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ACTION OF MULTIPLICATIVE (GENERALIZED)-DERIVATIONS AND RELATED MAPS ON SQUARE CLOSED LIE IDEALS IN PRIME RINGS

B. Dhara. *Action of multiplicative (generalized)-derivations and related maps on square closed Lie ideals in prime rings*, Mat. Stud. **63** (2025), 3–13.

Let $\mathcal{R}$ be a prime ring and L a nonzero square closed Lie ideal of $\mathcal{R}$. Suppose $F, G, H\colon \mathcal{R} \to \mathcal{R}$ are three multiplicative (generalized)-derivations associated with the maps $\delta, g, h\colon \mathcal{R} \to \mathcal{R}$ respectively which are not necessarily additive or derivations. Assume that $E, T\colon \mathcal{R} \to \mathcal{R}$ be any two maps (not necessarily additive).

Let $d\colon \mathcal{R} \to \mathcal{R}$ be a nonzero derivation of $\mathcal{R}$. In the present article, following identities are studied
(1) $d(x)F(y)+G(y)d(x)\pm(E(x)y+yT(u))=0$, (2) $H(xy)+G(y)F(x)\pm(E(y)x+xT(y))=0$,
(3) $T(xy)+G(x)y\pm(yx+xy)=0$, (4) $F(x)F(y)+T(x)y\pm yx=0$,
(5) $d(x)d(y)+T(x)y+F(yx)=0$, for all $x, y \in L$.

1. Introduction. Throughout this paper $\mathcal{R}$ will be a prime ring with its center $Z(\mathcal{R})$. As usual we denote by $[\xi, \eta]$ the commutator operator for $\xi, \eta \in \mathcal{R}$ and by $\xi \circ \eta$ the anti-commutator operator for $\xi, \eta \in \mathcal{R}$, which are defined as $[\xi, \eta] = \xi\eta - \eta\xi$ and $\xi \circ \eta = \xi\eta + \eta\xi$. An ideal L is said to be a Lie ideal of $\mathcal{R}$ if L is an additive subgroup of $\mathcal{R}$ and $[L, \mathcal{R}] \subseteq L$. The Lie ideal L is said to be a square closed, if $u^2 \in L$ for all $u \in L$.

A linear map $F\colon \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation of $\mathcal{R}$, if there exists a derivation $d\colon \mathcal{R} \to \mathcal{R}$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. Generalized derivation generalizes the concept of derivation.

In the literature, several authors investigated many identities in prime and semiprime rings admitting derivations and generalized derivations and obtained commutativity of rings. For example, we refer to [4, 12–14, 19, 24], where further references can be found.

Suppose that F, G denote generalized derivations, d denotes nonzero derivation, I denotes nonzero two-sided ideal, U denotes nonzero right ideal and L denotes square closed Lie ideal of $\mathcal{R}$. In the prime ring $\mathcal{R}$, Ashraf et al. [4] have studied the following situations:
(i) $F(\xi\eta) - \xi\eta \in Z(\mathcal{R})$, (ii) $F(\xi\eta) + \xi\eta \in Z(\mathcal{R})$, (iii) $F(\xi\eta) - \eta\xi \in Z(\mathcal{R})$, (iv) $F(\xi\eta) + \eta\xi \in Z(\mathcal{R})$, (v) $F(\xi)F(\eta) - \xi\eta \in Z(\mathcal{R})$, (vi) $F(\xi)F(\eta) + \xi\eta \in Z(\mathcal{R})$, for all $\xi, \eta \in I$, and obtained the commutativity of prime ring $\mathcal{R}$.

Dhara et al. [14] studied the situations

(i) $F(\xi)F(\eta) - \eta\xi \in Z(\mathcal{R})$ and (ii) $F(\xi)F(\eta) + \eta\xi \in Z(\mathcal{R})$ for all ξ, η in L.

Recently, Tiwari et al. [24] considered the identities taking three items together, that is,

$$(i)\ G(\xi\eta) \pm F(\xi)F(\eta) \pm \xi\eta \in Z(\mathcal{R}), \quad (ii)\ G(\xi\eta) \pm F(\xi)F(\eta) \pm \eta\xi \in Z(\mathcal{R}),$$
$$(iii)\ G(\xi\eta) \pm F(\eta)F(\xi) \pm \xi\eta \in Z(\mathcal{R}), \quad (iv)\ G(\xi\eta) \pm F(\eta)F(\xi) \pm \eta\xi \in Z(\mathcal{R}),$$
$$(v)\ G(\xi\eta) \pm F(\eta)F(\xi) \pm [\xi, \eta] \in Z(\mathcal{R}), \quad (vi) G(\xi\eta) \pm F(\xi)F(\eta) \pm [\alpha(\xi), \eta] \in Z(\mathcal{R}),$$

2020 *Mathematics Subject Classification:* 16N60, 16W25, 16R50.
Keywords: prime ring; derivation; multiplicative (generalized)-derivation; Lie ideal.
doi:10.30970/ms.63.1.3-13

for all $\xi, \eta \in I$, where $\alpha\colon \mathcal{R} \to \mathcal{R}$ is any mapping and obtained commutativity of prime rings. It is also interesting to consider identities with commutator and anti-commutator operator. In [6], Bell and Daif proved for semiprime ring $\mathcal{R}$ that if $[d(\xi), d(\eta)] = [\xi, \eta]$ for all $\xi, \eta \in U$, then $U \subseteq Z(\mathcal{R})$. In [5], Ashraf et al. examined the identities: (i) $d(\xi) \circ F(\eta) = 0$, (ii) $[d(\xi), F(\eta)] = 0$, (iii) $d(\xi) \circ F(\eta) = \xi \circ \eta$, (iv) $d(\xi) \circ F(\eta) + \xi \circ \eta = 0$, (v) $[d(\xi), F(\eta)] = [\xi, \eta]$, (vi) $[d(\xi), F(\eta)] + [\xi, \eta] = 0$, (vii) $d(\xi)F(\eta) \pm \xi\eta \in Z(\mathcal{R})$ for all $\xi, \eta \in I$, where d is associated to F. Huang [16] examined the above identities in L and obtained that $L \subseteq Z(\mathcal{R})$. Then Dhara et al. [15] investigated the same identities in semiprime ring.

In a recent paper, Dhara et al. [19] have studied more general situations involving three items, that is: (i) $F(\xi) \circ \eta \pm d(\xi) \circ F(\eta) \pm \xi \circ \eta = 0$; (ii) $[F(\xi), \eta] \pm [d(\xi), F(\eta)] \pm [\xi, \eta] = 0$; (iii) $F([\xi, \eta]) \pm [d(\xi), F(\eta)] \pm [\xi, \eta] = 0$, (iv) $F(\xi \circ \eta) \pm [d(\xi), F(\eta)] \pm \xi \circ \eta = 0$, (v) $F(\xi)G(\eta) \pm d(\xi)F(\eta) \pm \xi\eta \in Z(\mathcal{R})$, (vi) $G(\xi\eta) \pm d(\xi)F(\eta) \pm F(\eta\xi) = 0$, (vii) $F(\xi v) \pm F(\eta)F(\xi) \pm \xi\eta \in Z(\mathcal{R})$ for all $\xi, \eta \in L$, where d is associated to F.

There is also ongoing interest to investigate the above situation replacing generalized derivations F, G with multiplicative (generalized)-derivations. The concept of multiplicative (generalized)-derivation was introduced by Dhara and Ali in [21]. The multiplicative (generalized)-derivation is a mapping $F\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $F(\xi\eta) = F(\xi)\eta + \xi g(\eta)$ holds for all $\xi, \eta \in \mathcal{R}$, where g is any mapping. Thus multiplicative (generalized)-derivation covers all the concepts of derivation and generalized derivations maps. We refer the articles [1], [2], [3], [10], [11], [20], [21], [23]; where maps considered are multiplicative (generalized)-derivations in prime ring $\mathcal{R}$. In [21], Dhara and Ali studied the following identities in semiprime rings: (1) $F(\xi\eta) \pm \xi\eta \in Z(\mathcal{R})$, (2) $F(\xi\eta) \pm \eta\xi \in Z(\mathcal{R})$, (4) $F(\xi)F(\eta) \pm \xi\eta \in Z(\mathcal{R})$, (6) $F(\xi)F(\eta) \pm \eta\xi \in Z(\mathcal{R})$ for all $\xi, \eta \in \lambda$; where λ is a nonzero left ideal in a semiprime ring $\mathcal{R}$ and F is a multiplicative (generalized)-derivation of $\mathcal{R}$.

Recently, in [10], Dhara et al. studied the following identities in semiprime rings: (1) $[d(\xi), F(\eta)] = \pm[\xi, \eta]$, (2) $[d(\xi), F(\eta)] = \pm\xi \circ y$, (3) $[d(\xi), F(\eta)] = 0$, (4) $F([\xi, \eta]) \pm [\delta(\xi), \delta(\eta)] \pm [\xi, \eta] = 0$, (5) $d'([\xi, \eta]) \pm [\delta(\xi), \delta(\eta)] \pm [\xi, \eta] = 0$, (6) $d'([\xi, \eta]) \pm [\delta(\xi), \delta(\eta)] = 0$, (7) $F(\xi \circ \eta) \pm \delta(\xi) \circ \delta(\eta) \pm \xi \circ \eta = 0$, (8) $d'(\xi \circ \eta) \pm \delta(\xi) \circ \delta(\eta) \pm \xi \circ \eta = 0$, (9) $d'(\xi \circ \eta) \pm \delta(\xi) \circ \delta(\eta) = 0$, for all $\xi, \eta \in \lambda$, where λ is a nonzero left ideal in $\mathcal{R}$ and F is a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map d, and δ, d' are multiplicative derivations of $\mathcal{R}$.

In a recent paper, Dhara et al. [20] have studied the following identities in semiprime rings:

(1) $F([\xi, \eta]) + G(\eta\xi) + d(\xi)F(\eta) + \xi\eta \in Z(\mathcal{R})$, (2) $F(\xi \circ \eta) + G(\eta\xi) + d(\xi)F(\eta) + \xi\eta \in Z(\mathcal{R})$,
(3) $F(\xi\eta) + G(\eta\xi) + d(\xi)F(\eta) \pm [\xi, \eta] \in Z(\mathcal{R})$, (4) $F([\xi, \eta]) + G(\xi\eta) + d(\xi)F(\eta) + \eta\xi \in Z(\mathcal{R})$,
(5) $F(\xi \circ \eta) + G(\xi\eta) + d(\xi)F(\eta) + \eta\xi \in Z(\mathcal{R})$, (6) $F([\xi, \eta]) + G(\eta\xi) + d(\eta)F(\xi) - \xi\eta \in Z(\mathcal{R})$,
(7) $F(\xi)F(\eta) - G(\eta\xi) - \xi\eta + \eta\xi \in Z(\mathcal{R})$,

for all $\xi, \eta \in \lambda$, where λ is a nonzero left ideal of $\mathcal{R}$ and F, G are multiplicative (generalized)-derivations of $\mathcal{R}$ associated to the maps d and g respectively.

In all the above results, authors considered some particular type maps. It is natural to ask if we replace any maps in place of some particular type maps, then what will happen. This question motivate us to consider identities with blended maps, that is, multiplicative (generalized)-derivations, derivations and any maps acting on a square closed Lie ideal in prime rings. Let $F, G, H\colon \mathcal{R} \to \mathcal{R}$ be three multiplicative (generalized)-derivations associated with the maps $\delta, g, h\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) respectively and $E, T\colon R \to R$ be any two maps. Let $d\colon \mathcal{R} \to \mathcal{R}$ be a nonzero derivation of $\mathcal{R}$. In the present article, following identities are studied

(1) $d(\xi)F(\eta)+G(\eta)d(\xi)\pm(E(\xi)\eta+\eta T(\xi))=0$, (2) $H(\xi\eta)+G(\eta)F(\xi)\pm(E(\eta)\xi+\xi T(\eta))=0$,
(3) $T(\xi\eta)+G(\xi)\eta\pm(\eta\xi+\xi\eta)=0$, (4) $F(\xi)F(\eta)+T(\xi)\eta\pm\eta\xi=0$;
(5) $d(\xi)d(\eta)+T(\xi)\eta+F(\eta\xi)=0$,

for all $\xi,\eta\in L$. We also present an example at the end of the paper to show that the primeness hypothesis is essential.

2. Preliminaries. We shall use frequently the following basic commutator and anti-commutator identities:
1. $[\xi\eta,\zeta]=\xi[\eta,\zeta]+[\xi,\zeta]\eta$, 2. $[\xi,\eta\zeta]=\eta[\xi,\zeta]+[\xi,\eta]\zeta$,
3. $(\xi\circ\eta\zeta)=(\xi\circ\eta)\zeta-\eta[\xi,\zeta]=\eta(\xi\circ\zeta)+[\xi,\eta]\zeta$,
4. $(\xi\eta\circ\zeta)=\xi(\eta\circ\zeta)-[\xi,\zeta]\eta=(\xi\circ\zeta)\eta+\xi[\eta,\zeta]$

for all $\xi,\eta,\zeta\in\mathcal{R}$.

To prove our Theorems, we need the following Facts. Unless specifically stated, we assume in all that follows that $\mathcal{R}$ be a prime ring of char$(\mathcal{R})\neq 2$, $Z(\mathcal{R})$ be its center, L a nonzero Lie ideal of $\mathcal{R}$ and d a nonzero derivation of $\mathcal{R}$.

Fact 1. *If L be a square closed Lie ideal, that is, $u^2\in L$ for all $u\in L$, then for any $u,v\in L$, $uv+vu=(u+v)^2-u^2-v^2\in L$. Moreover, we know by definition of Lie ideal that $[u,v]=uv-vu\in L$. Adding these two relation, we have $2uv\in L$ for all $u,v\in L$.*

Fact 2 ([8], Lemma 4)**.** *If $L\not\subseteq Z(\mathcal{R})$ such that $aLb=0$ for some $a,b\in\mathcal{R}$, then either $a=0$ or $b=0$.*

Fact 3 ([8], Lemma 5)**.** *If $d(L)=(0)$, then $L\subseteq Z(\mathcal{R})$.*

Fact 4 ([18], Theorem 5)**.** *If $[u,d(u)]\in Z(\mathcal{R})$ for all $u\in L$, then $L\subseteq Z(\mathcal{R})$.*

Fact 5 ([9], Theorem 1)**.** *If $u[d(u),u]=0$ for all $u\in L$, then $L\subseteq Z(\mathcal{R})$.*

Fact 6 ([22], Lemma 1)**.** *Let $\mathcal{R}$ be a 2-torsion free semiprime ring. If $[L,L]=0$, then $L\subseteq Z(\mathcal{R})$.*

Fact 7 ([14], Lemma 2.5)**.** *Assume the set $V=\{u\in L\mid d(u)\in L\}$. Then V is also a nonzero Lie ideal of $\mathcal{R}$. Moreover, if L is noncentral, then V is also noncentral.*

Fact 8. *If L is square closed and for some fixed $a,b\in L$, $aub+bua=0$ holds for all $u\in L$, then either $a=0$ or $b=0$.*

Proof. Given that $aub=-bua$ for all $u\in L$ and for some fixed $a,b\in L$. Therefore, for any $u,w\in L$, we have

$$4aubwaub=-4buawaub=\{-b(4uaw)a\}ub. \tag{1}$$

Since $2ua\in L$, $4uaw\in L$. Thus again by using the equality $aub=-bua$, the relation (1) can be written as

$$4aubwaub=\{a(4uaw)b\}ub\ =4au(awb)ub. \tag{2}$$

Again by the equalities $aub=-bua$, (2) we get $4aubwaub=-4au(bwa)ub$ which gives $8aubwaub=0$ for all $u,w\in L$. Since char$(\mathcal{R})\neq 2$, by Fact 2, $aub=0$ for all $u\in L$. Again by Fact 2, either $a=0$ or $b=0$. □

Fact 9. *Let $\mathcal{R}$ be a semiprime ring. If $F\colon\mathcal{R}\to\mathcal{R}$ is a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map d of $\mathcal{R}$, then d must be multiplicative derivation.*

Proof. For any $\xi, \eta, \zeta \in \mathcal{R}$,

$$F(\xi\eta\zeta) = F((\xi\eta)\zeta) = F(\xi\eta)\zeta + \xi\eta d(\zeta) = F(\xi)\eta\zeta + \xi d(\eta)\zeta + \xi\eta d(\zeta)$$

and $F(\xi\eta\zeta) = F(\xi(\eta\zeta)) = F(\xi)\eta\zeta + \xi d(\eta\zeta)$, hence, $\xi\{d(\eta\zeta) - d(\eta)\zeta - \eta d(\zeta)\} = 0$.

Since $\mathcal{R}$ is semiprime ring, it yields $d(\eta\zeta) - d(\eta)\zeta - \eta d(\zeta) = 0$ implying $d(\eta\zeta) = d(\eta)\zeta + \eta d(\zeta)$ for all $\eta, \zeta \in \mathcal{R}$. Therefore, d is a multiplicative derivation. □

Fact 10. *Let $\mathcal{R}$ be a ring and $d\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative derivation of $\mathcal{R}$. Then*

$$d(Z(\mathcal{R})) \subseteq Z(\mathcal{R}).$$

Proof. For any $\xi \in \mathcal{R}$ and $\zeta \in Z(\mathcal{R})$,

$$0 = d(\zeta\xi) - d(\xi\zeta) = d(\zeta)\xi + \zeta d(\xi) - d(\xi)\zeta - \xi d(\zeta) = [d(\zeta), \xi].$$

This implies $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$. □

3. Results on square closed Lie ideals.

Theorem 1. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $F, G\colon R \to R$ be two multiplicative (generalized)-derivations of $\mathcal{R}$ associated to the maps $\delta, g\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) respectively and $H, T\colon R \to R$ be any two maps. If $d\colon \mathcal{R} \to \mathcal{R}$ be a nonzero derivation of $\mathcal{R}$ such that*

$$d(x)F(y) + G(y)d(x) \pm (H(x)y + yT(x)) = 0$$

for all $x, y \in L$, then either $L \subseteq Z(\mathcal{R})$ or $\delta(2L) = (0)$ and $F(2uv) = F(u)2v$, $G(2uv) = 2uG(v)$ for all $u, v \in L$.

Proof. We assume that $L \not\subseteq Z(\mathcal{R})$. By assumption

$$d(x)F(y) + G(y)d(x) \pm (H(x)y + yT(x)) = 0, \quad \forall\ x, y \in L. \tag{3}$$

Substituting $2yt$ for y in (3), where $t \in L$, and then using $F(2yt) = F(y2t) = F(y)2t + y\delta(2t)$, we get

$$\left\{d(x)\{2F(y)t + y\delta(2t)\} + \{2G(y)t + yg(2t)\}d(x) \pm (2H(x)yt + 2ytT(x))\right\} = 0, \quad \forall\ x, y, t \in L. \tag{4}$$

Post multiplying (3) by $2t$ and then subtracting from (4), we get

$$d(x)y\delta(2t) + 2G(y)[t, d(x)] + yg(2t)d(x) \pm 2y[t, T(x)] = 0, \quad \forall\ x, y, t \in L. \tag{5}$$

Writing $2uy$ in place of y in (5), we have

$$2d(x)uy\delta(2t) + 2G(2uy)[t, d(x)] + 2uyg(2t)d(x) \pm 4uy[t, T(x)] = 0, \quad \forall\ x, y, t, u \in L. \tag{6}$$

Pre-multiplying (5) by $2u$ and then subtracting from (6), one can see that

$$2[d(x), u]y\delta(2t) + 2(G(2uy) - 2uG(y))[t, d(x)] = 0, \quad \forall\ x, y, t, u \in\ L. \tag{7}$$

Replacing t by $2tw$ in (7), we observe that

$$2[d(x),u]y(2\delta(2t)w+2t\delta(2w))+4(G(2uy)-2uG(y))[t,d(x)]w+ \\ +4(G(2uy)-2uG(y))t[w,d(x)]=0,\quad \forall\ x,y,t,u,w\in L. \tag{8}$$

Post multiplying (7) by $2w$ and then subtracting from (8), we get

$$4[d(x),u]yt\delta(2w)+4(G(2uy)-2uG(y))t[w,d(x)]=0,\quad \forall\ x,y,t,u,w\in L. \tag{9}$$

For $p\in L$, $[p,d(v)]\in L$ and hence as above $2[p,d(v)]t\in L$. Thus we put $t=2[p,d(v)]t$ in (9) and get

$$8(G(2uy)-2uG(y))[p,d(v)]t[w,d(x)]+8[d(x),u]y[p,d(v)]t\delta(2w)=0,\quad \forall\ x,y,p,t,u,v,w\in L.$$

By using (7), above relation yields

$$-8[d(v),u]y\delta(2p)t[d(x),w]-8[d(x),u]y[d(v),p]t\delta(2w)=0,\quad \forall\ x,y,p,t,u,v,w\in L. \tag{10}$$

Let $V=\{u\in L\mid d(u)\in L\}$. By Fact 7, V is also noncentral Lie ideal of $\mathcal{R}$ such that $V\subseteq L$. Then for $v\in V$, $d(v)\in L$. Thus for $u,d(v)\in L$, we have $2d(v)u\in L$. Hence replacing u with $2d(v)u$ in (10) and then using (10), we get

$$8[d(x),d(v)]uy[d(v),p]t\delta(2w)=0,\quad \forall\ x,y,u,w,p,t\in L, v\in V. \tag{11}$$

Above expression yields by Fact 2 that either $\delta(2L)=(0)$ or $8[d(x),d(v)]uy[d(v),p]=0$ for all $x,y,u,p\in L, v\in V$. Again by using $\mathrm{char}(\mathcal{R})\neq 2$ and Fact 2, last expression yields that for each $v\in V$, either $[d(x),d(v)]=0$ for all $x\in L$ or $[d(v),p]=0$ for all $p\in L$. Since the sets $T_1=\{v\in V|[d(x),d(v)]=0\forall x\in L\}$ and $T_2=\{v\in V|[d(v),p]=0\forall p\in L\}$ forms two additive subgroup of V such that $T_1\bigcup T_2=V$. Since a group can not be union of its two proper subgroups, either $T_1=V$ or $T_2=V$, that is, either $[d(x),d(v)]=0$ for all $x\in L$ and $v\in V$ or $[d(v),p]=0$ for all $p\in L$ and $v\in V$. Now $[d(v),p]=0$ for all $p\in L$ and $v\in V$ implies $[d(v),v]=0$ for all $v\in V$. Then by Fact 4, V is central. This is a contradiction. Thus we are to consider the following two cases:

Case-1: $\delta(2L)=(0)$.
By (9), we have

$$4(G(2uy)-2uG(y))t[w,d(x)]=0,\quad \forall\ x,y,t,u,w\in L. \tag{12}$$

By Fact 2, above expression yields either $4(G(2uy)-2uG(y))=0$ for all $u,y\in L$ or $[w,d(x)]=0$ for all $w,x\in L$. By Fact 4, last expression yields $d=0$. It is a contradiction. Thus by using $\mathrm{char}(\mathcal{R})\neq 2$, we have $G(2uy)-2uG(y)=0$ for all $u,y\in L$.

Case-2: $[d(x),d(v)]=0$ for all $x\in L$ and $v\in V$.
Replacing x with $2wv$, where $w\in L$, $v\in V$, we have

$$2[d(w)v+wd(v),d(v)]=0 \tag{13}$$

for all $w\in L$ and $v\in V$ which gives

$$2d(w)[v,d(v)]+2[w,d(v)]d(v)=0 \tag{14}$$

for all $w \in L$ and $v \in V$. Replacing w with $2pw$ in (14), where $p \in L$ and then using it, we have

$$4(d(p)w[v, d(v)] + [p, d(v)]wd(v)) = 0 \tag{15}$$

for all $p, w \in L$ and $v \in V$. In particular, for $p = v \in V$, we have by using $\mathrm{char}(\mathcal{R}) \neq 2$ that

$$d(v)w[v, d(v)] + [v, d(v)]wd(v) = 0 \tag{16}$$

for all $w \in L$ and $v \in V$. By Fact 8, this implies $d(v)w[v, d(v)] = 0$ for all $v \in V$ and $w \in U$. Again by Fact 2, $d(v) = 0$ or $[v, d(v)] = 0$. In any case we have $[v, d(v)] = 0$ for all $v \in V$. By Fact 4, $V \subseteq Z(\mathcal{R})$. It is a contradiction. □

Corollary 1. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $F\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map $\delta\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation). If $\delta(2L) \neq (0)$ and there exists a nonzero derivation $d\colon \mathcal{R} \to \mathcal{R}$ such that any one of the following holds:*
(1) $[d(x), F(y)] \pm [x, y] = 0$ for all $x, y \in L$;
(2) $(d(x) \circ F(y)) \pm (x \circ y) = 0$ for all $x, y \in L$;
then $L \subseteq Z(\mathcal{R})$.

Theorem 2. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $F, G, H\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivations of $\mathcal{R}$ associated to the maps $d, d, \delta\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) respectively and $E, T\colon \mathcal{R} \to \mathcal{R}$ be any two maps. If*

$$H(xy) + G(y)F(x) \pm (T(y)x + xE(y)) = 0$$

for all $x, y \in L$, then $x[d(x), x] = 0$ for all $x \in 2L$.

Proof. If $L \subseteq Z(\mathcal{R})$, the conclusion holds trivially. Therefore, we assume that $L \not\subseteq Z(\mathcal{R})$. By assumption

$$H(xy) + G(y)F(x) \pm (T(y)x + xE(y)) = 0 \tag{17}$$

for all $x, y \in L$. Replacing x with $2xz$, where $z \in L$ in the above relation, we have

$$H(2xzy) + G(y)(2F(x)z + xd(2z)) \pm (2T(y)xz + 2xzE(y)) = 0 \tag{18}$$

for all $x, y, z \in L$. Right multiplying (17) by $2z$ and then subtracting from (18), we have

$$H(2xzy) - 2H(xy)z + G(y)xd(2z) \pm 2x[z, E(y)] = 0 \tag{19}$$

for all $x, y, z \in L$. Putting $y = 2yz$ and $x = 2zx$ respectively in the above relation, we have the following two relations

$$H(4xzyz) - 2H(2xyz)z + (2G(y)z + yd(2z))xd(2z) \pm 2x[z, E(2yz)] = 0 \tag{20}$$

for all $x, y, z \in L$ and

$$H(4zxzy) - 2H(2zxy)z + 2G(y)zxd(2z) \pm 4zx[z, E(y)] = 0 \tag{21}$$

for all $x, y, z \in L$. Subtracting (21) from (20), we get

$$H(4xzyz) - 2H(2xyz)z - H(4zxzy) + 2H(2zxy)z + yd(2z)xd(2z) \\ \pm(2x[z, E(2yz)] - 4zx[z, E(y)]) = 0 \tag{22}$$

for all $x, y, z \in L$. In particular, for $y = z$, we have

$$H(4xz^3) - 2H(2xz^2)z - H(4zxz^2) + 2H(2zxz)z + zd(2z)xd(2z) \\ \pm(2x[z, E(2z^2)] - 4zx[z, E(z)]) = 0 \tag{23}$$

that is

$$2xz^2\delta(2z) - 2zxz\delta(2z) + zd(2z)xd(2z) \pm (2x[z, E(2z^2)] - 4zx[z, E(z)]) = 0 \tag{24}$$

for all $x, z \in L$. Putting $x = 2zx$, above relation yields

$$4zxz^2\delta(2z) - 4z^2xz\delta(2z) + 2zd(2z)zxd(2z) \pm 4z(x[z, E(2z^2)] - 2zx[z, E(z)]) = 0 \tag{25}$$

for all $x, z \in L$. Left multiplying (24) by $2z$ and then subtracting from (25), we get

$$2z[d(2z), z]xd(2z) = 0 \tag{26}$$

for all $x, z \in L$. This implies $2z[d(2z), 2z]x2z[d(2z), 2z] = 0$ for all $x, z \in L$. By Fact 2, it yields $2z[d(2z), 2z] = 0$ for all $z \in L$. □

Corollary 2. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $H\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map $\delta\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation). If there exists a nonzero derivation d of $\mathcal{R}$ such that any one of the following equalities holds:*
(1) $H(xy) + d(y)d(x) + yx = 0$, (2) $H(xy) + d(y)d(x) + xy = 0$,
(3) $H(xy) + d(y)d(x) + [x, y] = 0$, (4) $H(xy) + d(y)d(x) + (x \circ y) = 0$
for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.

Proof. By Theorem 2, we have $x[d(2x), x] = 0$ for all $x \in L$. Since d is additive and $\mathrm{char}(\mathcal{R}) \neq 2$, $x[d(x), x] = 0$ for all $x \in L$. Then by Fact 5, conclusion follows. □

Theorem 3. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $G\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map $d\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) and $T\colon R \to R$ be any map. If*

$$T(xy) + G(x)y \pm (yx + xy) = 0$$

for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$ and $d(2L) = (0)$, $G(2uv) = 2G(u)v$ for all $u, v \in L$.

Proof. By assumption

$$H(xy) + G(x)y \pm (yx + xy) = 0 \tag{27}$$

for all $x, y \in L$. Replacing x with $2xz$, where $z \in L$ in the above relation, we have

$$H(2xzy) + 2G(x)zy + xd(2z)y \pm 2(yxz + xzy) = 0 \tag{28}$$

for all $x, y, z \in L$. Replacing y with $2zy$ in (27), we have

$$H(2xzy) + 2G(x)zy \pm 2(zyx + xzy) = 0 \tag{29}$$

for all $x, y \in L$. Subtracting (29) from (28), we get

$$xd(2z)y \pm 2[yx, z] = 0 \tag{30}$$

for all $x, y, z \in L$. Replacing x with $2zx$ in (30) and then using (30), we obtain the relation $4[[y, z]x, z] = 0$ for all $x, y, z \in L$. Putting $x = 2xy$ in above relation we obtain $8[y, z]x[y, z] = 0$ for all $x, y, z \in L$.

Since $\mathrm{char}(\mathcal{R}) \neq 2$, we have $[y, z]x[y, z] = 0$ for all $x, y, z \in L$. If $L \nsubseteq Z(\mathcal{R})$, then by Fact 2, it yields $[L, L] = (0)$. It implies $L \subseteq Z(\mathcal{R})$ by Fact 6. This is a contradiction. Thus $L \subseteq Z(\mathcal{R})$. Then by (30), one has $d(2L) = (0)$, because the center of a prime ring contains no divisor of zero. Hence $G(2uv) = 2G(u)v$ for all $u, v \in L$. □

By Fact 3, the following corollary is straightforward.

Corollary 3. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a nonzero derivation $d\colon \mathcal{R} \to \mathcal{R}$ and $T\colon \mathcal{R} \to \mathcal{R}$ be any map. If*

$$T(xy) + d(x)y \pm (yx + xy) = 0$$

for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.

Theorem 4. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $F\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map $d\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) and $T\colon \mathcal{R} \to \mathcal{R}$ be any map. If*

$$F(x)F(y) + T(x)y \pm yx = 0$$

for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$ and $d(2L) = (0)$, $F(2uv) = 2F(u)v$ for all $u, v \in L$.

Proof. By assumption

$$F(x)F(y) + T(x)y \pm yx = 0 \tag{31}$$

for all $x, y \in L$. Replacing y with $2yz$, where $z \in L$ in the above relation, we have

$$F(x)(2F(y)z + yd(2z)) + 2T(x)yz \pm 2yzx = 0 \tag{32}$$

for all $x, y, z \in L$. Right multiplying (31) by $2z$ and then subtracting from (32), we get

$$F(x)yd(2z)) \pm 2y[z, x] = 0 \tag{33}$$

for all $x, y, z \in L$. Replace x with $2xt$ and y with $2ty$ respectively, $t \in L$, in above equation we have the following two equations:

$$(2F(x)t + xd(2t))yd(2z)) \pm 4y[z, xt] = 0 \tag{34}$$

for all $x, y, z, t \in L$ and

$$2F(x)tyd(2z)) \pm 4ty[z, x] = 0 \tag{35}$$

for all $x, y, z, t \in L$. Subtracting (35) from (34), we have the following

$$xd(2t)yd(2z)) \pm 4(y[z, xt] - ty[z, x]) = 0 \tag{36}$$

for all $x, y, z, t \in L$. Put $x = 2ux$, where $u \in L$, in (36) and then using (36), we get

$$8\{y[z, uxt] - ty[z, ux] - uy[z, xt] + uty[z, x]\} = 0 \tag{37}$$

for all $x, y, z, t \in L$. Put $x = 2xz$ in (37) and then using (37), we get

$$16\{y[z, uxzt] - uy[z, xzt] - y[z, uxt]z + uy[z, xt]z\} = 0 \tag{38}$$

for all $x, y, z, t \in L$. Put $y = 2py$, $p \in L$ in (38) and then using (38), we get

$$32\{-[u, p]y[z, xzt] + [u, p]y[z, xt]z\} = 0 \tag{39}$$

for all $x, y, z, p, t \in L$. Since $\mathrm{char}(\mathcal{R}) \neq 2$, above equation yields $[u, p]y[z, x[z, t]] = 0$ for all $x, y, z, p, t \in L$.

If $L \not\subseteq Z(\mathcal{R})$, then by Fact 2, either $[L, L] = (0)$ or $[z, x[z, t]] = 0$ for all $x, y, z, p, t \in L$. Now $[L, L] = (0)$ implies $L \subseteq Z(\mathcal{R})$ by Fact 6. It is a contradiction. In this last expression again replacing x with $2tx$ we get $[z, t]x[z, t] = 0$ for all $x, z, t \in L$. Again by Fact 2, it gives $[L, L] = (0)$. It implies $L \subseteq Z(\mathcal{R})$ by Fact 6. This is a contradiction.

Therefore, $L \subseteq Z(\mathcal{R})$. Then by (36), we have $d(2L) = (0)$, since center of a prime ring contains no divisor of zero and $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$ (see Fact 10). Hence $F(2uv) = 2F(u)v$ for all $u, v \in L$. □

By Fact 3, following corollary is straightforward.

Corollary 4. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a nonzero derivation $d\colon \mathcal{R} \to \mathcal{R}$ and $T\colon \mathcal{R} \to \mathcal{R}$ be any map. If*

$$d(x)d(y) + T(x)y \pm yx = 0$$

for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.

Theorem 5. *Let $\mathcal{R}$ be a prime ring of $\mathrm{char}(\mathcal{R}) \neq 2$ and L a nonzero square closed Lie ideal of $\mathcal{R}$. Let $F\colon \mathcal{R} \to \mathcal{R}$ be a multiplicative (generalized)-derivation of $\mathcal{R}$ associated to the map $\delta\colon \mathcal{R} \to \mathcal{R}$ (not necessarily additive nor derivation) and $T\colon \mathcal{R} \to \mathcal{R}$ be any map. If there exists a nonzero derivation d of $\mathcal{R}$ such that*

$$d(x)d(y) + T(x)y + F(yx) = 0$$

for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.

Proof. We assume that $L \not\subseteq Z(\mathcal{R})$. By assumption

$$d(x)d(y) + T(x)y + F(yx) = 0 \tag{40}$$

for all $x, y \in L$. Replacing y with $2yx$ in the above relation, we have

$$2d(x)(d(y)x + yd(x)) + 2T(x)yx + 2F(yx)x + yx\delta(2x) = 0 \tag{41}$$

for all $x, y \in L$. Right multiplying (40) by $2x$ and then subtracting from (41), we get

$$2d(x)yd(x) + yx\delta(2x) = 0 \tag{42}$$

for all $x, y \in L$. Replacing y with $2xy$ in above equation, we obtain

$$4d(x)xyd(x) + 2xyx\delta(2x) = 0 \tag{43}$$

for all $x, y \in L$. Pre-multiplying (42) by $2x$ and then subtracting from (43), we have the following $4[d(x), x]yd(x) = 0$ for all $x, y \in L$. Above relation implies $8[d(x), x]y[d(x), x] = 0$ for all $x, y \in L$. By Fact 2 and $\mathrm{char}(\mathcal{R}) \neq 2$, it yields $[d(x), x] = 0$ for all $x \in L$. By Fact 4, it gives $L \subseteq Z(\mathcal{R})$. But this is a contradiction. □

The following example shows that the primeness hypothesis in the theorems is not superfluous.

Example 1. Let $\mathcal{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in Z \right\}$. Since

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{R} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathcal{R}$ is not prime. Define the mappings $F, d \colon \mathcal{R} \to \mathcal{R}$ by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that d is a derivation, F is not additive map in R, but $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Hence, F is a multiplicative (generalized)-derivation associated with the map d. Assuming

$$L = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b \in Z \right\},$$

a nonzero square closed Lie ideal of R, we have $[d(x), F(y)] \pm [x, y] = 0$ and $d(x) \circ F(y) \pm x \circ y = 0$ for all $x, y \in L$. Since $d(2L) \neq (0)$ and L is noncentral, the primeness hypothesis is not superfluous in Corollary 3.2.

Acknowledgements. This research is supported by a grant from Science and Engineering Research Board (SERB), New Delhi, India. Grant No. is MTR/2022/000568.

REFERENCES

1. S. Ali, B. Dhara, N.A. Dar, A.N. Khan, *On Lie ideals with multiplicative (generalized)-derivations in prime and semiprime rings*, Beitr. Algebra Geom., **56** (2015), №1, 325–337.

2. S. Ali, B. Dhara, N.A. Dar, A.N. Khan, *On Lie ideals with multiplicative (generalized)-derivations in prime and semiprime rings*, Beitr. Algebra Geom., **56** (2015), №1, 325–337.
3. Asma Ali, B. Dhara, S. Khan, F. Ali, *Multiplicative (Generalized)-derivations and left ideals in semiprime rings*, Hacettep J. Math. Stat., **44** (2015), №6, 1293–1306.
4. M. Ashraf, A. Ali, S. Ali, *Some commutativity theorem for prime rings with generalized derivations*, Southeast Asian Bull. Math., **31** (2007), 415–421.
5. M. Ashraf, A. Ali, R. Rani, *On generalized derivations of prime rings*, Southeast Asian Bull. Math., **29** (2005), №4, 669–675.
6. H.E. Bell, M.N. Daif, *On commutativity and strong commutativity-preserving maps*, Canad. Math. Bull., **37** (1994), №4, 443–447.
7. H.E. Bell, W.S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull., **30** (1987), 92–101.
8. J. Bergen, I.N. Herstein, J.W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra, **71** (1981), 259–267.
9. V. De Filippis, F. Rania, *Commuting and centralizing generalized derivations on lie ideals in prime rings*, Math. Notes, **88** (2010), №5–6, 748–758.
10. B. Dhara, S. Kar, S. Kuila, *A note on multiplicative (generalized)-derivations and left ideals in semiprime rings*, Rend. Circ. Mat. Palermo, II Ser., **70** (2021), №2, 631–640.
11. B. Dhara, K.G. Pradhan, *A note on multiplicative (generalized)-derivations with annihilator conditions*, Georgian Math. J., **23** (2016), №2, 191–198.
12. B. Dhara, N. Rehman, A. Raza, *Lie ideals and action of generalized derivations in rings*, Miskolc Math. Notes, **16** (2015), №2, 769–779.
13. B. Dhara, S. Kar, K.G. Pradhan, *Generalized derivations acting as Homomorphisms or Anti-homomorphisms with central values in semiprime rings*, Miskolc Math. Notes, **16** (2015), №2, 781–791.
14. B. Dhara, S. Kar, S. Mondal, *A result on generalized derivations on Lie ideals in prime rings*, Beitr. Algebra Geom., **54** (2013), №2, 677–682.
15. B. Dhara, S. Ali, A. Pattanayak, *Identities with generalized derivations in semiprime rings*, Demonstratio Mathematica, **46** (2013), №3, 453–460.
16. S. Huang, *Generalized derivations of prime rings*, Internat. J. Math. & Math. Sci., **2007**, Article ID 85612.
17. T.K. Lee, W.K. Shiue, *A result on derivations with Engel conditions in prime rings*, Southeast Asian Bull. Math., **23** (1999), 437–446.
18. P.H. Lee, T.K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica, **11** (1983), 75–79.
19. B. Dhara, S. Ghosh, S. Sandhu, *On Lie ideals satisfying certain differential identities in prime rings*, Extracta Mathematicae, **38** (2023), №1, 67–84.
20. B. Dhara, S. Kar, N. Bera, *Some identities related to multiplicative (generalized)-derivations in prime and semiprime rings*, Rend. Circ. Mat. Palermo, Ser. 2, **72** (2023), 1497–1516.
21. B. Dhara, S. Ali, *On multiplicative(generalized)-derivations in prime and semiprime rings,* Aequat. Math., **86** (2013), №1–2, 65–79.
22. L. Oukhtite, S. Salhi, L. Taoufiq, *Strong commutativity preserving maps on Lie ideals of semiprime rings*, Beitr. Algebra Geom., **49** (2008), №2, 441–447.
23. S.K. Tiwari, R.K. Sharma, B. Dhara, *Multiplicative (generalized)-derivations in semiprime rings*, Beitr. Algebra Geom., **58** (2017), №1, 211–225.
24. S.K. Tiwari, R.K. Sharma, B. Dhara, *Identities related to generalized derivations on ideal in prime rings*, Beitr. Algebra Geom., **57** (2016), №4, 809–821.

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Received 15.08.2024
Revised 25.01.2025