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STOCHASTIC EVOLUTION SYSTEM WITH MARKOV-MODULATED POISSON PERTURBATIONS IN THE AVERAGING SCHEMA

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This paper discusses the asymptotic behavior of the stochastic evolutionary system under the Markov-modulated Poisson perturbations in an averaging schema. Such a perturbation process combines the Poisson process with the Markov process that modulates the intensity of jumps. This allows us to model systems with transitions between different modes or rare but significant jumps. Initially, the asymptotic properties of the Markov-modulated Poisson perturbation are investigated. For this purpose, we build the generator for the limit process solving the singular perturbation problem for the original process. Then we introduce a compensated Poisson process with a zero mean value, and it is used to center the jumps. The stochastic evolutionary system perturbed by the compensated Poisson process with an additional jump size function is described. We build the generator for an evolution process and investigate its asymptotic properties. Solving the singular perturbation problem we obtain the form of the limit process and its generator. This allows us to formulate and prove the theorem about weak convergence of the evolution process to the averaged one. The limit process for the stochastic evolutionary system at increasing time intervals is determined by the solution of a deterministic differential equation. The obtained result makes it possible to study the rate of convergence of the perturbed process to the limit one, as well as to consider stochastic approximation and optimization procedures for problems in which the system is described by an evolutionary equation with the Markov-modulated Poisson perturbation.

Introduction. The study of stochastic evolutionary systems is becoming increasingly important in various fields of science, including network traffic modeling, financial modeling, modeling of service and reliability systems [1], epidemiology [2], etc., since these systems can effectively describe the randomness and uncertainty inherent in the real-world phenomena. One important aspect of stochastic evolutionary systems is their asymptotic behavior, which can exhibit various dynamical patterns, from stable equilibrium to complex oscillations and even chaos.

This work investigates the asymptotic behavior of stochastic evolutionary systems under the influence of perturbations caused by the Markov-modulated Poisson process (MMPP) [3] in the averaging schema. This type of perturbation is a combination of the Poisson process with the Markov process that modulates the intensity of jumps. This allows us to model systems with the transitions between the different modes or rare but significant jumps. They are often observed in the natural and technical processes.

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MMPP Properties. Let us consider MMP as a two-component process $(x(t), N(t))$, where $x(t), t \geq 0$ is a uniformly ergodic Markov process in the standard phase space $(X, \boldsymbol{X})$ defined by the generator [4]

$$Q\varphi(x) = q(x) \int_X P(x, dy)\big(\varphi(y) - \varphi(x)\big), \quad \varphi \in \mathbf{B}(X), \tag{1}$$

where $\mathbf{B}(X)$ is the Banach space of bounded functions with the sup-norm

$$\|\varphi\| = \max\{|\varphi(x)| \colon x \in X\}.$$

Stochastic kernel $P(x, B), x \in X, B \in \boldsymbol{X}$ defines a uniformly ergodic embedded Markov chain $x_n = x(\tau_n)$, $n \geq 0$, with the stationary distribution $\rho(B), B \in \boldsymbol{X}$. Stationary distribution $\pi(B), B \in \boldsymbol{X}$ of the Markov process $x(t), t \geq 0$, is defined by the representation

$$\pi(dx)q(x) = q\rho(dx), \quad q = \int_X \pi(dx)q(x).$$

Let us denote by R_0 the potential operator of the generator Q, which is defined by equality

$$R_0 = \Pi - (\Pi + Q)^{-1},$$

where $\Pi\varphi(x) = \int_X \pi(dy)\varphi(y)$ is the projector onto the subspace of zeros $N_Q = \{\varphi : Q\varphi = 0\}$ of the operator Q.

The Poisson process $N(t)$ is the process of counting the number of events that have occurred up to time t. The rate of arrival of events at the moment t is defined as $\lambda(x(t))$, that is, the intensity parameter is modulated by the Markov process of the state.

The probability that the random variable $N(t)$ is equal to k is defined by [5]

$$P\big(N(t) = k\big) = \frac{\Lambda^k(t)}{k!} e^{-\Lambda(t)},$$

where $\Lambda(t) = \int_0^t \lambda(x(s))ds$ is a measure describing the intensity and distribution of the jumps to time t, where $\lambda(x(t))$ is the rate of arrival of events at the moment t, when the Markov process is in the state $x(t)$. Such a process has the following properties:

$$P\big(N(t+h) - N(t) = 0\big) = 1 - \Lambda(h) + o(h),$$
$$P\big(N(t+h) - N(t) = 1\big) = \Lambda(h) + o(h),$$
$$P\big(N(t+h) - N(t) > 1\big) = o(h), \quad h \to 0.$$

Lemma 1. *Generator of the two-component process* $(x(t), N(t))$ *has the form:*

$$L\varphi(x, N) = Q\varphi(x, N) + \lambda(x)\big(R_+ - I\big)\varphi(x, N), \tag{2}$$

where $R_+\varphi(x, N) = \varphi(x, N+1)$ *and* $I\varphi(x, N) = \varphi(x, N)$.

Proof. Similar to [6], the process generator definition is used. Then the conditional expectation is calculated using the properties of the Poisson process and the definition of the generator of the Markov process (1), and thus the resulting formula (2) is obtained. □

To study the asymptotic properties of the MMPP, we will consider the process with a small parameter scheme $(x(t/\varepsilon), N^\varepsilon(t))$. Here $N^\varepsilon(t)$ is a Poisson process with the intensity $\lambda(x(t/\varepsilon))$.

Lemma 2. *The generator of two-component process* $(x(t/\varepsilon), N^\varepsilon(t))$ *has the form*

$$L^\varepsilon\varphi(x,N) = \varepsilon^{-1}Q\varphi(x,N) + \lambda(x)\big(R_+ - I\big)\varphi(x,N). \tag{3}$$

Proof. It is carried out similarly to the generator (2) in Lemma 1. □

Lemma 3. *The singular perturbation problem for the operator* (3) *on the test functions*

$$\varphi^\varepsilon(x,N) = \varphi(N) + \varepsilon\varphi_0(x,N)$$

has the solution in the form

$$L^\varepsilon\varphi^\varepsilon(x,N) = L_N\varphi(N) + \varepsilon\theta_N(x)\varphi(N), \tag{4}$$

where the remaining term $\theta_N(x)$ *is uniformly bounded on* x.

Limit operator L_N *is defined by*

$$L_N\Pi = \Lambda_\Pi\Pi(R_+ - I)\Pi, \tag{5}$$

where $\Lambda_\Pi = \Pi\lambda(x) = \int_X \pi(dx)\lambda(x)$.

Proof. Let us conduct the similar terms with respect to ε to proof equality (4)

$$L^\varepsilon\varphi^\varepsilon(x,N) = \varepsilon^{-1}Q\varphi(N) + \big(Q\varphi_0(x,N) + \lambda(x)(R_+ - I)\varphi(N)\big) + \varepsilon\lambda(x)(R_+ - I)\varphi_0(x,N).$$

Since $\varphi(N)$ doesn't depend on x, then $Q\varphi(N) = 0 \Longleftrightarrow \varphi(N) \in N_Q$.

Next term would be written in the form

$$Q\varphi_0(x,N) + \lambda(x)(R_+ - I)\varphi(N) = L_N\varphi(N).$$

We can obtain limit process L_N in the form (5) using the solution condition of the last equation. Then

$$\varphi_0(x,N) = R_0\big(\lambda(x)(R_+ - I) - L_N\big)\varphi(N), \tag{6}$$

and taking into account that $R_0L_N = 0$, we obtain $\varphi_0(x,N) = R_0\lambda(x)(R_+ - I)\varphi(N)$.

Using (6) we can bring the last term to the form

$$\varepsilon\lambda(x)(R_+ - I)\varphi_0(x,N) = \varepsilon\lambda(x)(R_+ - I)R_0\big(\lambda(x)(R_+ - I)\big)\varphi(N) = \varepsilon\theta_N(x)\varphi(N).$$

We can prove that the operator $\theta_N(x)$ on the functions $\varphi(N)$ is bounded using the form of operators R_+ and R_0. □

Let us consider the compensated Poisson process $\tilde N(t)$ defined as

$$\tilde N(t) = N^\varepsilon(t) - \Lambda(t) = N^\varepsilon(t) - \int_0^t \lambda(x(s))ds. \tag{7}$$

The compensated Poisson process is used to center the jumps, i.e. $E\big(\tilde N(t)\big) = 0$.

Lemma 4. *The generator of the process* $\tilde N(t)$ (7) *has the form*

$$\tilde L^\varepsilon\varphi(N) = \lambda(x)\big(R_+ - I\big)\varphi(N) - \lambda(x)\varphi'(N).$$

Proof. For a Poisson process in a small interval $\big(0,t\big)$, the probability of more than one jump is $o(t^2)$ so we consider only scenarios with 0 or 1 jump. In this case we can write

$$\tilde N(t) = \begin{cases} -\Lambda(t) & \text{with probability} \quad (1 - \Lambda(t) + o(t^2)),\ t \to 0; \\ 1 - \Lambda(t) & \text{with probability} \quad \Lambda(t). \end{cases}$$

Let $\Delta\tilde{N}(t) = \tilde{N}(t) - \tilde{N}(0)$, when there are no jumps. That is,

$$\Delta\tilde{N}(t) = (\tilde{N}(t) - \tilde{N}(0))|_{N^{\varepsilon}(t)=0} = -\int_0^t \lambda(x(s))ds.$$

So that $\mathbb{E}\big(\Delta\tilde{N}(t)\big) = -\Lambda(t)$ and $\mathbb{E}\big(\Delta\tilde{N}(t)\big)^k = 0$, when $k > 1$. Now we can rewrite both cases using Taylor's series,

$$\varphi(\tilde{N}(t))|_{N^{\varepsilon}(t)=0} = \varphi(\tilde{N}(0)) + \varphi'(\tilde{N}(0))\Delta\tilde{N}(t) + \frac{1}{2}\varphi''(\tilde{N}(0))(\Delta\tilde{N}(t))^2 + \ldots,$$

and

$$\varphi(\tilde{N}(t))|_{N^{\varepsilon}(t)=1} = \varphi(\tilde{N}(0)+1) + \varphi'(\tilde{N}(0)+1)\Delta\tilde{N}(t) + \frac{1}{2}\varphi''(\tilde{N}(0)+1)(\Delta\tilde{N}(t))^2 + \ldots.$$

Combining with jump probabilities, we get

$$\begin{aligned}
\mathbb{E}\big(\varphi(\tilde{N}(t))|_{\tilde{N}(0)=N}\big) &= (1-\Lambda(t)+o(t^2))\mathbb{E}\big(\varphi(\tilde{N}(t))_{N^{\varepsilon}(t)=0}\big) + \Lambda(t)\mathbb{E}\big(\varphi(\tilde{N}(t))_{N^{\varepsilon}(t)=1}\big) = \\
&= \mathbb{E}\big(\varphi(\tilde{N}(t))_{N^{\varepsilon}(t)=0}\big) + \Lambda(t)\{\mathbb{E}\big(\varphi(\tilde{N}(t))_{N^{\varepsilon}(t)=1}\big) - \mathbb{E}\big(\varphi(\tilde{N}(t))_{N^{\varepsilon}(t)=0}\big)\} + o(t^2) = \\
&= \mathbb{E}\big(\varphi(\tilde{N}(0))\big) + \varphi'(N)\mathbb{E}\big(\Delta\tilde{N}(t)\big) + \frac{1}{2}\varphi''(N)\mathbb{E}\big((\Delta\tilde{N}(t))^2\big) + o(t^2) + \\
&+\Lambda(t)\Big\{\big(\varphi(N+1)-\varphi(N)\big) + \big(\varphi'(N+1)-\varphi'(N)\big)\mathbb{E}\big(\Delta\tilde{N}(t)\big) + \\
&+\frac{1}{2}\big(\varphi''(N+1)-\varphi''(N)\big)\mathbb{E}\big(\Delta\tilde{N}(t)\big)^2 + o(t^2)\Big\}
\end{aligned}$$

as $t \to 0$. Thus, the calculation is completed knowing the conditional moments of the $\Delta\tilde{N}(t)$

$$\begin{aligned}
\mathbb{E}\big(\varphi(\tilde{N}(t)) - \varphi(\tilde{N}(0))\big) &= -\Lambda(t)\varphi'(N) + \Lambda(t)\big(\varphi(N+1)-\varphi(N)\big) + \\
&+\Lambda^2(t)\big(\varphi'(N+1)-\varphi'(N)\big) + o(t^2) \quad (t \to 0).
\end{aligned} \tag{8}$$

In view of the following limits $\lim_{t\to 0}\Lambda(t) = 0$ and $\lim_{t\to 0}\frac{\Lambda(t)}{t} = \lambda(x)$ (using L'Hospital's Rule) we divide (8) by t and obtain as $t \to 0$

$$\tilde{L}^{\varepsilon}\varphi(N) = \lambda(x)\big(R_+ - I\big)\varphi(N) - \lambda(x)\varphi'(N).$$

□

Remark 1. Combining results of Lemma 2 and Lemma 4 it can be shown that the generator of two-component process $(x(t/\varepsilon), \tilde{N}(t))$ has the form

$$\tilde{L}^{\varepsilon}\varphi(x,N) = \varepsilon^{-1}Q\varphi(x,N) + \lambda(x)\big(R_+ - I\big)\varphi(x,N) - \lambda(x)\varphi'(N). \tag{9}$$

Evolution system. A stochastic evolutionary system in an ergodic Markov environment is given by the evolution equation

$$du^{\varepsilon}(t) = C(u^{\varepsilon}(t), x(t/\varepsilon))dt + a(t)\tilde{N}(t)dt, \quad u^{\varepsilon}(t) \in R, \tag{10}$$

where $a(t)$ represents the size and effect of the jump at the time t.

Lemma 5. *The generator of three-component process* $(u^\varepsilon(t), \tilde{N}(t), x(t/\varepsilon))$, $t \geq 0$ *has the form*

$$G^\varepsilon(x)\varphi(u,N,x) = \varepsilon^{-1}Q\varphi(u,N,x) + \big(C(x)+A\big)\varphi(u,N,x)+ \\ +\lambda(x)(R_+ - I)\varphi(u,N,x) - W(x)\varphi(u,N,x), \tag{11}$$

where

$$C(x)\varphi(u,N,x) = C(u,x)\varphi'_u(u,N,x), \quad A\varphi(u,N,x) = a(t)N\varphi'_u(u,N,x), \\ W(x)\varphi(u,N,x) = \lambda(x)\varphi'_N(u,N,x).$$

Proof. Let us denote $u^\varepsilon(t) = u_t$, $\tilde{N}(t) = w_t$, $x(t/\varepsilon) = x_t$ and then calculate the conditional mathematical expectation

$$E\big(\varphi(u^\varepsilon(t+\Delta), \tilde{N}^\varepsilon(t+\Delta), x((t+\Delta)\varepsilon)) - \varphi(u^\varepsilon(t), \tilde{N}(t), x(t/\varepsilon))\big| \\ u^\varepsilon(t) = u, \tilde{N}(t) = w, x(t/\varepsilon) = x\big) = E\big(\varphi(u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(u,w,x)\big) = \\ = E\big(\varphi(u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(u, w_{t+\Delta}, x_{t+\Delta})\big) + E\big(\varphi(u, w_{t+\Delta}, x_{t+\Delta}) - \varphi(u,w,x)\big).$$

According to (10) and Taylor's decomposition, $u(t+\Delta) = u + C(u,x)\Delta + a(t)w\Delta + o(\Delta)$. Therefore,

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E\big(\varphi(u_{t+\Delta}, w_{t+\Delta}, x_{t+\Delta}) - \varphi(u, w_{t+\Delta}, x_{t+\Delta})\big) = \\ = \lim_{\Delta\to 0}\frac{1}{\Delta}E\big(\varphi(u + C(u,x)\Delta + a(t)w\Delta + o(\Delta)), w_{t+\Delta}, x_{t+\Delta}) - \varphi(u, w_{t+\Delta}, x_{t+\Delta})\big) = \\ = \lim_{\Delta\to 0}\frac{1}{\Delta}E\big(\varphi'_u(u, w_{t+\Delta}, x_{t+\Delta})(C(u,x)\Delta a(t)w\Delta + o(\Delta))\big) = \big(C(u,x) + a(t)w\big)\varphi'_u(u,w,x).$$

Also from (9) we have

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E\big(\varphi(u, w_{t+\Delta}, x_{t+\Delta}) - \varphi(u,w,x)\big) = \varepsilon^{-1}Q\varphi(u,w,x)+ \\ +\lambda(x)\big(R_+ - I\big)\varphi(u,w,x) + \lambda(x)\varphi'_w(u,w,x).$$

Combining these results we obtain (11). □

Lemma 6. *The singular perturbation problem for the operator* (11) *on the test functions*

$$\varphi^\varepsilon(u,w,x) = \varphi(u,w) + \varepsilon\varphi_0(u,w,x),$$

has the solution in the form

$$G^\varepsilon(x)\varphi^\varepsilon(u,w,x) = G\varphi(u,w) + \varepsilon\theta(x)\varphi(u,w), \tag{12}$$

where the remaining term $\theta(x)$ *is uniformly bounded on* x.

The limit operator G *is defined by the equality*

$$G\Pi = \Pi C(x)\Pi + \Pi A\Pi + \Lambda_\Pi\Pi(R_+ - I)\Pi - \Pi W(x)\Pi. \tag{13}$$

Proof. Let us conduct the similar terms with respect to ε to proof the equality (12)

$$G^\varepsilon(x)\varphi^\varepsilon(u,w,x) = \varepsilon^{-1}Q\varphi(u,w)+ \\ +Q\varphi_0(u,w,x) + \big(C(x) + A + \lambda(x)(R_+ - I) - W(x)\big)\varphi(u,w)+$$

$$+\varepsilon\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)\varphi_0(u,w,x).$$

Since $\varphi(u,w)$ doesn't depend on x, then $Q\varphi(u,w)=0 \iff \varphi(u,w)\in N_Q$.

Next term would be written in the form

$$Q\varphi_0(u,w,x)+\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)\varphi(u,w)=G\varphi(u,w).$$

We can obtain the limit process G in the form (13) using the solution condition of the last equation. Then

$$\varphi_0(u,w,x)=R_0\big(C(x)+A+\lambda(x)(R_+-I)-W(x)-G\big)\varphi(u,w), \tag{14}$$

and taking into account that $R_0G=0$, we obtain

$$\varphi_0(u,w,x)=R_0\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)\varphi(u,w).$$

Using (14) we can substitute the last term to the form

$$\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)\varphi_0(x,N)=$$
$$=\varepsilon\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)R_0\big(C(x)+A+\lambda(x)(R_+-I)-W(x)\big)\varphi(u,w)=$$
$$=\varepsilon\theta(x)\varphi(u,w).$$

We can proof that the operator $\theta(x)$ on the functions $\varphi(u,w)$ is bounded using the form of operators $C(x)$, A, R_+, $W(x)$ and R_0. □

Theorem 1. *The weak convergence takes place* $(u^\varepsilon(t),N^\varepsilon(t))\to(\widehat{u}(t),\widehat{N}(t))$ *as* $\varepsilon\to 0$.

The limit process $(\widehat{u}(t),\widehat{N}(t))$ *is defined by the generator*

$$G\varphi(u,w)=\big(\widehat{C}(u)+\widehat{A}\big)\varphi'_u(u,w)+\Lambda_\Pi\Pi(R_+-I)\varphi(u,w)-\Lambda_\Pi\varphi'_w(u,w), \tag{15}$$

where

$$\widehat{C}(u)=\Pi C(x)=\int_X \pi(dx)C(u,x)$$

and $\widehat{A}=a(t)w$. *The limit process* $\widehat{u}(t)$ *can be obtained as a solution of the differential equation* $d\widehat{u}(t)=\big(\widehat{C}(u)+\widehat{A}\big)\widehat{u}(t)dt$.

Proof. Let's calculate the right-hand part of Equation (15). In this case, we obtain

$$G\varphi(u,w)=\big(\Pi C(x)+\Pi A\big)\varphi'_u(u,w)+\Lambda_\Pi\Pi(R_+-I)\varphi(u,w)-\Lambda_\Pi\varphi'_w(u,w).$$

In view of Theorem 4.2 from [4], the proof of the theorem is completed. □

Remark 2. There are the following explanations of the terms in the formula (15):

a) $\widehat{C}(u)\varphi'_u(u,w)$ corresponds to the deterministic evolution of the process;

b) $\widehat{A}\varphi'_u(u,w)$ is an evolutionary process drift term caused by the perturbation;

c) $\Lambda_\Pi\Pi(R_+-I)\varphi(u,w)$ accounts for the contribution from the jumps, which are modeled by MMPP;

d) $-\Lambda_\Pi\varphi'_w(u,w)$ is a correction that compensates for the linear approximation, ensuring that the generator accurately captures the effect of jumps.

Conclusions. The limit process for a stochastic evolutionary system at increasing time intervals is determined by the solution of a deterministic differential equation. The obtained result makes it possible to study the rate of convergence of the perturbed process to the limit, as well as to consider stochastic approximation [8] and optimization procedures for problems in which the system is described by an evolutionary equation with the Markov-modulated Poisson perturbation.

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