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SOME INEQUALITIES FOR ENTIRE FUNCTIONS

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Let $\mathcal{L}_p$ be the subspace of the space $L_p(\mathbb{R})$ consisting of the restriction to the real axis of all entire functions of exponential type $\le \pi$. In this paper, for any function $f \in \mathcal{L}_p$ $(1 \le p \le \infty)$, we obtain estimates for the norm of f in terms of the sequence $(f(n/2))_{n\in\mathbb{Z}}$, namely

$$\frac{1}{2}\|f\|_{p,1} \le \|f\|_{\mathcal{L}_p} \le 2\|f\|_{p,1},$$

where $\|f\|_{p,1} := \frac{1}{2}(\|Jf\|_{\ell_p(\mathbb{Z})} + \|JT_{1/2}f\|_{\ell_p(\mathbb{Z})})$. Here $J : \mathcal{L}_p \to \ell_p(\mathbb{Z})$ is the linear operator given by the formula

$$(Jf)(n) := (-1)^n f(n), \quad n \in \mathbb{Z},$$

and T_τ is the shift by $\tau \in \mathbb{R}$ of the function f,

$$(T_\tau f)(z) := f(z + \tau), \quad z \in \mathbb{C}.$$

1. Introduction. Let us denote by $\mathcal{B}$ the linear space of all entire functions of exponential type $\le \pi$ such that

$$\sup_{x,y\in\mathbb{R}} |f(x+iy)|e^{-\pi|y|} < \infty.$$

The linear space $\mathcal{B}$ becomes a Banach space with the norm defined by the formula

$$\|f\|_{\mathcal{B}} := \sup_{x,y\in\mathbb{R}} |f(x+iy)|e^{-\pi|y|}, \quad f \in \mathcal{B}.$$

Let us consider the linear subspaces of the space $\mathcal{B}$

$$\mathcal{L}_p := \mathcal{B} \cap L_p(\mathbb{R}), \quad 1 \le p \le \infty,$$

$$\|f\|_{\mathcal{L}_\infty} := \sup_{x\in\mathbb{R}} |f(x)|, \quad \|f\|_{\mathcal{L}_p} := \Big(\int_{\mathbb{R}} |f(x)|^p\, dx\Big)^{1/p} \quad (p \in [1,\infty)).$$

A lot of mathematicians have studied the spaces $\mathcal{L}_p$, $1 \le p \le \infty$. The main results of their researches are presented in the monograph of Levin [1].

Let us denote by J the linear operator acting from the space $\mathcal{L}_p$ to the space $\ell_p := \ell_p(\mathbb{Z})$ by the formula

$$(Jf)(n) := (-1)^n f(n), \quad n \in \mathbb{Z}, \quad f \in \mathcal{L}_p.$$

It follows from the results of Levin [1] that the operator $J\colon \mathcal{L}_p \to \ell_p$ is continuous if $1 \le p \le \infty$ and has a continuous inverse operator if $1 < p < \infty$. Denote by $\|\cdot\|_p$ the norm on ℓ_p, namely

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$$\|u\|_\infty := \sup_{n\in\mathbb{Z}} |u(n)|, \quad \|u\|_p := \Big(\sum_{n\in\mathbb{Z}} |u(n)|^p\Big)^{1/p} \quad (u = (u(n))_{n\in\mathbb{Z}} \in \ell_p).$$

We also denote by T_τ ($\tau \in \mathbb{R}$) the shift operator acting on $\mathcal{L}_p$ by the formula

$$(T_\tau f)(z) := f(z+\tau), \quad z \in \mathbb{C}, \quad f \in \mathcal{L}_p.$$

Obviously, the operator T_τ maps isometrically the space $\mathcal{L}_p$ onto itself.

The main result of this paper is the following theorem.

Theorem 1. *Let $f \in \mathcal{L}_p$ ($1 \le p \le \infty$) and $\|f\|_{p,1} := \frac{1}{2}(\|Jf\|_p + \|JT_{1/2}f\|_p)$. Then*

$$\frac{1}{2}\|f\|_{p,1} \le \|f\|_{\mathcal{L}_p} \le 2\|f\|_{p,1}. \tag{1}$$

The inequalities (1) have the same nature as the phenomenon of interference discovered by S. N. Bernstein and for space $\mathcal{L}_\infty$ is described as follows.

Theorem (S. N. Bernstein [1])**.** *Let $f \in \mathcal{L}_\infty$. Then*

$$|f(x+1) + f(x)| \le C \sup_{n\in\mathbb{Z}} |f(n)|, \quad x \in \mathbb{R},$$

where C is an absolute constant.

2. The proof of Theorem 1. First, we prove the following auxiliary lemma.

Lemma 1. *Let $\tau \in [-1/4, 1/4]$ and the sequence $(g_\tau(k))_{k\in\mathbb{Z}}$ be given by the formula*

$$g_\tau(k) := \frac{1}{\pi}\left(\frac{1}{1/2-k} - \frac{1}{1/2+\tau-k}\right), \quad k \in \mathbb{Z}.$$

Then

$$\|g_\tau\|_1 \le 1. \tag{2}$$

Proof. It is easy to see that $-g_{-\tau}(-k+1) = g_\tau(k)$, $k \in \mathbb{Z}$. Hence $\|g_{-\tau}\|_1 = \|g_\tau\|_1$, $\tau \in [-1/4, 1/4]$. Therefore, it suffices to prove the estimate (2) for $\tau \in [0, 1/4]$. Let us consider the function $h(\tau) := \pi\|g_\tau\|_1$, $\tau \in [0, 1/4]$. Since the function $x \mapsto x^{-1}$ is decreasing on $\mathbb{R}_+$ and $\mathbb{R}_-$, we have $0 \le g_\tau(k) \le g_{1/4}(k)$, $k \in \mathbb{Z}$, $\tau \in [0, 1/4]$, thus

$$h(\tau) \le h(1/4) = \sum_{k\in\mathbb{Z}} \left(\frac{1}{1/2-k} - \frac{1}{3/4-k}\right).$$

Note that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{1/2-k} = 0$$

and

$$-\sum_{k=-N}^{N} \frac{1}{3/4-k} = 4 + \sum_{k=1}^{N-1} \frac{1}{k+1/4} - \sum_{k=1}^{N+1} \frac{1}{k-1/4}.$$

This implies that

$$h(1/4) = 4 - \sum_{k=1}^{\infty}\left(\frac{1}{k-1/4} - \frac{1}{k+1/4}\right) = 4 - 8\sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k+1)}.$$

Using the equality (see [2, p.11])

$$\sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8},$$

we obtain that $h(1/4) = \pi$, i.e. $\|g_\tau\|_1 \le 1$. □

The proof of Theorem 1. Let $p \in [1,\infty)$ and $f \ne 0$, $f \in \mathcal{L}_p$. Since $J\colon \mathcal{L}_p \to \ell_p$ is continuous, we get that $\sum\limits_{n\in\mathbb{Z}} |f(x+n)|^p < \infty$ for every $x \in \mathbb{R}$. Thus, the function

$$\mathbb{R} \ni x \mapsto \psi(x,f) := \Big(\sum_{n\in\mathbb{Z}} |f(x+n)|^p\Big)^{1/p}$$

is Lebesgue measurable on $\mathbb{R}$. Moreover, it is a periodic function with period 1. Let us show that $\psi(\cdot,f)$ is bounded. According to Plancherel-Polya's theorem (see [1]), each function $f \in \mathcal{L}_p$ admits the representation

$$f(z) = \sum_{k\in\mathbb{Z}}(-1)^k f(k)\frac{\sin \pi z}{\pi(z-k)}, \quad z \in \mathbb{C}.$$

Then

$$f(\tau+1/2) - \cos(\pi\tau)f(1/2) = -\cos(\pi\tau)\sum_{k\in\mathbb{Z}}(-1)^k f(k)g_\tau(k), \quad \tau \in [-1/4,1/4]. \tag{3}$$

Substituting $T_n f$ $(n \in \mathbb{Z})$ for f in (3), we obtain

$$f(\tau+1/2+n) - \cos(\pi\tau)f(1/2+n) = -\cos(\pi\tau)\sum_{k\in\mathbb{Z}}(-1)^k f(k+n)g_\tau(k),$$

and thus

$$|f(\tau+1/2+n)| \le |f(1/2+n)| + \sum_{k\in\mathbb{Z}}|f(k+n)|\,g_\tau(k), \quad n \in \mathbb{Z}. \tag{4}$$

Taking into account the convolution properties, we deduce that

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f(k+n)|g_\tau(k)\Big)^p \le \Big(\sum_{n\in\mathbb{Z}}|f(n)|^p\Big)\|g_\tau\|_1^p \le \sum_{n\in\mathbb{Z}}|f(n)|^p.$$

Combining (4) and the triangle inequality for the norm $\|\cdot\|_p$, we get

$$\psi(\tau+1/2,f) \le \psi(1/2,f) + \psi(0,f), \quad \tau \in [-1/4,1/4]. \tag{5}$$

Replacing f by $T_{1/2}f$ in (5), we can write the inequality

$$\psi(\tau+1,f) \le \psi(1,f) + \psi(1/2,f), \quad \tau \in [-1/4,1/4]. \tag{6}$$

Combining (5), (6) and taking into account the periodicity of $\psi(\cdot,f)$, we obtain

$$\psi(x,f) \le \psi(0,f) + \psi(1/2,f), \quad x \in \mathbb{R}. \tag{7}$$

It means that

$$\int_0^1 \psi^p(x,f)\,dx \le (\psi(0,f)+\psi(1/2,f))^p.$$

Applying the monotone convergence theorem, we conclude that

$$\int_0^1 \psi^p(x,f)\,dx = \int_0^1 \sum_{n\in\mathbb{Z}} |f(x+n)|^p\,dx = \int_{\mathbb{R}} |f(x)|^p\,dx, \tag{8}$$

in other words

$$\|f\|_{\mathcal{L}_p} \le \psi(0,f)+\psi(1/2,f) = 2\|f\|_{p,1}. \tag{9}$$

The inequality (7) yields

$$\psi(x,f)+\psi(x+1/2,f) \le 2(\psi(0,f)+\psi(1/2,f)), \quad x\in\mathbb{R}. \tag{10}$$

Since $\psi(t,T_xf)=\psi(t+x,f)$, $x,t\in\mathbb{R}$, we have

$$\psi(0,f)+\psi(1/2,f) = \psi(-x,T_xf)+\psi(-x+1/2,T_xf) \le$$
$$\le 2(\psi(0,T_xf)+\psi(1/2,T_xf)) = 2(\psi(x,f)+\psi(x+1/2,f)).$$

Taking into account the inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$, $a,b\ge 0$, we get

$$\left(\frac{\psi(0,f)+\psi(1/2,f)}{4}\right)^p \le \left(\frac{\psi(x,f)+\psi(x+1/2,f)}{2}\right)^p \le$$
$$\le \frac{1}{2}\left(\psi^p(x,f)+\psi^p(x+1/2,f)\right).$$

Thus, in view of (8), we obtain

$$\|f\|^p_{\mathcal{L}_p} = \frac{1}{2}\int_0^1 [\psi^p(x,f)+\psi^p(x+1/2,f)]\,dx \ge \left(\frac{\psi(0,f)+\psi(1/2,f)}{4}\right)^p,$$

i.e. $\frac{1}{2}\|f\|_{p,1} \le \|f\|_{\mathcal{L}_p}$. Therefore, the inequalities (1) are proved for $p\in[1,\infty)$.

In the case $p=\infty$ the proof is similar. □

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