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SYMMETRIC LINEAR FUNCTIONALS ON THE BANACH SPACE GENERATED BY PSEUDOMETRICS

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In this work we consider the notion of B-equivalence of pseudometrics. Two pseudometrics d_1 and d_2 on a set X are called B-equivalent, where B is a subgroup of the group of all bijections on X, if there exists an element b of B such that $d_1(x,y)=d_2(b(x),b(y))$ for every $x,y\in X$, that is, d_1 can be obtained from d_2 by permutating elements of X with the aid of the bijection b. The group B generates the group $\widehat{B}$ of transformations of the set of all pseudometrics on X, elements of which act as $d(\cdot,\cdot)\mapsto d(b(\cdot),b(\cdot))$, where d is a pseudometrics on X and $b\in B$. A function f on the set of all pseudometrics on X is called $\widehat{B}$-symmetric if f is invariant under the action on its argument of elements of the group $\widehat{B}$. If two pseudometrics d_1 and d_2 are B-equivalent, then $f(d_1)=f(d_2)$ for every $\widehat{B}$-symmetric function f.

In general, the technique of symmetric functions is well-developed for the case of symmetric continuous polynomials and, in particular, for the case of symmetric continuous linear functionals on Banach spaces. To use this technique for the construction of $\widehat{B}$-symmetric functions on sets of pseudometrics, we map these sets to some appropriate Banach space V, which is isometrically isomorphic to the Banach space ℓ_1 of all absolutely summing real sequences. We investigate symmetric (with respect to an arbitrary group of symmetry, elements of which map the standard Schauder basis of ℓ_1 into itself) linear continuous functionals on ℓ_1. We obtain the complete description of the structure of these functionals. Also we establish analogical results for symmetric linear continuous functionals on the space V. These results are used for the construction of $\widehat{B}$-symmetric functionals on the set of all pseudometrics on an arbitrary set X for the following case: the group B of bijections on X, that generates the group $\widehat{B}$, is such that the set of all $x\in X$, for which there exists $b\in B$ such that $b(x)\neq x$, is finite.

Introduction. Pseudometrics are a generalization of metrics that relaxes the non-degeneracy requirement (i.e., distinct points are not necessarily separated by a positive distance). The notion of pseudometrics is important in various fields of mathematics and applied sciences. The possibility that distinct points can have zero distance between them gives researchers the ability to apply pseudometrics to problems where the difference between distinct but similar objects can be ignored. Note that the choice of an appropriate pseudometric from the set of all pseudometrics on a given fixed set can play a decisive role in solving theoretical or practical problems. Therefore, it is important to study the properties of the set of all pseudometrics and certain subsets of this set (such as the set of all pseudoultrametrics) on a fixed set (see [12, 13]).

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Among the wide variety of pseudometrics on a fixed set there are some similar pseudometrics that can be obtained from one another by the action of some bijection on the underlying set. In section 2 we establish some properties of such pseudometrics. In particular, we define the notion of B-equivalence of pseudometrics. For the investigation of classes of B-equivalence it is useful to construct functions of pseudometrics that take the same values on B-equivalent pseudometrics. Such functions are symmetric (invariant) under the action of some group of operators on the set of all pseudometrics, which is also constructed in section 2. In general, the technique of symmetric functions is well-developed for the case of symmetric continuous polynomials and, in particular, for the case of symmetric continuous linear functionals on Banach spaces (see, e.g., [2, 8, 11]). Unfortunately, the set of all pseudometrics on a fixed set with point-wise operations of addition and multiplication to scalars is not a vector space. So, to use the above-mentioned technique to symmetric mappings on the set of pseudometrics, we need to map this set to some appropriate Banach space and to construct the group of operators on this space consistent with the above-mentioned group on the set of pseudometrics. We use this approach in section 4, where we construct the appropriate Banach space and establish the structure of symmetric continuous linear functionals on it. We show that this Banach space is isometrically isomorphic to the Banach space ℓ_1 of all absolutely summing real sequences. This result gave us the opportunity to use the general result on the structure of symmetric continuous linear functionals on ℓ_1, which is established in section 3.

1. Preliminaries.

1.1. Symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f\colon A \to B$ is called *S-symmetric* if $f(s(a)) = f(a)$ for every $a \in A$ and $s \in S$.

1.2. The space ℓ_1. Let ℓ_1 be the Banach space of all absolutely summing sequences of real numbers with the norm

$$\|x\|_1 = \sum_{m=1}^{\infty} |x_m|,$$

where $x = (x_1, x_2, \ldots) \in \ell_1$. Let

$$e_m = (\underbrace{0, \ldots, 0}_{m-1}, 1, 0, \ldots) \text{ for } m \in \mathbb{N}.$$

It is well-known that the set $\{e_m\}_{m=1}^{\infty}$ is a Schauder basis in ℓ_1. Also it is well-known that the mapping

$$f \in \ell_1' \mapsto \big(f(e_1), f(e_2), \ldots\big) \in \ell_\infty$$

is an isometrical isomorphism, where ℓ_1' is the Banach space of all continuous linear functionals on ℓ_1 and ℓ_∞ is the Banach space of all bounded sequences of real numbers with norm

$$\|x\|_\infty = \sup\{|x_m|\colon m \in \mathbb{N}\},$$

where $x = (x_1, x_2, \ldots) \in \ell_\infty$. For every $f \in \ell_1'$ and $x = (x_1, x_2, \ldots) \in \ell_1$, by the continuity and the linearity of f,

$$f(x) = \sum_{m=1}^{\infty} x_m f(e_m). \tag{1}$$

2. B-equivalence of pseudometrics. Let X be a nonempty set. Let $Ps(X)$ be the set of all pseudometrics on X. Let B be some subgroup of the group of all bijections on X. Let us call two pseudometrics $d_1, d_2 \in Ps(X)$ B-equivalent if there exists $b \in B$ such that

$$d_1(x, y) = d_2(b(x), b(y)) \tag{2}$$

for every $x, y \in X$. Since B is the group, it follows that the relation of B-equivalence is an equivalence relation on $Ps(X)$. Also note that the equality (2) can be interpreted in the following way: d_1 can be obtained from d_2 by "renaming" or "permutating" of elements of X with the aid of the bijection b.

Let a function $f\colon Ps(X) \to A$, where A is some set, has the following property:

$$\text{if } d_1, d_2 \in Ps(X) \text{ are } B\text{-equivalent, then } f(d_1) = f(d_2). \tag{3}$$

In other words, the restriction of f to every class of B-equivalence is constant. Functions with the property (3) are important in the investigations of B-equivalence. Let us rephrase the property (3) in terms of symmetric mappings. For $b \in B$ and $d \in Ps(X)$, let us define the pseudometrics $\widehat{b}(d)$ on X by

$$\widehat{b}(d)(x, y) = d(b(x), b(y)) \tag{4}$$

for every $x, y \in X$. Let the mapping $\widehat{b}$ be defined by

$$\widehat{b}\colon d \in Ps(X) \mapsto \widehat{b}(d) \in Ps(X), \tag{5}$$

where $\widehat{b}(d)$ is defined by (4). Let

$$\widehat{B} = \big\{\widehat{b}\colon b \in B\big\}. \tag{6}$$

It can be checked that $\widehat{B}$ is a group with respect to the operation of composition. Note that d and $\widehat{b}(d)$ are B-equivalent for every $d \in Ps(X)$.

Lemma 1. *A function $f\colon Ps(X) \to A$, where A is some set, has the property* (3) *if and only if f has the following property:*

$$f\big(\widehat{b}(d)\big) = f(d) \textit{ for every } \widehat{b} \in \widehat{B} \textit{ and } d \in Ps(X). \tag{7}$$

Proof. Suppose f has the property (3). Let $\widehat{b} \in \widehat{B}$ and $d \in Ps(X)$. Let us show that $f\big(\widehat{b}(d)\big) = f(d)$. Let $d_1 = d$ and $d_2 = \widehat{b}(d)$. Since d_1 and d_2 are B-equivalent, by the property (3), $f(d_1) = f(d_2)$, i.e., $f(d) = f\big(\widehat{b}(d)\big)$. So, f has the property (7).

Suppose f has the property (7). Let $d_1, d_2 \in Ps(X)$ be B-equivalent. Let us show that $f(d_1) = f(d_2)$. Since d_1 and d_2 are B-equivalent, there exists $b \in B$ such that the equality (2) holds. Consequently, taking into account (4), $d_1 = \widehat{b}(d_2)$. Therefore, taking into account (7), $f(d_1) = f\big(\widehat{b}(d_2)\big) = f(d_2)$. So, f has the property (3). □

Note that the property (7) is the property of $\widehat{B}$-symmetry of f. Thus, by Lemma 1, a function f on $Ps(X)$ has the property (3) if and only if f is $\widehat{B}$-symmetric.

3. Symmetric linear functionals on ℓ_1. Let us construct the reordering of the representation (1), connected to some partition of the set $\mathbb{N}$ of positive integers.

Lemma 2. *Let f be a continuous linear functional on ℓ_1. Let $\mathcal{M}$ be a partition of $\mathbb{N}$. Let us have some linear order on $\mathcal{M}$. Then*

$$f(x) = \sum_{M \in \mathcal{M}} \sum_{m \in M} x_m f(e_m) \tag{8}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$.

Proof. Let $M \in \mathcal{M}$. Consider the sum $\sum_{m\in M} x_m f(e_m)$. In the case $|M| < \infty$, this sum is finite. Consider the case $|M| = \infty$. Let us show that, in this case, the series $\sum_{m\in M} x_m f(e_m)$ is convergent. For $k \in \mathbb{N}$, let

$$h_k = \begin{cases} f(e_k), & \text{if } k \in M; \\ 0, & \text{otherwise.} \end{cases}$$

Let $h = (h_1, h_2, \ldots)$. Since $(f(e_1), f(e_2), \ldots) \in \ell_\infty$, it follows that $h \in \ell_\infty$. Consequently, the functional $f_M \colon \ell_1 \to \mathbb{R}$, defined by $f_M(x) = \sum_{m=1}^\infty x_m h_m$, where $x = (x_1, x_2, \ldots) \in \ell_1$, belongs to ℓ_1'. On the other hand, $\sum_{m=1}^\infty x_m h_m = \sum_{m\in M} x_m f(e_m)$, that is,

$$\sum_{m\in M} x_m f(e_m) = f_M(x) \tag{9}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$. Consequently, since f_M is well defined, it follows that the series $\sum_{m\in M} x_m f(e_m)$ is convergent for every $x = (x_1, x_2, \ldots) \in \ell_1$.

Let us show that the equality (8) holds.

Consider the case $|\mathcal{M}| < \infty$. For definiteness, let $\mathcal{M} = \{M_1, \ldots, M_s\}$. In this case, taking into account (9),

$$\sum_{M\in\mathcal{M}} \sum_{m\in M} x_m f(e_m) = f_{M_1}(x) + \ldots + f_{M_s}(x) \tag{10}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$. On the other hand,

$$f_{M_1}(x) + \ldots + f_{M_s}(x) = f(x) \tag{11}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$. Thus, by (10) and (11), the equality (8) holds.

Consider the case $|\mathcal{M}| = \infty$. For definiteness, let $\mathcal{M} = \{M_1, M_2, \ldots\}$. Fix some element $x = (x_1, x_2, \ldots) \in \ell_1$. The right-hand side of the equality (8) can be rewritten in the form

$$\sum_{j=1}^\infty \sum_{m\in M_j} x_m f(e_m). \tag{12}$$

Let us show that the series (12) converges to $f(x)$. Let $\varepsilon > 0$. By (1), there exists $r_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{k=1}^r x_k f(e_k) \right| < \varepsilon/2$$

for every $r \geq r_0$. On the other hand, since $x \in \ell_1$ and $(f(e_1), f(e_2), \ldots) \in \ell_\infty$, it follows that

$$\sum_{k=1}^\infty |x_k f(e_k)| \leq \|(f(e_1), f(e_2), \ldots)\|_\infty \sum_{k=1}^\infty |x_k| = \|f\| \|x\|_1.$$

Thus, the series $\sum_{k=1}^\infty |x_k f(e_k)|$ is convergent. Consequently, there exists $l_0 \in \mathbb{N}$ such that $\sum_{k=l+1}^\infty |x_k f(e_k)| < \varepsilon/2$ for every $l \geq l_0$. Let $k_0 = \max\{r_0, l_0\}$. Then

$$\left| f(x) - \sum_{k=1}^{k_0} x_k f(e_k) \right| < \varepsilon/2 \quad \text{and} \quad \sum_{k=k_0+1}^\infty |x_k f(e_k)| < \varepsilon/2. \tag{13}$$

Let $j_0 \in \mathbb{N}$ be such that $M_1 \cup \ldots \cup M_{j_0} \supset \{1, \ldots, k_0\}$. Let $J \geq j_0$. Then $M_1 \cup \ldots \cup M_J \supset \{1, \ldots, k_0\}$. Consequently,

$$\sum_{j=1}^{J}\sum_{m\in M_j} x_m f(e_m) = \sum_{k=1}^{k_0} x_k f(e_k) + \sum_{k\in M_1\cup\ldots\cup M_J\setminus\{1,\ldots,k_0\}} x_k f(e_k).$$

Therefore, taking into account (13),

$$\left|f(x) - \sum_{j=1}^{J}\sum_{m\in M_j} x_m f(e_m)\right| = \left|f(x) - \sum_{k=1}^{k_0} x_k f(e_k) - \sum_{k\in M_1\cup\ldots\cup M_J\setminus\{1,\ldots,k_0\}} x_k f(e_k)\right| \le$$

$$\le \left|f(x) - \sum_{k=1}^{k_0} x_k f(e_k)\right| + \sum_{k\in M_1\cup\ldots\cup M_J\setminus\{1,\ldots,k_0\}} |x_k f(e_k)| \le \left|f(x) - \sum_{k=1}^{k_0} x_k f(e_k)\right| +$$

$$+ \sum_{k=k_0+1}^{\infty} |x_k f(e_k)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$\left|f(x) - \sum_{j=1}^{J}\sum_{m\in M_j} x_m f(e_m)\right| < \varepsilon$$

for every $J \ge j_0$. So, the series (12) converges to $f(x)$. Therefore the equality (8) holds. □

Let A be a nonempty set, A_0 be some nonempty subset of A and

$$B_{A_0}(A) = \big\{b\colon A \to A\colon b \text{ is a bijection and } b(A_0) = A_0\big\}. \tag{14}$$

Lemma 3. *The set $B_{A_0}(A)$, defined by* (14), *with the operation of composition is a group.*

Proof. The proof is straightforward. □

Let us describe the structure of continuous linear functionals on ℓ_1 that are symmetric with respect to a group of symmetry, elements of which map the basis of ℓ_1 into itself.

Theorem 1. *Let S be a subgroup of the group $B_{\{e_m\}_{m=1}^\infty}(\ell_1)$, where $B_{\{e_m\}_{m=1}^\infty}(\ell_1)$ is defined by* (14). *Then*
1) the relation "$\sim$" on $\mathbb{N}$, defined as

$$i \sim j \Leftrightarrow \text{ there exists } s \in S \text{ such that } s(e_i) = e_j, \tag{15}$$

is an equivalence relation;
2) every continuous linear S-symmetric functional f on ℓ_1 can be represented as

$$f(x) = \sum_{M\in\mathbb{N}/\sim} \gamma_M \sum_{m\in M} x_m \tag{16}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$, where the relation "$\sim$" is defined by (15) *and $\gamma_M = f(e_m)$ for $m \in M$ (the value $f(e_m)$ does not depend on the choice of $m \in M$).*

Proof. Let us show that the relation "$\sim$", defined by (15), is an equivalence relation on $\mathbb{N}$. Since S is a subgroup of the group $B_{\{e_m\}_{m=1}^\infty}(\ell_1)$, it follows that S has the following properties:

1. S contains the identity mapping on ℓ_1;
2. $s^{-1} \in S$ for every $s \in S$;

3. $s \circ t \in S$ for every $s, t \in S$.

Properties 1, 2 and 3 imply the reflexivity, the symmetry and the transitivity of the relation "$\sim$" resp. So, the relation "$\sim$" is an equivalence relation.

Fix some linear order on $\mathbb{N}/_\sim$. For example, let $M \prec N$ if $\min M \leq \min N$ for $M, N \in \mathbb{N}/_\sim$. Let f be a continuous linear S-symmetric functional on ℓ_1. By Lemma 2,

$$f(x) = \sum_{M \in \mathbb{N}/_\sim} \sum_{m \in M} x_m f(e_m) \tag{17}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$.

Let $M \in \mathbb{N}/_\sim$. Let $i, j \in M$. Since $i \sim j$, there exists $s \in S$ such that $s(e_i) = e_j$. Consequently, taking into account that f is S-symmetric, $f(e_i) = f(s(e_i)) = f(e_j)$. So, $f(e_m)$ does not depend on the choice of $m \in M$. Let $\gamma_M = f(e_m)$, where $m \in M$. Then

$$\sum_{m \in M} x_m f(e_m) = \gamma_M \sum_{m \in M} x_m. \tag{18}$$

Equalities (17) and (18) imply the equality (16). □

4. Symmetric linear functionals on the space generated by pseudometrics. Let us construct the Banach space, elements of which have similar structure as pseudometrics. Let V be the set of all mappings $a\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that

1. $a(i, i) = 0$ for every $i \in \mathbb{N}$;
2. $a(i, j) = a(j, i)$ for every $i, j \in \mathbb{N}$;
3. The series $\sum_{i,j \in \mathbb{N}} |a(i, j)|$ is convergent.

We endow V with point-wise operations of addition and scalar multiplication and with norm

$$\|a\| = \sum_{(i,j) \in \mathcal{N}_2^{(+)}} |a(i, j)|,$$

where $a \in V$ and $\mathcal{N}_2^{(+)} = \{(i, j) \in \mathbb{N}^2 \colon i < j\}$. It can be checked that V is a normed space.

Let us show that V is isometrically isomorphic to ℓ_1.

Let $\varkappa\colon \mathcal{N}_2^{(+)} \to \mathbb{N}$ be defined by

$$\varkappa((i, j)) = \frac{(j-2)(j-1)}{2} + i, \tag{19}$$

where $(i, j) \in \mathcal{N}_2^{(+)}$. It can be checked that $\varkappa$ is a bijection.

Let us define the mapping $J\colon V \to \ell_1$ by

$$J(a) = \sum_{(i,j) \in \mathcal{N}_2^{(+)}} a(i, j) e_{\varkappa((i,j))}. \tag{20}$$

Proposition 1. *The mapping J, defined by* (20), *is an isometrical isomorphism.*

Proof. Since, for every $a \in V$, the series $\sum_{i,j\in\mathbb{N}} |a(i,j)|$ is convergent, it follows that $J(a)$ belongs to ℓ_1. Moreover,

$$\|J(a)\|_1 = \|a\| \tag{21}$$

for every $a \in V$. It can be checked that J is linear. Consequently, taking into account (21), J is injective. Also it can be checked that J is surjective. Thus, J is an isometrical isomorphism. □

For $(i,j) \in \mathcal{N}_2^{(+)}$, let $c_{ij} \in V$ be defined by

$$c_{ij}(k,l) = \begin{cases} 1, & \text{if } (k,l) = (i,j) \text{ or } (l,k) = (i,j); \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

where $k, l \in \mathbb{N}$. Note that

$$J(c_{ij}) = e_{\varkappa((i,j))} \tag{23}$$

for every $(i,j) \in \mathcal{N}_2^{(+)}$. Let

$$C = \{c_{ij} \colon (i,j) \in \mathcal{N}_2^{(+)}\}, \tag{24}$$

where c_{ij} is defined by (22). Since $\varkappa$ is a bijection, it follows that

$$J(C) = \{e_m\}_{m=1}^{\infty} \quad \text{and} \quad J^{-1}\big(\{e_m\}_{m=1}^{\infty}\big) = C. \tag{25}$$

Consequently, taking into account that $\{e_m\}_{m=1}^{\infty}$ is the Schauder basis in ℓ_1 and J is an isometrical isomorphism, it follows that C is the Schauder basis in V.

Theorem 1 and Proposition 1 imply the following theorem.

Theorem 2. *Let S be a subgroup of the group $B_C\big(V\big)$, where C is defined by (24) and $B_C\big(V\big)$ is defined by (14). Then*

1) the relation "$\sim$" on $\mathcal{N}_2^{(+)}$, defined as

$$(i_1, j_1) \sim (i_2, j_2) \Leftrightarrow \text{ there exists } s \in S \text{ such that } s(c_{i_1 j_1}) = c_{i_2 j_2}, \tag{26}$$

is an equivalence relation;

2) every continuous linear S-symmetric functional f on V can be represented as

$$f(a) = \sum_{M \in \mathcal{N}_2^{(+)}/\sim} \gamma_M \sum_{(i,j)\in M} a(i,j) \tag{27}$$

for every $a \in V$, where the relation "$\sim$" is defined by (26) and $\gamma_M = f(c_{ij})$ for $(i,j) \in M$ (the value $f(c_{ij})$ does not depend on the choice of $(i,j) \in M$).

Proof. Let

$$\tilde{S} = \{J \circ s \circ J^{-1} \colon s \in S\},$$

where J is defined by (20). Since S is a group, it follows that $\tilde{S}$ is a group too. For every $s \in S$, by (25), taking into account that S is a subgroup of $B_C\big(V\big)$,

$$\big(J \circ s \circ J^{-1}\big)\big(\{e_m\}_{m=1}^{\infty}\big) = \big(J \circ s\big)(C) = J(C) = \{e_m\}_{m=1}^{\infty}.$$

Consequently, $\tilde{S}$ is a subgroup of $B_{\{e_m\}_{m=1}^{\infty}}(\ell_1)$.

Let "$\simeq$" be the equivalence relation on $\mathbb{N}$, defined by (15), where we set $\tilde{S}$ instead of S. Let us prove the following properties:

a) if $(i_1, j_1) \sim (i_2, j_2)$ with respect to the relation (26), then $\varkappa((i_1, j_1)) \simeq \varkappa((i_2, j_2))$, where $\varkappa$ is defined by (19);

b) if $i \simeq j$, then $\varkappa^{-1}(i) \sim \varkappa^{-1}(j)$ with respect to the relation (26).

Let us prove the property a). Let $(i_1, j_1) \sim (i_2, j_2)$ with respect to the relation (26). Then there exists $s \in S$ such that $s(c_{i_1 j_1}) = c_{i_2 j_2}$. Let $\tilde{s} = J \circ s \circ J^{-1}$. Note that $\tilde{s} \in \tilde{S}$ and, taking into account (23),

$$\tilde{s}\big(e_{\varkappa((i_1,j_1))}\big) = \big(J \circ s \circ J^{-1}\big)\big(e_{\varkappa((i_1,j_1))}\big) = \big(J \circ s\big)(c_{i_1 j_1}) = J(c_{i_2 j_2}) = e_{\varkappa((i_2,j_2))}.$$

So, by (15), $\varkappa((i_1, j_1)) \simeq \varkappa((i_2, j_2))$.

Let us prove the property b). Let $i \simeq j$. Then there exists $\tilde{s} \in \tilde{S}$ such that $\tilde{s}(e_i) = e_j$. Since $\tilde{s} \in \tilde{S}$, it follows that there exists $s \in S$ such that $\tilde{s} = J \circ s \circ J^{-1}$. Consequently, $s = J^{-1} \circ \tilde{s} \circ J$. Therefore, taking into account (23),

$$s(c_{\varkappa^{-1}(i)}) = \big(J^{-1} \circ \tilde{s} \circ J\big)\big(c_{\varkappa^{-1}(i)}\big) = \big(J^{-1} \circ \tilde{s}\big)(e_i) = J^{-1}\big(\tilde{s}(e_i)\big) = J^{-1}(e_j) = c_{\varkappa^{-1}(j)}.$$

So, by (26), $\varkappa^{-1}(i) \sim \varkappa^{-1}(j)$.

The reflexivity, the symmetry and the transitivity of the relation "$\simeq$" and the properties a) and b) imply the reflexivity, the symmetry and the transitivity of the relation "$\sim$." So, "$\sim$" is an equivalence relation.

Let f be a continuous linear S-symmetric functional on V. Let $\tilde{f} \colon \ell_1 \to \mathbb{R}$ be defined by

$$\tilde{f} = f \circ J^{-1}. \tag{28}$$

Since mappings f and J^{-1} are continuous and linear, it follows that $\tilde{f}$ is continuous and linear. Let us show that $\tilde{f}$ is $\tilde{S}$-symmetric. Let $\tilde{s} \in \tilde{S}$ and $x \in \ell_1$. Let $s \in S$ be such that $\tilde{s} = J \circ s \circ J^{-1}$. Then, taking into account that f is S-symmetric,

$$\tilde{f}\big(\tilde{s}(x)\big) = \big(f \circ J^{-1}\big)\big((J \circ s \circ J^{-1})(x)\big) = \big(f \circ s \circ J^{-1}\big)(x) = f\big(s\big(J^{-1}(x)\big)\big) = f\big(J^{-1}(x)\big) = \tilde{f}(x).$$

So, $\tilde{f}$ is $\tilde{S}$-symmetric. Thus, $\tilde{f}$ is a continuous linear $\tilde{S}$-symmetric functional on ℓ_1. Consequently, by Theorem 1,

$$\tilde{f}(x) = \sum_{N \in \mathbb{N}/_\simeq} \beta_N \sum_{m \in N} x_m \tag{29}$$

for every $x = (x_1, x_2, \ldots) \in \ell_1$, where

$$\beta_N = \tilde{f}(e_m) \tag{30}$$

for $m \in N$, and $\tilde{f}(e_m)$ does not depend on the choice of m.

Since $\tilde{f} = f \circ J^{-1}$, it follows that $f = \tilde{f} \circ J$. Consequently, taking into account (29),

$$f(a) = \tilde{f}(J(a)) = \sum_{N \in \mathbb{N}/_\simeq} \beta_N \sum_{m \in N} a\big(\varkappa^{-1}(m)\big) = \sum_{N \in \mathbb{N}/_\simeq} \beta_N \sum_{(i,j) \in \varkappa^{-1}(N)} a(i,j) \tag{31}$$

for every $a \in V$. By the properties a) and b), $\varkappa^{-1}(N) \in \mathcal{N}_2^{(+)}/_\sim$ for every $N \in \mathbb{N}/_\simeq$ and $\varkappa(M) \in \mathbb{N}/_\simeq$ for every $M \in \mathcal{N}_2^{(+)}/_\sim$. Consequently, by (31),

$$f(a) = \sum_{M \in \mathcal{N}_2^{(+)}/_\sim} \beta_{\varkappa(M)} \sum_{(i,j) \in M} a(i,j)$$

for every $a \in V$, where $\gamma_M = \beta_{\varkappa(M)}$. Fix some $m \in \varkappa(M)$. Let $(i,j) = \varkappa^{-1}(m)$. Note that $(i,j) \in M$. By (30), $\beta_{\varkappa(M)} = \tilde{f}(e_m)$. Consequently, taking into account equalities (28) and (23),

$$\beta_{\varkappa(M)} = f\big(J^{-1}(e_m)\big) = f\big(c_{\varkappa^{-1}(m)}\big) = f(c_{ij}).$$

Since f is S-symmetric, $f(c_{i_1 j_1}) = f(c_{ij})$ for every $(i_1, j_1) \sim (i,j)$, that is, for every $(i_1, j_1) \in M$. Let $\gamma_M = f(c_{ij})$. Then $\beta_{\varkappa(M)} = \gamma_M$ and, consequently,

$$f(a) = \sum_{M \in \mathcal{N}_2^{(+)}/\sim} \gamma_M \sum_{(i,j)\in M} a(i,j)$$

for every $a \in V$. □

Thus, the isomorphism of the spaces ℓ_1 and V gave us the opportunity to obtain the complete description of the structure of symmetric continuous linear functionals on V analogical to the description of the structure of symmetric continuous linear functionals on ℓ_1, obtained in Theorem 1. Similar results on isomorphisms of algebras of symmetric polynomials and analytic functions on Banach spaces were established in [17–19].

Let B be some subgroup of the group of all bijections on $\mathbb{N}$. For every $b \in B$ and $a \in V$, let $s_b(a)\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ be defined by

$$s_b(a)(i,j) = a(b(i), b(j)), \tag{32}$$

where $i, j \in \mathbb{N}$. Note that $s_b(a) \in V$. For $b \in B$, let us define $s_b\colon V \to V$ by

$$s_b\colon a \mapsto s_b(a), \tag{33}$$

where $a \in V$ and $s_b(a)$ is defined by (32). It can be checked that s_b is an isometrical isomorphism. Let

$$S_B = \{s_b\colon b \in B\}. \tag{34}$$

Note that S_B is a group.

Let us consider the equivalence relation (26), where we set S_B instead of S. Let $b \in B$ and $(i,j) \in \mathcal{N}_2^{(+)}$. For $k, l \in \mathbb{N}$, by (32) and (22), taking into account that b is a bijection,

$$s_b(c_{ij})(k,l) = c_{ij}(b(k), b(l)) = \begin{cases} 1, & \text{if } (b(k), b(l)) = (i,j) \text{ or } (b(l), b(k)) = (i,j); \\ 0, & \text{otherwise} \end{cases} =$$

$$= \begin{cases} 1, & \text{if } (k,l) = \big(b^{-1}(i), b^{-1}(j)\big) \text{ or } (l,k) = \big(b^{-1}(i), b^{-1}(j)\big); \\ 0, & \text{otherwise} \end{cases} = c_{uv}(k,l),$$

where $u = \min\{b^{-1}(i), b^{-1}(j)\}$ and $v = \max\{b^{-1}(i), b^{-1}(j)\}$. Therefore, by (26), $(i,j) \sim (u,v)$. It can be checked that the class of equivalence that contains $(i,j) \in \mathcal{N}_2^{(+)}$ has the following form $[(i,j)] = \big\{\big(\min\{b^{-1}(i), b^{-1}(j)\}, \max\{b^{-1}(i), b^{-1}(j)\}\big)\colon b \in B\big\}$ or, equivalently, taking into account that B is a group,

$$[(i,j)] = \big\{\big(\min\{b(i), b(j)\}, \max\{b(i), b(j)\}\big)\colon b \in B\big\}. \tag{35}$$

Consider some examples.

Example 1. Let B be the group of all bijections on $\mathbb{N}$. This group is widely used in investigations of symmetric polynomials and symmetric analytic functions on Banach spaces of sequences with symmetric Schauder basis (see [1,5,14]). Taking into account (35), $[(i,j)] = \mathcal{N}_2^{(+)}$ for every $(i,j) \in \mathcal{N}_2^{(+)}$. That is, $(i_1,j_1) \sim (i_2,j_2)$ for every $(i_1,j_1),(i_2,j_2) \in \mathcal{N}_2^{(+)}$. Therefore, by Theorem 2, every S_B-symmetric continuous linear functional on V can be represented in the form

$$f(a) = \gamma \sum_{(i,j)\in\mathcal{N}_2^{(+)}} a(i,j), \tag{36}$$

where $a \in V$ and $\gamma \in \mathbb{R}$.

Example 2. Let $\mathcal{P}$ be a partition of $\mathbb{N}$. Let B be the group of all bijections $b\colon \mathbb{N} \to \mathbb{N}$ such that $b(E) = E$ for every $E \in \mathcal{P}$. Let $(i,j) \in \mathcal{N}_2^{(+)}$. By (35), if there exists $E \in \mathcal{P}$ such that $i,j \in E$, then $[(i,j)] = E^2 \cap \mathcal{N}_2^{(+)}$, else $[(i,j)] = (E_i \times E_j \cup E_j \times E_i) \cap \mathcal{N}_2^{(+)}$, where $E_i, E_j \in \mathcal{P}$ are such that $i \in E_i$ and $j \in E_j$. Consequently, by Theorem 2, every S_B-symmetric continuous linear functional on V can be represented in the form

$$f(a) = \sum_{E\in\mathcal{P}} \gamma_E \sum_{(i,j)\in E^2\cap\mathcal{N}_2^{(+)}} a(i,j) + \sum_{\{E_1,E_2\}\in\mathcal{E}} \gamma_{\{E_1,E_2\}} \sum_{i\in E_1, j\in E_2} a(i,j),$$

where $a \in V$, $\gamma_E, \gamma_{\{E_1,E_2\}} \in \mathbb{R}$ and $\mathcal{E} = \big\{\{E_1,E_2\}\colon E_1,E_2 \in \mathcal{P}, E_1 \neq E_2\big\}$.

Example 3. Let $n \in \mathbb{N}$. Let B_n be the group of all bijections $b\colon \mathbb{N} \to \mathbb{N}$ such that $b(j) = j$ for every $j > n$. This example can be considered as a partial case of the previous example, where we set $\mathcal{P} = \big\{\{1,\ldots,n\},\{n+1\},\{n+2\},\ldots\big\}$. Therefore every S_{B_n}-symmetric continuous linear functional on V can be represented in the form

$$f(a) = \gamma \sum_{1\le i<j\le n} a(i,j) + \sum_{j=n+1}^{\infty} \gamma_j \sum_{i=1}^{n} a(i,j) + \sum_{i,j=n+1}^{\infty} \gamma_{ij} a(i,j),$$

where $a \in V$ and $\gamma, \gamma_j, \gamma_{ij} \in \mathbb{R}$.

Example 4. Let $B = \bigcup_{n=1}^{\infty} B_n$, where B_n is defined in Example 3. The group B is used in investigations of the so-called finitely symmetric functions on infinite-dimensional spaces [6,7]. By (35), $[(i,j)] = \mathcal{N}_2^{(+)}$ for every $(i,j) \in \mathcal{N}_2^{(+)}$. Consequently, every S_B-symmetric continuous linear functional on V can be represented in the form (36).

Example 5. Let $n \in \mathbb{N}$. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a bijection. Let $b_\sigma\colon \mathbb{N} \to \mathbb{N}$ be defined by

$$b_\sigma(k) = n\sigma(q) + r, \tag{37}$$

where q and r are the quotient and the remainder of the division of k by n resp. In other words, b_σ permutes "blocks" $\{1,2,\ldots,n\},\{n+1,n+2,\ldots,2n\},\ldots$ of the length n. For example, let σ be defined by

$$\sigma(j) = \begin{cases} 2, & \text{if } j = 1; \\ 1, & \text{if } j = 2; \\ j, & \text{otherwise.} \end{cases}$$

Then $b_\sigma\big(\{1,2,\ldots,n\}\big) = \{n+1,n+2,\ldots,2n\}$, $b_\sigma\big(\{n+1,n+2,\ldots,2n\}\big) = \{1,2,\ldots,n\}$ and $b_\sigma(j) = j$ for $j > 2n$.

Let $B = \{b_\sigma\colon \sigma\colon \mathbb{N} \to \mathbb{N} \text{ is a bijection}\}$, where b_σ is defined by (37). This group is used for investigations of symmetric functions on Cartesian products of Banach spaces with symmetric Schauder basis or, equivalently, of the so-called block-symmetric functions on these Banach spaces [3, 9, 10] (see also [4, 15, 16], where symmetric functions on Cartesian products of rearrangement invariant Banach spaces were studied).

For $i, j \in \mathbb{N}$ such that $1 \le i < j \le n$, let

$$M_{ij} = \big\{(n(k-1)+i, n(k-1)+j) \colon k \in \mathbb{N}\big\}.$$

By (35), $[(i,j)] = M_{ij}$ for every $(i,j) \in \mathcal{N}_2^{(+)}$ such that $1 \le i < j \le n$. Since

$$\bigcup_{1\le i<j\le n} M_{ij} = \mathcal{N}_2^{(+)},$$

it follows that $\mathcal{N}_2^{(+)}/_\sim = \{M_{ij} \colon 1 \le i < j \le n\}$. Consequently, by Theorem 2, every S_B-symmetric continuous linear functional on V can be represented in the form

$$f(a) = \sum_{1\le i<j\le n} \gamma_{ij} \sum_{k=1}^{\infty} a\big(n(k-1)+i, n(k-1)+j\big),$$

where $a \in V$ and $\gamma_{ij} \in \mathbb{R}$.

Let us apply the results of this section to the construction of symmetric functions on sets of pseudometrics. Let X be a nonempty set. Let B be some group of bijections on X such that the set

$$X_B = \big\{x \in X \colon \text{there exists } b \in B \text{ such that } b(x) \ne x\big\}$$

is finite. Let us construct $\widehat{B}$-symmetric functionals on $Ps(X)$, where the group $\widehat{B}$ is defined by (6). Let X_0 be an arbitrary finite set such that $X_B \subset X_0 \subset X$. Fix some bijection $h \colon X_0 \to \{1, \ldots, n\}$, where $n = |X_0|$. Let us define the mapping $p_h \colon Ps(X) \to V$ by

$$p_h(d)(i,j) = \begin{cases} d\big(h^{-1}(i), h^{-1}(j)\big), & \text{if } i, j \in \{1, \ldots, n\}; \\ 0, & \text{otherwise,} \end{cases}$$

where $d \in Ps(X)$ and $i, j \in \mathbb{N}$. For $b \in B$, let $\sigma_b \colon \mathbb{N} \to \mathbb{N}$ be defined by

$$\sigma_b(j) = \begin{cases} \big(h \circ b \circ h^{-1}\big)(j), & \text{if } j \in \{1, \ldots, n\}; \\ j, & \text{otherwise,} \end{cases}$$

where $j \in \mathbb{N}$. It can be checked that σ_b is a bijection. Let $B_0 = \{\sigma_b \colon b \in B\}$. Note that B_0 is a group of bijections on $\mathbb{N}$. It can be checked that $p_h\big(\widehat{b}(d)\big) = s_{\sigma_b}(p_h(d))$ for every $b \in B$, where s_{σ_b} is defined by (33). Consequently, the function $f \circ p_h$ is $\widehat{B}$-symmetric for every S_{B_0}-symmetric function f on V, where S_{B_0} is defined by (34). Indeed,

$$(f \circ p_h)\big(\widehat{b}(d)\big) = f(s_{\sigma_b}(p_h(d))) = f(p_h(d)) = (f \circ p_h)(d)$$

for every $b \in B$ and $d \in Ps(X)$. By Theorem 2, all S_{B_0}-symmetric continuous linear functionals on V are given by (27). Let us denote by F the set of all such functionals. Then every element of the set $\{f \circ p_h \colon f \in F\}$ is a $\widehat{B}$-symmetric function on $Ps(X)$.

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REFERENCES

1. R. Alencar, R. Aron, P. Galindo, A. Zagorodnyuk, *Algebras of symmetric holomorphic functions on* ℓ_p, Bull. Lond. Math. Soc., **35** (2003), №2, 55–64. doi:10.1112/S0024609302001431
2. R. Aron, P. Galindo, D. Pinasco, I. Zalduendo, *Group-symmetric holomorphic functions on a Banach space*, Bull. Lond. Math. Soc., **48** (2016), №5, 779–796. doi:10.1112/blms/bdw043
3. A. Bandura, V. Kravtsiv, T. Vasylyshyn, *Algebraic basis of the algebra of all symmetric continuous polynomials on the Cartesian product of* ℓ_p*-spaces*, Axioms, **11** (2022), №2, 41. doi:10.3390/axioms11020041
4. I.V. Burtnyak, Yu.Yu. Chopyuk, S.I. Vasylyshyn, T.V. Vasylyshyn, *Algebras of weakly symmetric functions on spaces of Lebesgue measurable functions*, Carpathian Math. Publ., **15** (2023), №2, 411–419. doi:10.15330/cmp.15.2.411-419
5. I. Chernega, P. Galindo, A. Zagorodnyuk, *On the spectrum of the algebra of bounded-type symmetric analytic functions on* ℓ_1, Math. Nachr., (2024) doi:10.1002/mana.202300415
6. J. Falcó, D. García, M. Jung, M. Maestre, *Group-invariant separating polynomials on a Banach space*, Publicacions Matematiques, **66** (2022), №1, 207–233. doi:10.5565/PUBLMAT6612209
7. P. Galindo, T. Vasylyshyn, A. Zagorodnyuk, *Symmetric and finitely symmetric polynomials on the spaces* ℓ_∞ *and* $L_\infty[0,+\infty)$, Math. Nachr., **291** (2018), №11–12, 1712–1726. doi:10.1002/mana.201700314
8. M. González, R. Gonzalo, J. A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. Lond. Math. Soc., **59** (1999), №2, 681–697. doi:10.1112/S0024610799007164
9. V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *On algebraic basis of the algebra of symmetric polynomials on* $\ell_p(\mathbb{C}^n)$, J. Funct. Spaces, **2017** (2017), 4947925, 8 p. doi:10.1155/2017/4947925
10. V.V. Kravtsiv, A.V. Zagorodnyuk, *Spectra of algebras of block-symmetric analytic functions of bounded type*, Mat. Stud., **58** (2022), №1, 69–81. doi:10.30970/ms.58.1.69-81
11. A.S. Nemirovskii, S.M. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. USSR Sbornik, **21** (1973), №2, 255–277. doi:10.1070/SM1973v021n02ABEH002016
12. S. Nykorovych, O. Nykyforchyn, *Metric and topology on the poset of compact pseudoultrametrics*, Carpathian Math. Publ., **15** (2023), №2, 321–330. doi:10.15330/cmp.15.2.321-330
13. S. Nykorovych, O. Nykyforchyn, A. Zagorodnyuk, *Approximation relations on the posets of pseudoultrametrics*, Axioms, **12** (2023), №5, 438. doi:10.3390/axioms12050438
14. S.I. Vasylyshyn, *Spectra of algebras of analytic functions, generated by sequences of polynomials on Banach spaces, and operations on spectra*, Carpathian Math. Publ., **15** (2023), №1, 104–119. doi:10.15330/cmp.15.1.104-119
15. T. Vasylyshyn, *Algebras of symmetric analytic functions on Cartesian powers of Lebesgue integrable in a power* $p \in [1,+\infty)$ *functions*, Carpathian Math. Publ., **13** (2021), №2, 340–351. doi:10.15330/cmp.13.2.340-351
16. T. Vasylyshyn, *Symmetric analytic functions on the Cartesian power of the complex Banach space of Lebesgue measurable essentially bounded functions on* $[0,1]$, J. Math. Anal. Appl., **509** (2022), №2, Article number 125977. doi:10.1016/j.jmaa.2021.125977
17. T. Vasylyshyn, *Algebras of symmetric and block-symmetric functions on spaces of Lebesgue measurable functions*, Carpathian Math. Publ., **16** (2024), №1, 174–189. doi:10.15330/cmp.16.1.174-189
18. T. Vasylyshyn, V. Zahorodniuk, *Weakly symmetric functions on spaces of Lebesgue integrable functions*, Carpatian Math. Publ. **14** (2022), №2, 437–441. doi:10.15330/cmp.14.2.437-441
19. T. Vasylyshyn, V. Zahorodniuk, *On isomorphisms of algebras of entire symmetric functions on Banach spaces*, J. Math. Anal. Appl., **529** (2024), №2, Article number 127370. doi:10.1016/j.jmaa.2023.127370

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