УДК 514.1+519.21+517.5

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# DISTRIBUTION OF UNIT MASS ON ONE FRACTAL SELF-SIMILAR WEB-TYPE CURVE

M. V. Pratsiovytyi, I. M. Lysenko, S. P. Ratushniak, O. A. Tsokolenko. *Distribution of unit mass on one fractal self-similar web-type curve*, Mat. Stud. **62** (2024), 21–30.

In the article, we study structural, spectral, topological, metric and fractal properties of distribution of complex-valued random variable $\tau = \sum_{n=1}^{\infty} \frac{2\varepsilon_\tau}{3^n} \equiv \Delta^g_{\tau_1 \ldots \tau_n \ldots}$, where $(\tau_n)$ is a sequence of independent random variables taking the values $0, 1, \cdots, 6$ with the probabilities $p_{0n}, p_{1n}, \cdots, p_{6n}$; $\varepsilon_6 = 0$, $\varepsilon_0$, $\varepsilon_1, \cdots, \varepsilon_5$ are 6th roots of unity.

We prove that the set of values of random variable $\tau$ is self-similar six petal snowflake which is a fractal curve $G$ of spider web type with dimension $\log_3 7$. Its outline is the Koch snowflake.

We establish that $\tau$ has either a discrete or a singularly continuous distribution with respect to two-dimensional Lebesgue measure. The criterion of discreteness for the distribution is found and its point spectrum (set of atoms) is described. It is proved that the point spectrum is a countable everywhere dense set of values of the random variable $\tau$, which is the tail set of the seven-symbol representation of the points of the curve $G$.

In the case of identical distribution of the random variables $\tau_n$ (namely: $p_{kn} = p_k$) we establish that the spectrum of distribution $\tau$ is a self-similar fractal and that the essential support of density is the fractal Besicovitch-Eggleston type set. The set is defined by terms digits frequencies and has the fractal dimension $\alpha_0(E) = \frac{\ln p_0^{p_0} \cdots p_6^{p_6}}{-\ln 7}$ with respect to the Hausdorff-Billingsley $\alpha$-measure. The measure is a probabilistic generalization of the Hausdorff $\alpha$-measure. In this case, the random variables $\tau = \Delta^g_{\tau_1 \cdots \tau_n \cdots}$ and $\tau' = \Delta^g_{\tau'_1 \cdots \tau'_n \cdots}$ defined by different probability vectors $(p_0, \cdots, p_6)$ and $(p'_0, \cdots, p'_6)$ have mutually orthogonal distributions.

**Introduction.** Fractal curves (lines) is an important object of research in the modern fractal geometry. The spider web-type curves with a branching index $n > 2$ play a significant role in the theory of antenna modeling [5]. These curves are interesting to physicists and engineers from the perspective of conserving valuable metals (materials). The curves with fractal properties have infinite „length“, zero „area“, and the fractional Hausdorff-Besicovitch dimension [8]. They are not easy to define analytically (they are defined by formulas involving an infinite number of operations or by a limiting process [12, 13]). The idea of self-similarity or the concept of auto-modeling as its generalization are fruitful for their study [14].

In theoretical researches, a little attention is paid to the probability measures whose supports are plane curves with fractal properties, such as the Vicsek fractal, the Koch snowflake [10], the Sierpinski triangle [7, 11], and so on.

2020 *Mathematics Subject Classification:* 28A80, 28A78, 30G35.
*Keywords:* self-similar figure; fractal curve; spectrum of distribution; Lebesgue purity of the distribution; discrete distribution with dense point spectrum; singular distribution; Hausdorff-Besicovitch dimension; fractal Besicovitch-Eggleston type set.
doi:10.30970/ms.62.1.21-30

The aim of this paper is to fill partially this gap. It is devoted to the distribution of a unit mass (probability) on a single plane curve $G \subset R^2$ of the spider web type. We assume that the curve has a maximum branching index of 6. It is closely related to the Koch snowflake. The snowflake is a part (contour) of the curve $G$.

Let's introduce the following notations used in the paper.

Let $A_s = \{0, 1, ..., s-1\}$ be an alphabet, $L_s = A_s \times A_s \times \dots$ be a space of sequence of elements of the alphabet, $\Delta^s_{\alpha_1(x)\dots\alpha_n(x)\dots} = \sum_{n=1}^{\infty} s^{-n}\alpha_n(x)$ be a $s$-adic representation of number $x \in [0; 1]$.

Note that we use parentheses to indicate the period in the representation of a number. There is a countable dense set of numbers in the segment $[0; 1]$ that have two different $s$-adic representations: $\Delta^s_{c_1\dots c_{m-1}c(0)} = \Delta^s_{c_1\dots c_{m-1}[c-1](s-1)}$. Such numbers are called *s-adic-rational*. The rest of the numbers have a single representation and are called *s-adic-irrational*.

Let $N_j(x, k)$ be the number of digits $j \in A_s$ among the first $k$ digits $\alpha_1, \alpha_2, ..., \alpha_k$ of the $s$-adic representation $\Delta^s_{\alpha_1\alpha_2\dots\alpha_n\dots}$ of number $x \in [0; 1]$. The limit (if it exists)

$$\lim_{k\to\infty} \frac{N_j(x, k)}{k} \equiv \nu_j(x)$$

is called the *frequency of the digit* $j$ in the $s$-representation of the number $x$.

The well-known Borel theorem [3] states that almost all (in the sense of the Lebesgue measure) numbers in the segment $[0; 1]$ in the $s$-adic representation have frequencies of all digits of the $s$-adic alphabet, and they are equal to $\frac{1}{s}$.

**Theorem 1** ([1, 4], Besicovitch-Eggleston)**.** *The set of numbers of segment* $[0; 1]$

$$E[s; p_0, p_1, ..., p_{s-1}] = \{x : \nu_j(x) = p_j,\ j =\in \{0, 1, \dots, s-1\}\}$$

*has fractal the Hausdorff-Besicovitch dimension*

$$\alpha_0(E) = \ln p_0^{p_0} p_1^{p_1} ... p_{s-1}^{p_{s-1}} / (-\ln s). \tag{1}$$

The set $E$ is called the Besicovitch-Eggleston set because its fractal properties for $s = 2$ were described by A.S. Besicovitch [1], and for $s > 2$ by Eggleston [4]. According to Borel's Theorem, its Lebesgue measure is either 0 or 1, and it is 1 only if $p_j = \frac{1}{s}$ for all $j \in N$.

**1. Main object.** Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ be an 6th roots of unity, i.e. $\varepsilon_k = \cos\frac{\pi k}{6} + i \sin\frac{\pi k}{6}$, $\varepsilon_6 = 0$, $(\tau_n)$ be a sequence of independent random variables taking the values $0, 1, 2, 3, 4, 5, 6$ with the probabilities $p_{0n}, p_{1n}, p_{2n}, p_{3n}, p_{4n}, p_{5n}, p_{6n}$, respectively ($p_{kn} \geq 0$, $p_{0n} + p_{1n} + ... + p_{6n} = 1$).

Let us consider the complex-valued random variable

$$\tau = 2\sum_{n=1}^{\infty} 3^{-n} \varepsilon_{\tau_n}. \tag{2}$$

It is evident that the properties of random variable $\tau$ are defined by series $\sum_{n=1}^{\infty} 2 \cdot 3^{-n}$ and an infinite seven-row matrix $||p_{kn}||$. We are interested in the Lebesgue structure (that is, the content of the discrete and continuous components, and in the case of continuity, singular and absolutely continuous components of the probability measure) and spectral properties of distribution of the random variable $\tau$ (topological, metric and fractal properties of its point and continuous spectra), and as well as the essential support of the probability density function.

**2. The set of values of the random variable $\tau$.** The set $E_\tau$ of values of the random variable $\tau$ is the set of complex numbers, which are images of the mapping from the sequence space $L_7$ to the set of complex numbers, which is analytically expressed by the formula $g((\alpha_n)) = 2\sum_{n=1}^{\infty} 3^{-n}\varepsilon_{\alpha_n}$.

Let us consider the function $g(t)$ defined by the equation

$$g(t = \Delta^7_{\alpha_1\alpha_2...\alpha_n...}) = 2\sum_{n=1}^{\infty} 3^{-n}\varepsilon_{\alpha_n(t)} = \Delta^g_{\alpha_1\alpha_2...\alpha_n...}. \tag{3}$$

The definition of the function $g$ at the seven-rational point is not well-defined, because Equation (3) gives different values for two different representations of the argument. This drawback can be easily eliminated by agreeing to use only one of the argument representations (let's say the one containing the period (0)).

**Remark 1.** The set of values of the random variable $\tau$ is the set of values of the function $g$, supplemented by the countable set of values of the expressions

$$g(\Delta^7_{c_1...c_{m-1}c_m(6)}) = \sum_{k=1}^{m} \frac{2\varepsilon_{c_k}}{3^k} + 0,\ (c_1, ..., c_{m-1}, c_m) \in A_7^m.$$

An alternative construction is the following. If we assume that the seven-rational point $z = \Delta^7_{c_1...c_{m-1}c(0)} = \Delta^7_{c_1...c_{m-1}[c-1](6)}$ has two components: $z^- = \Delta^7_{c_1...c_{m-1}[c-1](6)}$ and $z^+ = \Delta^7_{c_1...c_{m-1}c(0)}$, then we get the extension $[0;1]^*$ of the segment $[0;1]$ with double points. Then the set of values of the random variable $\tau$ is the set of values of the complex-valued function $g(t) = 2\sum_{n=1}^{\infty} \frac{\varepsilon_{\alpha_n(t)}}{3^n}$ of the real argument $t = \Delta^7_{\alpha_1\alpha_2...\alpha_n...} \in [0;1]^*$ defined on the set $[0;1]^*$.

**Theorem 2.** *The function $g$, well-defined by Equation* (3), *is continuous on the set of seven-irrational points and discontinuous at each seven-rational point, where the function has a non-removable discontinuity. The value of the discontinuity at the point $\Delta^7_{c_1c_2...c_m(0)}$ is equal to $\frac{2}{3^m}$.*

*Proof.* Let $\Delta^7_{c_1c_2...c_n...} = t_0$ be a seven-irrational point, i.e., it has a single representation; $t_0 \neq t$ be a point close to $t_0$. Then $t = \Delta^7_{c_1c_2...c_m\alpha_{m+1}\alpha_{m+2}...}$.

Let us consider the difference module

$$|g(t) - g(t_0)| = 2\left|\sum_{k=m+1}^{\infty} \frac{\varepsilon_{\alpha_k}}{3^k} - \sum_{k=m+1}^{\infty} \frac{\varepsilon_{c_k}}{3^k}\right| = 2\left|\sum_{k=m+1}^{\infty} \frac{\varepsilon_{\alpha_k} - \varepsilon_{c_k}}{3^k}\right| =$$
$$= 2\sum_{k=m+1}^{\infty} \frac{|\varepsilon_{\alpha_k} - \varepsilon_{c_k}|}{3^k} \leq 2\sum_{k=m+1}^{\infty} \frac{2}{3^k} \to 0 (m \to \infty).$$

Therefore $\lim\limits_{t\to t_0} g(t) = g(t_0)$. Hence, the function $g$ is continuous at the point $t_0$.

Let $t_0 = \Delta^7_{c_1...c_{m-1}c_m(0)}$ be a seven-rational point, $t_0 \neq t$ be a point close enough to $t$. There are possible cases:

1) $t_0 < t = \Delta^7_{c_1...c_{m-1}c_m0...0\alpha_{m+k+1}\alpha_{m+k+2}...}$;
2) $t_0 > t = \Delta^7_{c_1...c_{m-1}c_m6...6\alpha_{m+k+1}\alpha_{m+k+2}...}$.

In the first case, one has $\lim\limits_{t\to t_0} |g(t) - g(t_0)| = 0$, and in the second case

$$\lim_{t\to t_0} |g(t) - g(t_0)| = \frac{2|\varepsilon_{c_m-1}|}{3^m} = \frac{2}{3^m}.$$

Thus, at the point $t_0$, the function $g$ has a discontinuity with a jump of $\frac{2}{3^m}$. □

**Corollary 1.** *The function $g$ has unbounded variation.*

**Theorem 3.** *The set $E_\tau$ of values of the random variable $\tau$ is a self-similar fractal curve $G$ of the space $R^2$ (complex plane) with the self-similarity structure $G = \bigcup_{k=0}^{6} G_k$, $G \overset{1/3}{\sim} G_k$, $G_k = \varphi_k(G)$, where*

$$\varphi_k(z) = \frac{2\varepsilon_k}{3} + \frac{1}{3}z.$$

*It is the set of values of the function $g$ of the real argument $t = \Delta^7_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \in [0;1]^*$, which is analytically expressed by Equation (3).*

*Proof.* 1. *Boundedness of the set $E_\tau$.* If $t_0 = \Delta^7_{(0)}$ then $g(t_0) = 0$. Since

$$|g(t) - g(t_0)| = 2\left|\sum_{n=1}^{\infty} \frac{\varepsilon_{\alpha_n}}{3^n}\right| \leq 2\sum_{n=1}^{\infty} \frac{|\varepsilon_{\alpha_n}|}{3^n} \leq \sum_{n=1}^{\infty} \frac{2}{3^n} = 1,$$

all the values of the function $g$ are concentrated in a closed circle of the radius 1 with the center $z_0 = 0 + i \cdot 0$.

Indeed, $g(\Delta^7_{(j)}) = \varepsilon_j \sum_{n=1}^{\infty} \frac{2}{3^n} = \varepsilon_j$, $|\varepsilon_j| = 1$, $j = \overline{0,5}$. If $t \neq \Delta^7_{(j)}$, then $|g(t)| < 1$, i.e. exactly six values of the function are distant from the point $z_0 = 0$ by a distance of 1, the rest by a distance less than 1.

2. *Self-similarity of a set $E_\tau$.* If $\alpha_n$ is the $n$th digit of the argument of seven-digit representation, then

$$g(\Delta^7_{\alpha_1\ldots\alpha_n\ldots}) = \frac{2\varepsilon_{\alpha_1}}{3} + \frac{1}{3}g(\Delta^7_{\alpha_2\ldots\alpha_n\ldots}),$$

and it is obvious that the set of values of the function has the structure $G = \bigcup_{k=0}^{6} G_k$, $G \overset{1/3}{\sim} G_k$. Moreover, $G_k = \varphi_k(G)$, $k = \overline{0,6}$.

3. *Closedness of the set.* The set $G$ is closed, because it is the limit of a monotonic sequence of nested closed sets (unions of non-overlapping circles: 7, $7^2$, $7^3$, etc.). Thus, $G$ is a compact of $R^2$.

4. *Null set.* The set $G$ belongs to a circle of radius 1. Thus, the Lebesgue measure is $\lambda_2(G) \leq \pi \cdot 1^2$. Given the self-similarity, it is easy to see that $G$ belongs to 7 circles of radius $\frac{1}{3}$. So, $\lambda_2(G) \leq 7 \cdot \pi(\frac{1}{3})^2$. Similarly, $G$ belongs to the union of $7^2$ circles of radius $3^{-2}$. So, $\lambda_2(G) \leq 7^2 \cdot \pi \cdot (\frac{1}{3})^2 = \frac{7^2\pi}{3^4}$. And so on $\lambda_2(G) \leq 7^n \cdot \pi \cdot (\frac{1}{3^n})^2 = (\frac{7}{9})^n\pi \to 0$ $(n \to \infty)$. So, $\lambda(G) = 0$.

5. *Connectivity of the set.* Each of the six sets $G_j$, $j = \overline{0,5}$, has three points in common with the curves of $G_{j+1}$ if $j < 5$, and $G_5 \cap G_0 = \Delta^g_{5(1)} = \Delta^g_{0(4)}$.

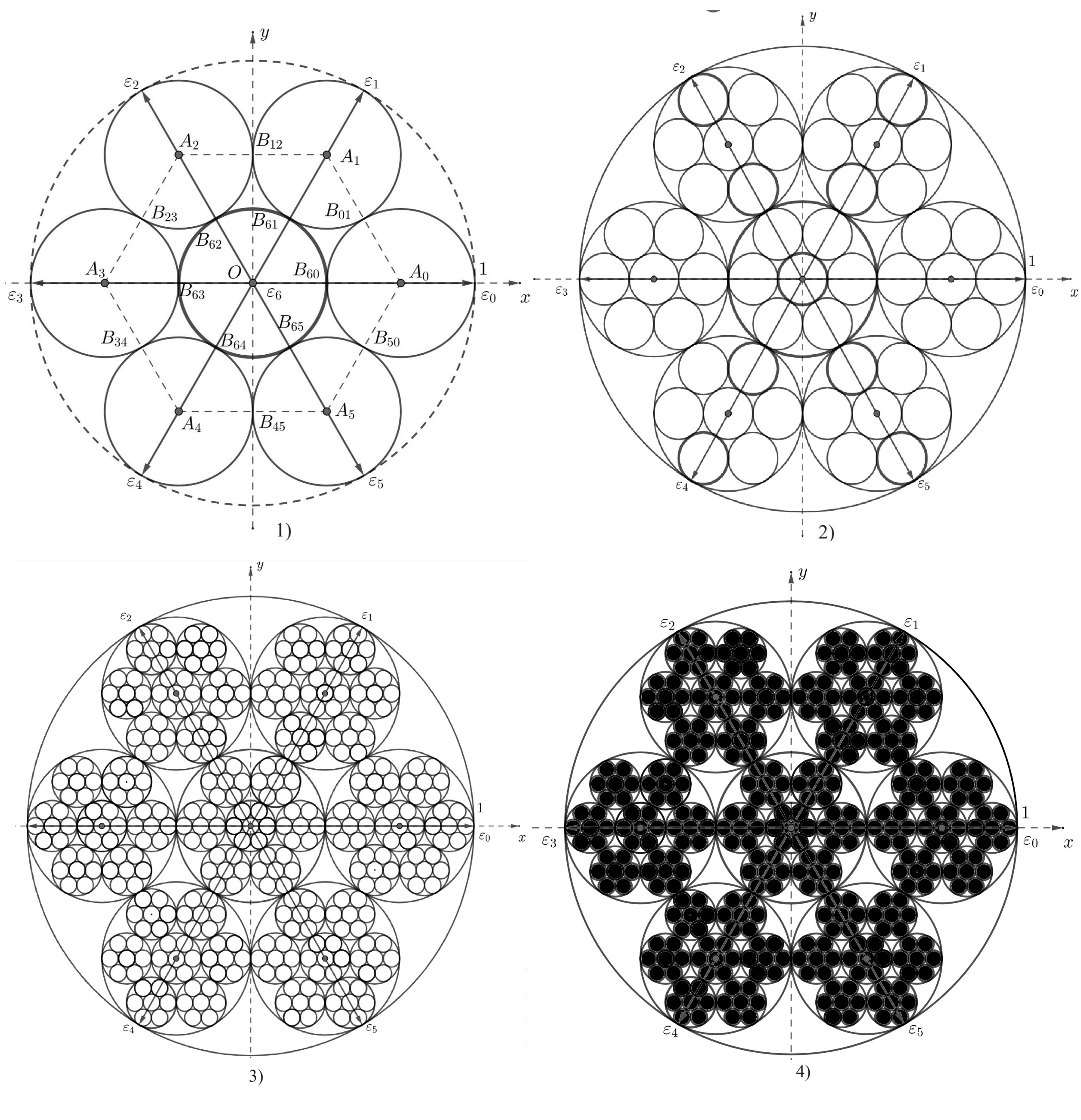
The connectivity of a set $G$ is a consequence of its self-similarity, closure, and the fact that (see Fig. 1))

$$G_6 \cap G_j = B_{6j} = \Delta^g_{6(j)} = \Delta^g_{(|j-3|)}, j = \overline{0,5};$$

$$G_0 \cap G_1 = \Delta^g_{0(2)} = \Delta^g_{1(5)},\ G_1 \cap G_2 = \Delta^g_{1(3)} = \Delta^g_{2(0)},\ G_2 \cap G_3 = \Delta^g_{2(4)} = \Delta^g_{3(1)},$$

$$G_3 \cap G_4 = \Delta^g_{3(5)} = \Delta^g_{4(2)},\ G_4 \cap G_5 = \Delta^g_{4(0)} = \Delta^g_{5(3)},\ G_5 \cap G_0 = \Delta^g_{5(1)} = \Delta^g_{0(4)}.$$

Let us prove that $G$ belongs to the whole segment $[z_0; \varepsilon_j]$, $j \in A_5$ and do it for $j = 0$, because it is done in the same way for other values of $j$. In addition to the ends of the segment $[z_0; \varepsilon_0]$, the set $G$ contains the points dividing the segment by three equal parts: $B_{60} = \Delta^g_{6(0)}$; $A_0 = \Delta^g_{0(6)}$. In view of the self-similarity of the figure $G$, the points dividing



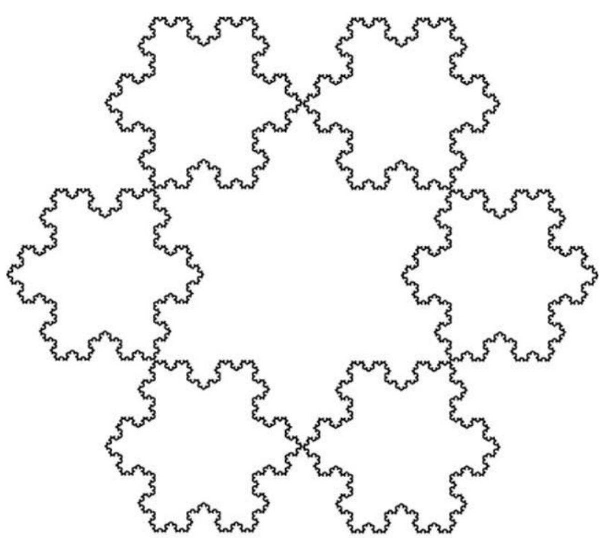

$[z_0; B_{60}]$ by three equal parts also belong to $G$. For the same reason, the points dividing the segment $[A_0; \varepsilon_0]$ by three equal parts also belong to $G$. Similarly, the points dividing the segment $[B_{60}; A_0]$ by three equal parts, namely $\Delta^g_{03(0)}$, $\Delta^g_{06(3)}$ belong to $G$. Thus, we have identified 10 points of the segment dividing it into 9 equal parts and belong to $G$. Given the self-similarity of $G$, the points that divide 9 resulting segments by three equal parts also belong to $G$, and so on. In view of the closedness of $G$, we conclude that the whole segment $[z_0; \varepsilon_0]$ belongs to $G$. Thus, the figure "spider„formed by the six segments $[z_0; \varepsilon_j]$ belongs entirely to $G$. The six similar shapes that belong to the shapes $G_j$, $j = \overline{0,6}$ due to the self-similarity of $G$ also belong to $G$, etc. Thus, the figure $G$ is connected.

6. *The outer boundary $K$ of the curve $G$ is the classical Koch snowflake.* Let $S$ be the Koch snowflake inscribed in a unit disk (circle), one of its vertices coinciding with the point $\varepsilon_0 = 1 + 0 \cdot i$. Then all points of $\varepsilon_i$ are common to $S$ and $G$.

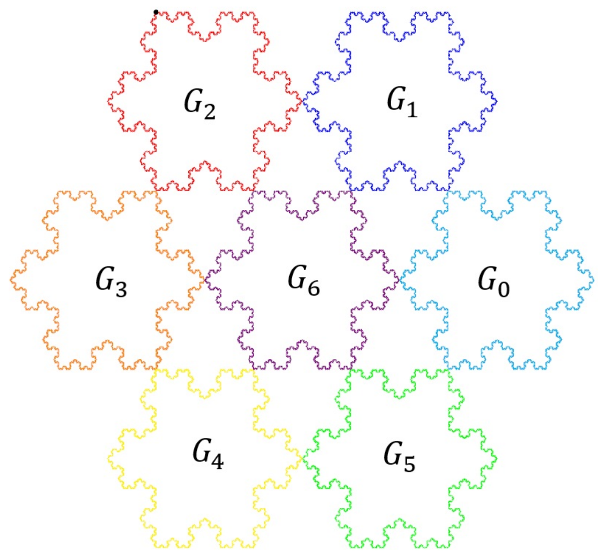
As we can see from Item 5 of this proof, the points $B_{jk}$ are the points of the peripheral boundary $K$ of the curve $G$ that are closest to the center $z_0$ of the curve $G$. They also belong to $S$. Given this, the self-similarity and closedness of the set $S$, we conclude that $G \cap S = S$. Thus, the outer boundary of the curve $G$ is the classic Koch snowflake (see Fig. 5,6)) $\square$

Note that the curve $G$ is the residual set of the space $R^2$. It is obtained as a result of an infinite sequence of removing parts of a circle according to an invariant procedure: the complement of the union of seven "inscribed„ circles of three times smaller radius is removed from the closed circle (see Fig. 1)–4))

Recall that the part of the plane bounded by the Koch snowflake is called the Koch island (see Fig. 5)). It is a self-similar figure. The peripheral contours of the curve $G$ and the Koch island coincide. At the same time, the set $G$ is nowhere dense on Koch island, i.e. every open circle belonging to the island contains a circle free of points of the set $G$.



5)




6)

**Definition 1.** The representation $\Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ of a point (number) $z \in G$ is called a *$g$-representation*, and $\alpha_n$ is called *its nth digit*.

As it follows from the proof of the previous theorem, there exist points of $G$ (we call them *$g$-binary*) having two $g$-representations:

$$\Delta^g_{6(j)} = \Delta^g_{j(|j-3|)}, i \in A_6;\ \Delta^g_{c_1\ldots c_m j(j+2)} = \Delta^g_{c_1\ldots c_m[j+1](j-1)}, j = 1,2,3$$

$$\Delta^g_{c_1\ldots c_m 0(2)} = \Delta^g_{c_1\ldots c_m 1(5)},\ \Delta^g_{c_1\ldots c_m 4(0)} = \Delta^g_{c_1\ldots c_m 5(3)},\ \Delta^g_{c_1\ldots c_m 5(1)} = \Delta^g_{c_1\ldots c_m 0(4)}.$$

The set of $g$-binary points is countable and dense in $G$. The rest of the points in $G$ have a single $g$-representation and are called $g$-unary.

A cylinder ($g$-cylinder) of rank $m$ with base $c_1 \ldots c_m$ on the curve $G$ is a set

$$\Delta^g_{c_1\ldots c_m} = \{z : z = \Delta^g_{c_1\ldots c_m\alpha_1\alpha_2\ldots}, (\alpha_n) \in A_7\}.$$

It follows directly from the definition that

1) $\Delta^g_{c_1\ldots c_m} = \bigcup\limits_{j=0}^{6} \Delta^g_{c_1\ldots c_m j}$;

2) $\Delta^g_{c_1...c_m} \overset{1/3}{\sim} \Delta^g_{c_1...c_m j}$ i $G \overset{1/3^m}{\sim} \Delta^g_{c_1...c_m}$.

As we can see from the proof of the theorem

3) $\Delta^g_{c_1...c_m k} \cap \Delta^g_{c_1...c_m j} = \begin{cases} \varnothing, & k \neq 6 \neq j \wedge |k-j| \neq 1, \\ \text{point}, & |k-j| = 1 \vee k = 6. \end{cases}$

The diameter of the cylinder $\Delta^g_{c_1...c_m}$ is calculated by the formula

4) $d(\Delta^g_{c_1...c_m}) = 2 \cdot 3^{-m}$;

For any sequence $(c_n) \in L_7$, the following equality holds

5) $\bigcap_{n=1}^{\infty} \Delta^g_{c_1...c_n} = \Delta^g_{c_1...c_n...}$,

that is, a point of $G$ is a $g$-cylinder of infinite rank.

At first, let us define the measure $m$ on the curve $G$ on the $g$-cylinders by the equation

$$m(\Delta^g_{c_1...c_m...}) = 7^{-m}$$

and extend it to the minimal $\sigma$-algebra containing all Borel subsets of the curve $G$.

**Theorem 4.** *A set of the Besicovitch-Eggleston type $M \subset G$, where*

$$M = M[g; p_0, p_1, ..., p_6] = \{x : x = \Delta^g_{\alpha_1\alpha_2...\alpha_n...}, \nu_j(x) = p_j, j = \overline{0,6}\},$$

*1) is the set of full measure $m$, if $p_j = \frac{1}{7}$, $j = \overline{0,6}$;*

*2) is the set of zero measure $m$, if there exists $p_j \neq \frac{1}{7}$.*

*Its fractal Hausdorff-Billingsley dimension [2] with respect to the probability measure $m$ and coverings of $g$-cylinders is calculated by Equation* (1).

*Proof.* Since the probability measure $m$ on the curve $G$ is a complete analogue of the Lebesgue measure on the segment $[0; 1]$, which is a self-similar set like the curve $G$ with the same self-similarity structure, then this theorem is a complete analogue of the Besicovitch-Eggleston Theorem.

An independent proof of this statement by means of analysis alone can be carried out in the same way as in [9] for $Q_2$-representation and $G_2$-representation of numbers. □

**3. Discrete distributions. Point spectrum.** We say that the $g$-representations of the points (numbers) $z_1 = \Delta^g_{\alpha_1\alpha_2...\alpha_n...}$ and $z_2 = \Delta^g_{\beta_1\beta_2...\beta_n...}$ of the curve $G$ have the same tail $(z_1 \sim z_2)$ if there exist numbers $k$ and $m$ such that $\alpha_{k+j} = \beta_{m+j}$ for any $j \in N$.

The binary relation „have the same tail“ is an equivalence relation. The set of all points of a curve $G$ that have the same tail of the $g$-representation is called the tail set. Each tail set is countable and everywhere dense in $G$.

Note that the point spectrum of a distribution is the set of its atoms.

**Theorem 5.** *The random variable $\tau$ has pure discrete distribution if and only if*

$$M \equiv \prod_{n=1}^{\infty} \max_k \{p_{kn}\} > 0. \tag{4}$$

*When the distribution is discrete, the point spectrum of the distribution $\tau$ is the tail set, which is represented by the atom $z_0 = \Delta^g_{a_1a_2...a_n...}$, where $p_{a_n n} = \max_k p_{kn}$ with the maximum mass $M$.*

*Proof.* We interpret the event $\{\tau = \Delta^g_{\alpha_1\alpha_2}...\}$ as $\{\tau_n = \alpha_n \forall n \in N\}$. In this context, the $g$-binary point $z$ with the representations: $\Delta^g_{c_1c_2...c_m i(i+2)} = \Delta^g_{c_1...c_m[i+1](i-1)}$ has two components (it is paired). However, only one of the two events: $\{\tau = \Delta^g_{c_1c_2...c_m i(i+2)}\}$ and $\{\tau = \Delta^g_{c_1c_2...c_m[i+1](i-1)}\}$ can have a positive probability.

Given the above, at $M = 0$ we have

$$P\{\tau = \Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\} \leq P\{\tau = z_0\} = M = 0 \;\; \forall z = \Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \in G.$$

Thus, the distribution of the random variable $\tau$ is continuous.

Let $M > 0$ and let $z = \Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ be a point belonging to the tail set $W_{z_0}$, to which the atom $z_0$ belongs, and $p_{\alpha_n n} > 0$ for $j \leq m$ and $\alpha_j = a_j$ for $j > m$. Then

$$P\{\tau = z\} = \left(\prod_{n=1}^{m} p_{\alpha_n n}\right) \prod_{n=m+1}^{\infty} p_{a_n n} > 0.$$

So, $z \in D_\tau$.

It remains to prove that the sum of the masses of all such atoms of distribution $\tau$ is 1. The condition $M > 0$ implies that the necessary condition for the convergence of an infinite product is satisfied, i.e. $\prod_{n=m+1}^{\infty} p_{a_n n} \to 1$ as $m \to \infty$. Let $B_m$ be the set of all such points $z \in E_\tau$ whose $g$-representation differs from the $g$-representation of $z_0$ by no more than first $m$ digits. Then $B_0 = \{z_0\}$, $B_0 \subset B_1 = \{z = \Delta^g_{\alpha_1\alpha_2\ldots}, \alpha_{1+j} = a_{1+j} \; \forall j \in N\}$ etc. We have $B_n \subset B_{n+1}$ $(n \geq 1)$,

$$P(B_n) = \left(\sum_{\alpha_1=0}^{6} \ldots \sum_{\alpha_m=0}^{6} \prod_{j=1}^{m} p_{\alpha_j j}\right) \cdot \prod_{n=m+1}^{\infty} p_{a_n n} = \prod_{n=m+1}^{\infty} p_{a_n n},$$

The value of the expression in the parentheses is 1. Then $P(B_m) \to 1$ as $m \to \infty$, and the distribution $\tau$ is purely discrete with the point spectrum belonging to the tail set $W_{z_0}$. □

**Corollary 2.** *A distribution of the random variable $\tau$ is continuous if and only if $M = 0$.*

**Corollary 3.** *The continuous distribution of the random variable $\tau$, being centered on the Lebesgue zero set, is singular (orthogonal to the two-dimensional Lebesgue measure).*

**Corollary 4.** *If $M > 0$ and there are no zeros among the elements of the matrix $||p_{kn}||$, then the point spectrum of the distribution $\tau$ is everywhere dense in the set of values of the distribution of the random variable $\tau$.*

**Remark 2.** Note that the first part of Theorem 5 is a consequence of the well-known Theorem of P. Levy [6].

**4. The full spectrum of the distribution of a random variable** $\tau$ Recall that the spectrum of the distribution of the random variable $\tau$ is a subset $S_\tau$ of the set of values $E_\tau$ containing all points $z$ whose $\varepsilon$-neighbourhood $O_\varepsilon(z)$ has a nonzero probability, i.e.

$$S_\tau = \{z \in E_\tau : P\{\tau \in O_\varepsilon(x)\} > 0 \, \forall \varepsilon > 0\}.$$

**Theorem 6.** *A spectrum $S_\tau$ of the distribution of the random variable $\tau$ is the set*

$$Q = \{z \in E_\tau : p_{\alpha_n(z) n} > 0 \, \forall n \in N\}. \tag{5}$$

*Proof.* 1. Let $\Delta^g_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = z \in S_\tau$. Then according to the definition of the spectrum for any $n \in N$ we have

$$0 < P\{\tau \in \Delta^g_{\alpha_1\alpha_2\ldots\alpha_n}\} = \prod_{j=1}^{n} p_{\alpha_j j}.$$

Indeed, taking the $\varepsilon$-neighborhood $O_\varepsilon(z)$ of $z$, it is easy to specify the cylinder $\Delta^g_{\alpha_1\ldots\alpha_k}$, which belongs entirely to the neighborhood $O_\varepsilon(z)$. If we assume that $P\{\tau \in \Delta^g_{\alpha_1\ldots\alpha_k}\} = 0$, then for $\varepsilon_1$-neighborhood $O_{\varepsilon_1}(z)$, where $\varepsilon_1 < \varepsilon$, and $O_{\varepsilon_1}(z) \cap E_\tau \subset \Delta^g_{\alpha_1\ldots\alpha_k}$ we have $P\{\tau \in O_{\varepsilon_1}(z)\} = 0$, which contradicts the condition $\tau \in S_\tau$. So, $p_{\alpha_n n} > 0 \; \forall n \in Z$ and $S_\tau \subset Q$.

2. Let $z \in Q$. Then $z \in \Delta^g_{\alpha_1 \ldots \alpha_n}$ and $P\{\tau \in \Delta^g_{\alpha_1 \ldots \alpha_n}\} > 0$ ($\forall n \in Z$). Consider an arbitrary number $\varepsilon > 0$. It is easy to point out the cylinder $\Delta^g_{\alpha_1 \ldots \alpha_n}$, which belongs completely to $O_\varepsilon(z)$. Then $P\{\tau \in O_\varepsilon(z)\} \geq P\{\tau \in \Delta^g_{\alpha_1 \ldots \alpha_n}\} > 0$. Thus, $z \in S_\tau$ is by definition and $Q \subset of S_\tau$. Therefore, $S_\tau = Q$. □

**Corollary 5.** *If there are no zeros among the elements of the matrix* $||p_{kn}||$, *then the distribution spectrum of the random variable* $\tau$ *is the fractal curve* $G$ *with dimension* $\alpha_0 = \log_3 7$.

**5. The Lebesgue structure of distribution $\tau$.** Next, we focus on the case of the continuity of the distribution $\tau$. In this situation, we can ignore countable sets, in particular, the set of points having two $g$-representations.

Note that if $p_{kn} = \frac{1}{7}$ for any $k \in A_7$ and $n \in N$, then the random variable $\tau$ have a uniform distribution on the curve $G$, because $P\{\tau \in \Delta^g_j\} = \frac{1}{7}$, $P\{\tau \in \Delta^g_{j_1 j_2 \ldots j_m}\} = \frac{1}{7^m}$, where $G \overset{7^{-m}}{\sim} \Delta^g_{j_1 j_2 \ldots j_m}$. In this case, one has $\tau = g(\xi)$, where $\xi$ is the random variable with a uniform distribution on the segment $[0;1]$, i.e. a random variable with independent identically distributed and uniform distributed digits of the seven-digits representation. In this situation, the probability measure $\mu_\tau$ on $G$ is equivalent to the one-dimensional Lebesgue measure $\lambda$ on $[0;1]$, since $\mu_\tau(\Phi) = \lambda(g^{-1}(\Phi))$ for $\forall\ \Phi \subset G$.

Next, we consider a special case where $p_{kn} = p_k\ \forall n \in N$.

**Remark 3.** Since the distribution of the random variable $\tau$ is equivalent to the distribution of the random variable $\xi = \Delta^7_{\xi_1 \xi_2 \ldots \xi_n \ldots}$ with independent identically distributed digits of the seven-digits representation: $P\{\xi_n = j\} = p_j$, $j = \overline{0,6}$, then the support of the distribution $\tau$ is the set $M[g; p_0, p_1, ..., p_6] = \{x : \nu_i(x) = p_i, i = \overline{0,6}\}$.

Let $m$ be a geometric probability measure on $G$ (equivalent to the Hausdorff measure). This class includes the measure that corresponds to the distribution of a random variable $\tau$ with probabilities $p_{kn} = \frac{1}{7}$.

**Theorem 7.** *The distributions of the random variables* $\tau$ *and* $\tau'$, *defined by the stochastic vectors* $\bar{p} = (p_0, p_1, ..., p_6)$ *and* $\bar{p}' = (p'_0, p'_1, ...., p'_6)$, *respectively, are mutually orthogonal if* $\bar{p} \neq \bar{p}'$ *and equivalent if* $\bar{p} = \bar{p}'$.

*Proof.* If $\bar{p} = \bar{p}'$, then the equivalence of the distributions of random variables $\tau$ and $\tau'$ is obvious.

Let $\bar{p} \neq \bar{p}'$. Taking into account Remark 3, the set $M[g; p_0, ..., p_6]$ is the set of the complete probability measure $\mu_\tau$ corresponding to the distribution $\tau$. In this case, $M[g; p'_0 ... p'_6]$ is the set of zero measure $\mu_\tau$, since $\bar{p} \neq \bar{p}'$.

Similarly, $\mu_{\tau'}(M[g; p'_0 ... p'_6]) = 1$ and $\mu_{\tau'}(M[g; p_0 ... p_6]) = 0$. Hence, $\mu_\tau \perp \mu_{\tau'}$. □

**Remark 4.** The random variable $\tau$, having a pure Lebesgue distribution, belongs to the Jessen-Wintner family of random variables [14, 15].

**6. Fractal properties of distribution $\tau$.** To summarize, the distribution of a random variable $\tau$ has a number of structurally and metrically fractal properties, since

1) the set of values of a random variable is a self-similar plane curve $G$ with a maximum branching index of 6 and fractal dimension $\log_3 7$; the peripheral contour of the curve $G$ is a fractal curve a Koch snowflake;

2) in the case of identical distribution of digits of $g$-representation of a random variable $\tau$, its spectrum is a self-similar fractal, which is a subset of the set of its values $E_\tau$;

3) if the continuous distribution of the random variable $\tau$ with independent equally distributed digits of the $g$-image is not uniform, then the support of the distribution is a fractal set of the Besicovitch-Eggleston type with fractional dimension of Hausdorff-Billingsley, which is calculated by Equation (1).

## REFERENCES


1. A.S. Besicovitch, *Sets of fractional dimension. 2: On the sum of digits of real numbers representet in diadic system*, Mathematische Annalen, **110** (1934), №3, 321–329.
2. P. Billingsley, Ergodic Theory and Information. John Wiley & Sons, Inc., New York, 1965. XIII, 193 p.
3. E. Borel, *Les probabilies denombrables et leurs applications arithmetiques*, Rend. Circ. Mat. Palermo, (1909), 247–271.
4. H.G. Eggleston, *The fractional dimension of a setdefined by desimal properties*, Quart. J. Math., **20** (1949), 31–36.
5. Y. Chen, *Fractal texture and structure of central place systems*, Fractals, **28** (2020), №01, 2050008.
6. P. Levy, *Sur les series dont les termes sont des variables independantes*, Studia Math., **3** (1931), 119–155.
7. B. Mandelbrot, The fractal geometry of nature, W. H. Freeman and Co., New York, 1982, 1922–1929.
8. M. Pratsiovytyi, N. Vasylenko, *Fractal properties of functions defined in terms of $Q$-representation*, Int. Journal of Math. Analysis, **7** (2013), №64, 3155–3167. doi: 10.12988/ijma.2013.311278
9. M.V. Pratsiovytyi, Two-symbol systems of encoding of real numbers and their aplications, Naukova Dumka, Kyiv, 2022. (in Ukrainian)
10. M.V. Pratsiovytyi, V.M. Kovalenko, *Koch snowflake as a parametrically defined plane curve*, Nauk. Chasop. Nats. Pedagog. Univ. Mykhaila Dragomanova. Ser 1. Fiz.-Mat. Nauky, **16** (2014), №2, 61–80. (in Ukrainian)
11. M.V. Pratsiovytyi, V.M. Kovalenko, *Probability measures on fractal curves (probability distributions on Vicsek fractal)*, Random Operators and Stochastic Equations, **23** (2015), №3, 161–168.
12. M.V. Pratsiovytyi, Ya.V. Goncharenko, I.M. Lysenko, S.P. Ratushnyak, *Fractal functions of exponential type that is generated by the $Q_2^*$-representation of argument*, Mat. Stud., **56** (2021), №2, 133–143.
13. M.V. Pratsiovytyi, Ya.V. Goncharenko, N.V. Dyvliash, S.P. Ratushnyak, *Inversor of digits of $Q_2^*$-representation of numbers*, Mat. Stud., **55** (2021), №1, 37–43.
14. M.V. Pratsiovytyi, Fractal approach in investigation of singular probability distribution, Kyiv, Nats. Pedagog. Mykhailo Dragomanov Univ., 1998. (in Ukrainian)
15. B. Jessen, A. Wintner, *Distribution functions and Rieman Zeta function*, Trans. Amer. Math. Soc., **38** (1935), №1, 48–88.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*Received 22.01.2024*
*Revised 22.08.2024*