta)$. Here sequence $\big(\xi_n(\omega)\big)$ may not be sub-gaussian and may be dependent. The main result of this paper is the following theorem.

Theorem 3. *Let $\delta > 0$. For $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0;1)$ of f.l.m. such that for all $r \in (r_0(\omega);1)\backslash E$ we have with probability $p \in (0;1)$*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r),\beta)\Big((1-r)^{-2} \cdot \ln \frac{\mu_f(r)\varphi(N(r),\beta)}{(1-p)(1-r)}\Big)^{1/4+\delta}.$$

Also we get the asymptotic estimates for maximum modulus of functions $f \in \mathcal{A}_1(\varphi, \beta)$.

Theorem 4. *Let $\delta > 0$. For $f \in \mathcal{A}_1(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0;1)$ of f.l.m. such that for all $r \in (r_0(\omega);1)\backslash E$ we get with probability $p \in (0;1)$*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/\beta}} \varphi(N(r),\beta)\Big((1-r)^{-2} \cdot \ln \frac{\mu_f(r)\varphi(N(r),\beta)}{(1-p)(1-r)}\Big)^{1/2+\delta}.$$

4. Some corollaries. If $\big(\xi_n(\omega)\big)$ is sub-exponential random variables (see [12]), i.e. there exist a constant $C_2 > 0$ such that for any $t \in [0;+\infty)$: $P(\omega\colon |Z_n(\omega)| \ge t) \le 2\exp\big(-\frac{t}{C_2}\big)$, and suppose that for any $n \in \mathbb{N}$ such that there exists $n \in \mathbb{N}$: $\mathbf{E}\xi_n = 0$, then we can choose $\beta = 1$ and prove that $\varphi(N,1) = \mathbf{E}\big(\max_{0\le n\le N} |\xi_n|\big) \le C_3 \ln N,\ C_3 > 0$.

Corollary 1. *Let $\delta > 0$ and $\big(\xi_n(\omega)\big)$ is centered sub-exponential random variables. Then for $f \in \mathcal{A}(\varphi, \beta)$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0;1)$ f.l.m. such that for all $r \in (r_0(\omega);1)\backslash E$ we have with probability $p \in (0;1)$*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{1-p}\Big((1-r)^{-2} \cdot \ln \frac{\mu_f(r)}{(1-p)(1-r)}\Big)^{1/4+\delta}.$$

If $\big(\xi_n(\omega)\big)$ satisfies the condition $\sup_{n\in\mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$ for some $a > 0$, then we can choose

$$\varphi(N,a) \le C_4 N^{1/a},\ C_4 > 0.$$

Corollary 2. *Let $\delta > 0$ and $\big(\xi_n(\omega)\big)$ is such that $(\exists a > 0)\colon \sup_{n\in\mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty$. Then for random entire function f of form (5) there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0;1)$ f.l.m. such that for all $r \in (r_0(\omega);1)\backslash E$ we obtain with probability $p \in (0;1)$*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/a}(1-r)^{m/a+1/2+\delta}} \ln^{m/a+1/4+\delta} \frac{\mu_f(r)}{(1-p)(1-r)}.$$

5. Auxiliary lemmas. Similarly to [11] one can prove the following lemma.

Lemma 1. *Let $l(r)$ be a continuous increasing to $+\infty$ function on $(0;1)$, $E \subset (0;1)$ be a open set such that its complement $\overline{E}$ is such that $\overline{E} \cap (y;1) \ne \varnothing$ for any $y \in (0;1)$. Then there is an infinite sequence $0 < r_1 \le ... \le r_n \uparrow 1$ $(n \to +\infty)$ such that 1) $(\forall n \in \mathbb{N})\colon r_n \notin E$; 2) $(\forall n \in \mathbb{N})\colon \ln l(r_n) \ge \frac{n}{2}$; 3) if $(r_n; r_{n+1}) \cap E \ne (r_n, r_{n+1})$, then $l(r_{n+1}) \le el(r_n)$; 4) the set of indices, for which 3) holds, is unbounded.*

Lemma 2. *Let $f \in \mathcal{A}(\varphi, \beta)$. For any $\delta > 0$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (0;1)$ f.l.m. such that for all $r \in (r_0(\omega);1)\backslash E$ we have $W_N(r,\omega) \le \mu_f(r)$.*

Proof. Let $f_k(z) = \sum_{n=0}^{+\infty} n^k a_n z^n$, $\mathfrak{M}_{f_k}(r) = \sum_{n=0}^{+\infty} n^k |a_n| r^n$, $k \in \mathbb{N}$. For $\delta > 0$ we denote $E_1 = \big\{r\colon r\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) > \frac{1}{1-r} \ln^{1+\delta} \mathfrak{M}_f(r),\ \ln \mathfrak{M}_f(r) > e\big\}$. Then

$$\int_{E_1} \frac{dr}{1-r} < \int_{E_1} \frac{dr}{r(1-r)} < \int_{E_1} \frac{\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) dr}{\ln^{1+\delta} \mathfrak{M}_f(r)} < \int_1^{+\infty} \frac{du}{u^{1+\delta}} < +\infty.$$

So, for $r \notin E_1$ we get $\mathfrak{M}_{f_1}(r) = \sum_{n=0}^{+\infty} n|a_n|r^n \le \frac{1}{1-r}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)$.

Also for $r \notin E_2$ ($\int_{E_2}(1-r)^{-1}dr < +\infty$) we obtain

$$\mathfrak{M}_{f_2}(r) \le \frac{1}{1-r}\mathfrak{M}_{f_1}(r)\ln^{1+\delta}\mathfrak{M}_{f_1}(r) \le \frac{1}{(1-r)^{2+\delta}}\mathfrak{M}_f(r)\ln^{2+3\delta}\mathfrak{M}_f(r).$$

Similarly for $r \notin E_k$ we have

$$\mathfrak{M}_{f_k}(r) \le \frac{1}{(1-r)^{k+\delta_2}}\mathfrak{M}_f(r)\ln^{k+\delta_2}\mathfrak{M}_f(r), \tag{8}$$

where the set E_k is a set of f.l.m., $\delta_2 > 0$.

For $n \ge N(r)$ denote $B_n = \{\omega\colon |\xi_n(\omega)|^\beta \ge n^{\alpha\beta+2+\delta_1}\}$, $\delta_1 > 0$. Then probabilities of these events we can estimate using Markov's inequality and (6). For some $C_1 > 0$ we have

$$\mathbb{P}(B_n) = \mathbb{P}\{\omega\colon |\xi_n|^\beta \ge n^{\alpha\beta+2+\delta_1}\} \le \frac{\mathbf{E}|\xi_n|^\beta}{n^{\alpha\beta+2+\delta_1}} \le \frac{1}{n^{\alpha\beta+2+\delta_1}}\mathbf{E}\Big(\max_{0\le k\le n}|\xi_k|^\beta\Big) \le C_1\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}}$$

as $r \uparrow 1$. So,

$$\sum_{n=N(r)}^{+\infty}\mathbb{P}(B_n) \le C_1\sum_{n=N(r)}^{+\infty}\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}} \le C_1\sum_{n=N(r)}^{+\infty}\frac{1}{n^{2+\delta_1/2}} \le \frac{1}{N^{1+\delta_1/3}(r)}, \quad r\uparrow 1.$$

Let $B = \bigcup_{n=N(r)}^{+\infty} B_n$. Then $\mathbb{P}(B) \le \frac{1}{N^{1+\delta_1/3}(r)}$, $r \uparrow 1$. For $\omega \notin B$ we get

$$W_N(r,\omega) = \sum_{n=N(r)}^{+\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n \le \sum_{n=N(r)}^{+\infty} n^{\alpha+(2+\delta_1)/\beta}\frac{n}{N(r)}|a_n|r^n \le$$

$$\le \frac{1}{N(r)}\sum_{n=0}^{+\infty} n^{\alpha+1+(2+\delta_1)/\beta}|a_n|r^n \le \frac{1}{N(r)}\sum_{n=0}^{+\infty} n^{[\alpha+2/\beta]+2}|a_n|r^n,\ r\uparrow 1,\ (r\notin E).$$

Then using (8), definition of $N(r)$ and Theorem 1, from [5] (in the case of $h(r) = (1-r)^{-1}$) we obtain

$$W_N(r,\omega) \le \frac{1}{N(r)}\frac{1}{(1-r)^{[\alpha+2/\beta]+2+\delta_2}}\mathfrak{M}_f(r)\ln^{[\alpha+2/\beta]+2+\delta_2}\mathfrak{M}_f(r) \le$$

$$\le \frac{1}{N(r)}\frac{\mu_f(r)}{(1-r)^{[\alpha+2/\beta]+3+3\delta_2}}\ln^{[\alpha+2/\beta]+5/2+3\delta_2}\frac{\mu_f(r)}{1-r} \le \frac{\mu_f(r)}{e}$$

as $r \uparrow 1$, $(r \notin E)$. Therefore, for $r \uparrow 1$ we obtain

$$\mathbb{P}\Bigg\{\omega\colon \sum_{n=N(r)}^{+\infty}|R_n(\omega)||\xi_n(\omega_2)||a_n|r^n \ge \mu_f(r)/e\Bigg\} \le N^{-1-\delta_1/3}(r).$$

Let us choose $l(r) = \frac{\mu_f(r)}{1-r}$, and a set E and a sequence $\{r_k\}$ from Lemma 1. We put $F_k := \{\omega\colon W_N(r_k,\omega) \ge \mu_f(r_k)/e\}$. By the definition of $N(r)$ we get $P(F_k) \le N^{-1-\delta_1/3}(r_k) \le \ln^{-1-\delta_1/3}\frac{\mu_f(r_k)}{1-r_k} \le k^{-1-\delta_1/3}$, thus $\sum_{k=1}^{+\infty}P(F_k) \le \sum_{k=1}^{+\infty}k^{-1-\delta_1/3} < +\infty$. Then by Borel-Cantelli's lemma for almost all $\omega \in [0,1]$ and for $k \ge k_0(\omega)$ we obtain $W_N(r_k,\omega) < \frac{\mu_f(r_k)}{e}$.

Let $r \ge r_{k_0(\omega)}$ be an arbitrary number outside set the E, $r \in (r_p, r_{p+1})$. By Lemma 1 $\frac{\mu_f(r_{p+1})}{1-r_{p+1}} \le e\frac{\mu_f(r_p)}{1-r_p} \le e\frac{\mu_f(r)}{1-r}$ and then $\mu_f(r_{p+1}) \le e\mu_f(r)$. Therefore for almost all $\omega \in [0;1]$ and $r \ge r_0(\omega)$ outside a set of f.l.m. E we have $W_N(r,\omega) < W_N(r_{p+1},\omega) \le \mu_f(r_{p+1})/e \le \mu_f(r)$. $\square$

6. Proofs.

Proof of Theorem 3. By Theorem 2, ω_1-almost surely there exists a set $E := E(\varepsilon, \omega, f) \subset (0;1)$ of f.l.m. such that for all $r \in (0;1) \setminus E$ we have

$$M_f(r,\omega) = M_f(r,\omega_1,\omega_2) \le \mu_f(r,\omega_2)\Big((1-r)^{-2} \cdot \ln \frac{\mu_f(r,\omega_2)}{1-r}\Big)^{1/4+\delta}. \tag{9}$$

Then by Lemma 2 we get

$$\mu_f(r,\omega_2) \le \max\Big\{\max_{0\le n\le N(r)} |\xi_n(\omega_2)||a_n|r^n; \sup_{N(r)<n<+\infty} |\xi_n(\omega_2)||a_n|r^n\Big\} \le$$

$$\le \max\Big\{\max_{0\le n\le N(r)} |\xi_n(\omega_2)| \cdot \mu_f(r); \mu_f(r)\Big\} = \max\Big\{\eta(\omega_2)\mu_f(r); \mu_f(r)\Big\}, \quad r \uparrow 1, \ (r \notin E),$$

where $\eta(\omega_2) = \max\limits_{0\le n\le N(r)} |\xi_n(\omega_2)|$ is non-negative random variable. Then by Markov's inequality we obtain $P\{\omega\colon \eta^\beta(\omega) < \frac{\mathbf{E}\eta^\beta}{1-p}\} \ge p$, $P\{\omega\colon \eta(\omega) < \big(\frac{\mathbf{E}\eta^\beta}{1-p}\big)^{1/\beta}\} \ge p$.

Remark that there exist $\delta > 0$ and a set $E \subset (0;1)$ of f.l.m. such that for all $r \in (0;1) \setminus E$ we have $(\mathbf{E}\eta^\beta)^{1/\beta} \le \varphi(N(r),\beta)$ and

$$\mu_f(r,\omega_2) \le \max\Big\{\Big(\frac{\mathbf{E}\eta^\beta}{1-p}\Big)^{1/\beta}\mu_f(r); \mu_f(r)\Big\} \le$$

$$\le \max\Big\{\frac{\mu_f(r)}{(1-p)^{1/\beta}}\varphi(N(r),\beta); \mu_f(r)\Big\} = \frac{\mu_f(r)}{(1-p)^{1/\beta}}\varphi(N(r),\beta). \tag{10}$$

Finally, it remains use inequalities (9) and (10). □

Proof of Corollary 1. It is enough to prove that we can choose $\beta = 1$ and $\varphi(N,1) = \mathbf{E}\Big(\max\limits_{0\le n\le N} |\xi_n|\Big) \le C_3 \ln N$, $C_3 > 0$. Remark that for sub-exponential random variables ([12, p.32]) there exists $b > 0$ such that for all $\lambda \in [0, 1/b]$ we have $\mathbf{E}(\mathbf{e}^{\lambda|\xi_n|}) \le e^{b\lambda}$. Then, for $\lambda = 1/b$ by Jensen's inequality we obtain

$$e^{\lambda \mathbf{E}\big(\max\limits_{0\le n\le N}|\xi_n|\big)} \le \mathbf{E}\Big(e^{\lambda \max\limits_{0\le n\le N}|\xi_n|}\Big) = \mathbf{E}\Big(\max_{0\le n\le N} e^{\lambda|\xi_n|}\Big) \le \mathbf{E}\Big(\sum_{n=0}^{N} e^{\lambda|\xi_n|}\Big) \le (N+1)e^{b\lambda},$$

$$\lambda\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|\Big) \le \ln(N+1) + b\lambda, \ \mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|\Big) \le \frac{\ln(N+1)}{\lambda} + b \le (b+2)\ln N, \ N \to +\infty.$$

□

Proof of Corollary 2. Here we can choose $\beta = a$. Then

$$\varphi(N,a) = \Big(\mathbf{E}\big(\max\{|\xi_n|^a \colon 0 \le n \le N\}\big)\Big)^{1/a} \le \Big(\mathbf{E}\Big(\sum_{n=0}^{N}|\xi_n|^a\Big)\Big)^{1/a} =$$

$$= \Big(\sum_{n=0}^{N}\mathbf{E}|\xi_n|^a\Big)^{1/a} = (N+1)^{1/a}(\mathbf{E}|\xi_n|^a)^{1/a} \le C(a)N^{1/a} \ (N \to +\infty), \ C(a) > 0.$$

□

Proof of Theorem 4. By Theorem 1, there exists a set $E := E(\varepsilon, f) \subset (0;1)$ f.l.m. such that for all $\omega \in [0;1]$ and all $r \in (0;1) \setminus E$ we have $M_f(r,\omega) \le \frac{\mu_f(r,\omega)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r,\omega)}{1-r}$.

It remans to use (10). □

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Ivan Franko National University of Lviv
andriykuryliak@gmail.com
kuryliakmariya@gmail.com

Lviv State University of Life Safety
trusev14@gmail.com

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