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RELATIVE GROWTH OF HADAMARD COMPOSITIONS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN A HALF-PLANE

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Let $\Lambda=(\lambda_n)$ be a positive sequence increasing to $+\infty$ and $S(\Lambda,A)$ be the class of Dirichlet series $F(s)=\sum\limits_{n=1}^{\infty}f_n\exp\{s\lambda_n\}$ with the abscissa of absolute convergence $A\in(-\infty,+\infty]$. A function F is called the Hadamard composition of the genus $m\ge 1$ of the functions $F_j(s)=$ $=\sum\limits_{n=0}^{\infty}a_{n,j}\exp\{s\lambda_n\}$ $(j\in\{1,2,\ldots,p\})$, if $f_n=\sum\limits_{k_1+\dots+k_p=m}c_{k_1\dots k_p}a_{n,1}^{k_1}\cdot\ldots\cdot a_{n,p}^{k_p}$ for all n. The growth of the function $F\in S(\Lambda,0)$ with respect to a function $G(s)=\sum\limits_{n=1}^{\infty}g_n\exp\{s\lambda_n\}\in$ $S(\Lambda,+\infty)$ is identified with the growth of the function $M_G^{-1}(M_F(\sigma))$ as $\sigma\uparrow 0$, where $M_F(\sigma)=$ $=\sup\{|F(\sigma+it)|:\,t\in\mathbb{R}\}$. The dependence of the growth of a function $M_G^{-1}(M_F(\sigma))$ on the growth of functions $M_G^{-1}(M_{F_j}(\sigma))$ is studied in terms of generalized orders and generalized convergence classes. In particular, there are proved the following statements:
Let $F\in S(\Lambda,0)$ be the Hadamard composition of genus $m\ge 1$ of the functions $F_j\in S(\Lambda,0)$; the positive increasing function α such that $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$, β and $\alpha(M_G^{-1}(e^x))$ are slowly increasing functions, i.e. $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. If $\varlimsup\limits_{n\to\infty}\ln n/\ln|f_n|<+\infty$, then $\varrho^0_{\alpha\beta}[F]_G\le\varrho^0:=\max\{\varrho^0_{\alpha\beta}[F_j]_G:\,1\le j\le p\}$. If there is a dominating function F_1 among the functions F_j then $\varrho^0_{\alpha,\beta}[F]_G=\varrho^0_{\alpha,\beta}[F_1]_G$ and $\lambda^0_{\alpha,\beta}[F]_G=\lambda^0_{\alpha,\beta}[F_1]_G$. (Theorem 1)

1. Introduction. Let $\Lambda=(\lambda_n)$ be an increasing to $+\infty$ sequence of non-negative numbers and by $S(\Lambda,A)$ we denote a class of Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}f_ne^{s\lambda_n},\quad s=\sigma+it, \tag{1}$$

with the abscissa of absolute convergence $A\in(-\infty,+\infty]$. We will assume that entire Dirichlet series (1) does not reduce to an exponential polynomial.

For $\sigma<A$ we put $M_F(\sigma)=\sup\{|F(\sigma+it)|:\,t\in\mathbb{R}\}$ and remark that for $F\in S(\Lambda,+\infty)$ the function $M_F(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty,+\infty)$ and, therefore, there

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exists the function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increase to $+\infty$ on $(x_0, +\infty)$. The growth of the function $F \in S(\Lambda, +\infty)$ with respect to a function $G \in S(\Lambda, +\infty)$,

$$G(s) = \sum_{n=1}^{\infty} g_n e^{s\lambda_n},$$

is identified [1–4] with the growth of the function $M_G^{-1}(M_F(\sigma))$ as $\sigma \to +\infty$.

Suppose that $F_j \in S(\Lambda, A)$,

$$F_j(s) = \sum_{n=1}^{\infty} f_{n,j} e^{s\lambda_n}, \quad j \in \{1, 2, \dots, p\},$$

and say [5, 6] that the function F is the Hadamard composition of genus $m \geq 1$ of the functions F_j if for all n

$$f_n = \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} f_{n,1}^{k_1} \cdot \ldots \cdot f_{n,p}^{k_p}. \tag{2}$$

The function F_1 is called dominant, if $|c_{m0\dots0}||f_{n,1}|^m \neq 0$ and $|f_{n,j}| = o(|f_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$. In [7] it is shown that if function F_1 is dominant then

$$f_n = (1 + o(1))|c_{m0\dots0}||f_{n,1}|^m, \quad n \to \infty. \tag{3}$$

To study the growth of function $F \in S(\Lambda, +\infty)$ with respect to function $G \in S(\Lambda, +\infty)$ in [7], the concepts of generalized $(\alpha\beta)$-order and generalized convergence $(\alpha\beta)$-class are used. For this purpose, L denotes the class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e. α is slowly increasing function. Clearly, $L_{si} \subset L^0$.

In [7] the generalized $(\alpha\beta)$-order $\varrho_{\alpha\beta}[F]_G$ and the generalized lower $(\alpha\beta)$-order $\lambda_{\alpha\beta}[F]_G$ of the function $F \in S(\Lambda, +\infty)$ with respect to the function $G \in S(\Lambda, +\infty)$ are defined as follows

$$\varrho_{\alpha\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}, \quad \lambda_{\alpha\beta}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)},$$

and the following theorem is proved.

Theorem A. *Let F be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, +\infty)$. If either $m = 1$, $\ln n = o(\lambda_n)$ as $n \to \infty$, $\alpha \in L^0$ and $\beta(\ln x) \in L^0$ or $m \geq 2$, $\ln n = O(\lambda_n)$ as $n \to \infty$, $\beta \in L$ and $\alpha(M_G^{-1}(e^x)) \in L_{si}$ then $\varrho_{\alpha\beta}[F]_G \leq$ $\leq \max\{\varrho_{\alpha\beta}[F_j]_G : 1 \leq j \leq p\}$.*

The relative generalized convergence $\alpha\beta$-class is defined [7] by condition

$$\int_{\sigma_0}^{\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)} d\sigma < +\infty.$$

Then the following theorem is true.

Theorem B. *Let F be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, +\infty)$, $\ln n = O(\lambda_n)$ as $n \to \infty$, $\beta \in L^0$ and $\alpha(M_G^{-1}(e^x)) \in L^0$. If F_j belongs to relative generalized convergence $\alpha\beta$-class for all j then F belongs to the same class.*

In the note proposed here we will obtain analogues of Theorems A and B in the case $G \in S(\Lambda, +\infty)$ and $F \in S(\Lambda, 0)$.

2. Growth estimates. If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, 0)$ then the quantities

$$\varrho^0_{\alpha\beta}[F] = \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)} \text{ and } \lambda^0_{\alpha\beta}[F] = \varliminf_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)}$$

are called [8] the *generalized* $(\alpha\beta)$*-order* and the *generalized lower* $(\alpha\beta)$*-order* of F, respectively. Similarly, the *generalized* $(\alpha\beta)$*-order* $\varrho^0_{\alpha\beta}[F]_G$ and the *generalized lower* $(\alpha\beta)$*-order* $\lambda^0_{\alpha\beta}[F]_G$ of the function $F \in S(\Lambda, 0)$ with respect to a function $G \in S(\Lambda, +\infty)$ we define as follows

$$\varrho^0_{\alpha\beta}[F]_G = \varlimsup_{\sigma\uparrow 0} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(1/|\sigma|)}, \quad \lambda^0_{\alpha\beta}[F]_G = \varliminf_{\sigma\uparrow 0} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(1/|\sigma|)}. \tag{4}$$

For $F \in S(\Lambda, 0)$ let $\mu_F(\sigma) = \max\{|f_n| \exp\{\sigma\lambda_n\} : n \geq 0\}$ be the maximal term of series (1). We remark that $\mu_F(\sigma) \uparrow +\infty$ as $\sigma \uparrow 0$ if and only if $\varlimsup_{n\to\infty} |f_n| = +\infty$. So, we will assume that this condition is satisfied. By the Cauchy inequality $\mu_F(\sigma) \leq M_F(\sigma)$, and the following lemma contains the estimate of $M_F(\sigma)$ from above.

Lemma 1 ([9])**.** *If $F \in S(\Lambda, 0)$ and*

$$h := \varlimsup_{n\to\infty} \frac{\ln n}{\ln |f_n|} < +\infty, \tag{5}$$

then for every $\varepsilon > 0$ and all $\sigma_0(\varepsilon) \leq \sigma < 0$

$$M_F(\sigma) \leq K(\varepsilon)\mu_F\Big(\frac{1-\varepsilon}{1+h}\sigma\Big)^{1+h+\varepsilon}, \quad K(\varepsilon) = \text{const} > 0.$$

Remark that if $G \in S(\Lambda, +\infty)$ then the function $\ln M_G(\sigma)$ is convex on $(-\infty, +\infty)$ and, thus, it has a continuous non-decreasing derivative $\frac{d \ln M_G(\sigma)}{d\sigma}$ on $(-\infty, +\infty)$ except of a countable number of points at which one-sided derivatives exist, and the left-sided derivative does not exceed the right-sided one. Since $\frac{\ln M_G(\sigma)}{\sigma} \to +\infty$ as $\sigma \to +\infty$, we have $\frac{d \ln M_G(\sigma)}{d\sigma} \nearrow +\infty$ as $\sigma \to +\infty$, i. e. $\frac{dM_G^{-1}(x)}{d \ln x} \searrow 0$ as $x \to +\infty$. Hence, it follows that $M_G^{-1} \in L_{si}$.

Now we prove the following theorem.

Theorem 1. *Let $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, 0)$, $\alpha \in L^0$, $\beta \in L_{si}$, $\alpha(M_G^{-1}(e^x)) \in L_{si}$ and (5) holds. Then*

$$\varrho^0_{\alpha\beta}[F]_G \leq \varrho^0 := \max\{\varrho^0_{\alpha\beta}[F_j]_G : 1 \leq j \leq p\}.$$

If among the functions F_j there is a dominant function F_1 then $\varrho^0_{\alpha,\beta}[F]_G = \varrho^0_{\alpha,\beta}[F_1]_G$ and $\lambda^0_{\alpha,\beta}[F]_G = \lambda^0_{\alpha,\beta}[F_1]_G$.

Proof. From (2) we obtain

$$|f_n| \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}||f_{n,1}|^{k_1}\cdot\ldots\cdot|f_{n,p}|^{k_p},$$

whence

$$|f_n|e^{m\sigma\lambda_n} \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|(|f_{n,1}|e^{\sigma\lambda_n})^{k_1}\cdot\ldots\cdot(|f_{n,p}|e^{\sigma\lambda_n})^{k_p}$$

and, thus,

$$\mu_F(m\sigma) \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|\mu_{F_1}(\sigma)^{k_1}\cdot\ldots\cdot\mu_{F_p}(\sigma)^{k_p}.$$

Let $\varrho^0 < +\infty$. Since $\varrho^0_{\alpha\beta}[F_j]_G \le \varrho^0$, for every $\varrho > \varrho^0$ and all $\sigma \in [\sigma_0(\varrho), 0)$ we have $M_{F_j}(\sigma) \le M_G(\alpha^{-1}(\varrho\beta(1/|\sigma|)))$ and, thus,

$$\mu_F(m\sigma) \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|M_{F_1}(\sigma)^{k_1}\cdot\ldots\cdot M_{F_p}(\sigma)^{k_p} \le CM^m_G(\alpha^{-1}(\varrho\beta(1/|\sigma|))),$$

where $C = \sum\limits_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|$, i.e.

$$\mu_F(\sigma) \le CM^m_G(\alpha^{-1}(\varrho\beta(m/|\sigma|))). \tag{6}$$

Therefore, by Lemma 1 for all $\sigma \in [\sigma_0(\varrho), 0)$ we get

$$M_F(\sigma) \le CK(\varepsilon)M_G^{m(1+h+\varepsilon)}\Big(\alpha^{-1}\Big(\varrho\beta\Big(\frac{m(1+h)}{(1-\varepsilon)|\sigma|}\Big)\Big)\Big). \tag{7}$$

The condition $\alpha(M_G^{-1}(e^x)) \in L_{si}$ implies $\alpha(M_G^{-1}(x^{m(1+h+\varepsilon)})) = (1+o(1))\alpha(M_G^{-1}(x))$ as $x \to +\infty$ and since $M_G^{-1} \in L_{si}$, $\alpha \in L^0$ and $\beta \in L_{si}$, from (7) we get

$$\varrho^0_{\alpha\beta}[F]_G \le \varlimsup_{\sigma\uparrow 0} \frac{\alpha\Big(M_G^{-1}\Big(CK(\varepsilon)M_G^{m(1+h+\varepsilon)}\Big(\alpha^{-1}\Big(\varrho\beta\Big(\frac{m(1+h)}{(1-\varepsilon)|\sigma|}\Big)\Big)\Big)\Big)\Big)}{\beta(1/|\sigma|)} = \varrho.$$

In view of the arbitrariness of ϱ we obtain the inequality $\varrho^0_{\alpha\beta}[F]_G \le \varrho^0$, which is obvious if $\varrho^0 = +\infty$. The first part of Theorem 1 is proved.

If function F_1 is dominant then (3) implies $\mu_F(\sigma) = (1+o(1))|c_{m0\dots0}|\mu_{F_1}(\sigma/m)^m$ as $\sigma \uparrow 0$. Therefore, $c_1\mu_{F_1}(\sigma/m)^m \le \mu_F(\sigma) \le c_2\mu_{F_1}(\sigma/m)^m$ for some $0 < c_1 < c_2 < \infty$, whence by Lemma 1

$$M_F(\sigma) \le K(\varepsilon)\mu_F\Big(\frac{1-\varepsilon}{1+h}\sigma\Big)^{1+h+\varepsilon} \le c_2K(\varepsilon)M_{F_1}\Big(\frac{1-\varepsilon}{m(1+h)}\sigma\Big)^{m(1+h+\varepsilon)}$$

and

$$M_F(\sigma) \ge \mu_F(\sigma) \ge c_1\mu_{F_1}(\sigma/m)^m \ge \frac{c_1}{K(\varepsilon)^m}M_{F_1}^{m/(1+h+\varepsilon)}\Big(\frac{1+h}{m(1-\varepsilon)}\sigma\Big).$$

Using the conditions $\alpha \in L^0$, $\beta \in L_{si}$ and $\alpha(M_G^{-1}(e^x)) \in L_{si}$ as above we obtain the equalities $\varrho^0_{\alpha\beta}[F]_G = \varrho^0_{\alpha\beta}[F_1]_G$ and $\lambda^0_{\alpha\beta}[F]_G = \lambda^0_{\alpha\beta}[F_1]_G$. □

Choosing $\alpha(x) = \beta(x) = \ln x$ for $x \ge x_0$ from (4) we obtain the definition of the relative logarithmic order $\varrho^0_l[F]_G$ and the relative lower logarithmic order $\lambda^0_l[F]_G$. Theorem 1 implies the following statement.

Corollary 1. *Let $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \ge 1$ of the functions $F_j \in S(\Lambda, 0)$, $\ln M_G^{-1}(e^x) \in L_{si}$ and (5) holds. Then $\varrho^0_l[F]_G \le \max\{\varrho^0_l[F_j]_G : 1 \le j \le p\}$. If among the functions F_j there is a dominant function F_1 then $\varrho^0_l[F]_G = \varrho^0_l[F_1]_G$ and $\lambda^0_l[F]_G = \lambda^0_l[F_1]_G$.*

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \geq 3$ then from (4) we obtain the definitions of the relative R-order $\varrho_R^0[F]_G = \overline{\lim\limits_{\sigma\uparrow 0}} |\sigma| \ln M_G^{-1}(M_F(\sigma))$, introduced by A.M. Gaisin [10]. Since here $\beta \notin L_{si}$, the corresponding corollary cannot be obtained from Theorem 1. However, from (9) it follows that

$$M_F(\sigma) \leq CK(\varepsilon) M_G^{m(1+h+\varepsilon)}\Big(\exp\Big\{\varrho\frac{m(1+h)}{(1-\varepsilon)|\sigma|}\Big\}\Big).$$

Therefore, if $\ln M_G^{-1}(e^x) \in L_{si}$ then as in the proof of Theorem 1 we get

$$\varrho_R^0[F]_G \leq \overline{\lim_{\sigma\uparrow 0}} |\sigma| \ln M_G^{-1}\Big(CK(\varepsilon) M_G^{m(1+h+\varepsilon)}\Big(\exp\Big\{\varrho\frac{m(1+h)}{(1-\varepsilon)|\sigma|}\Big\}\Big)\Big) = \frac{m(1+h)}{(1-\varepsilon)}\varrho.$$

In view of the arbitrariness of ε and ϱ, we get the following statement.

Proposition 1. *Let $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, 0)$, $\ln M_G^{-1}(e^x) \in L_{si}$ and (5) holds. Then*

$$\varrho_R^0[F]_G \leq m(1+h)\max\{\varrho_R^0[F_j]_G : 1 \leq j \leq p\}.$$

To replace condition (5) by a condition on the growth of the sequence Λ, we need the following lemma.

Lemma 2. *If $F \in S(\Lambda, 0)$, $G \in S(\Lambda, +\infty)$, $\alpha \in L_{si}$, $\beta \in L_{si}$ and*

$$\alpha(M_G^{-1}(n)) = o\Big(\beta\Big(\lambda_n/\ln n\Big)\Big), \quad n \to \infty, \tag{8}$$

then for every $\varepsilon \in (0, 1)$, $\delta > 0$ and all $\sigma_0(\varepsilon, \delta) \leq \sigma < 0$

$$M_F(\sigma) \leq \mu_F((1-\varepsilon)\sigma) M_G(\alpha^{-1}(\delta\beta(1/|\sigma|))). \tag{9}$$

Proof. It is clear that

$$M_F(\sigma) \leq \sum_{n=1}^{\infty} |f_n| e^{\sigma\lambda_n} = \sum_{n=1}^{\infty} |f_n| e^{(1-\varepsilon)\sigma\lambda_n} e^{\varepsilon\sigma\lambda_n} \leq \mu_F((1-\varepsilon)\sigma)\sum_{n=1}^{\infty} e^{-\varepsilon|\sigma|\lambda_n}. \tag{10}$$

From (8) it follows that $\alpha(M_G^{-1}(n)) \leq \delta_1\beta(\lambda_n/\ln n)$ for every $\delta_1 > 0$ and all $n \geq n_0(\delta_1)$ and, since $\beta \in L_{si}$, $\alpha(M_G^{-1}(n)) \leq \delta\beta(\varepsilon\lambda_n/(2\ln n))$ for every $\varepsilon > 0$, $\delta > 0$ and all $n \geq n_0(\varepsilon, \delta)$. Hence, we get

$$\lambda_n/\ln n \geq \frac{2}{\varepsilon}\beta^{-1}\Big(\alpha(M_G^{-1}(n))/\delta\Big). \tag{11}$$

Put $N(\sigma) = [M_G(\alpha^{-1}(\delta\beta(1/|\sigma|)))] + 1$. Then $N(\sigma) \geq n_0(\varepsilon, \delta)$ for all σ sufficiently close to 0 and

$$|\sigma| \geq 1/\beta^{-1}\Big(\alpha(M_G^{-1}(N(\sigma)))/\delta\Big) \geq 1/\beta^{-1}\Big(\alpha(M_G^{-1}(n))/\delta\Big)$$

for $n \geq N(\sigma)$. Therefore, in view of (11)

$$\sum_{n=N(\sigma)}^{\infty} e^{-\varepsilon|\sigma|\lambda_n} \leq \sum_{n=N(\sigma)}^{\infty} \exp\Big\{-\frac{\varepsilon\lambda_n}{\beta^{-1}\Big(\alpha(M_G^{-1}(n))/\delta\Big)}\Big\} \leq \sum_{n=N(\sigma)}^{\infty}\frac{1}{n^2}$$

and, thus,

$$\sum_{n=1}^{\infty} e^{-\varepsilon|\sigma|\lambda_n} \leq N(\sigma) - 1 + \sum_{n=N(\sigma)}^{\infty} e^{-\varepsilon|\sigma|\lambda_n} \leq M_G(\alpha^{-1}(\delta\beta(1/|\sigma|))) \tag{12}$$

for all σ sufficiently close to 0. From (10) and (12) we obtain (9). □

Using Lemma 2 we prove the following theorem.

Theorem 2. *Let $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, 0)$, $\alpha \in L_{si}$, $\beta \in L_{si}$, $\alpha(M_G^{-1}(e^x)) \in L_{si}$ and (8) holds. Then $\varrho^0_{\alpha\beta}[F]_G \leq \varrho^0 = \max\{\varrho^0_{\alpha\beta}[F_j]_G : 1 \leq j \leq p\}$.*

If among the functions F_j there is a dominant function F_1 then $\varrho^0_{\alpha\beta}[F]_G = \varrho^0_{\alpha\beta}[F_1]_G$ and $\lambda^0_{\alpha\beta}[F]_G = \lambda^0_{\alpha\beta}[F_1]_G$.

Proof. As in the proof of Theorem 1, for every $\varrho > \varrho^0$ and all $\sigma \in [\sigma_0(\varrho), 0)$ we obtain (6). Therefore, by Lemma 2 for all σ sufficiently close to 0 we get

$$M_F(\sigma) \leq CM_G^m\Big(\alpha^{-1}\Big(\varrho\beta\Big(\frac{m}{(1-\varepsilon)|\sigma|}\Big)\Big)\Big)M_G\Big(\alpha^{-1}\Big(\delta\beta\Big(\frac{1}{|\sigma|}\Big)\Big)\Big).$$

Due to the arbitrariness of δ, we can assume that $\delta < \varrho$. Hence, we obtain the following analogue of inequality (7)

$$M_F(\sigma) \leq CM_G^{m+1}\Big(\alpha^{-1}\Big(\varrho\beta\Big(\frac{m}{(1-\varepsilon)|\sigma|}\Big)\Big)\Big).$$

The further proof of the first part of Theorem 2 is the same as the proof of the first part of Theorem 1. The second part of Theorem 2 is proven in a similar way to the second part of Theorem 1. □

Note also that in view of Theorem 1 we can obtain analogs of Corollary 1 and Proposition 1.

3. Relative convergence classes. For $F \in S(\Lambda, 0)$ and functions $\alpha \in L$, $\beta \in L$ the *generalized convergence $\alpha\beta$-class* is defined [11] (see also [12, p.30]) by the condition

$$\int_{\sigma_0}^{0} \frac{\alpha(\ln M_F(\sigma))}{|\sigma|^2\beta(1/|\sigma|)}d\sigma < +\infty.$$

By analogy, we define a *relative generalized convergence $\alpha\beta$-class* $\mathfrak{C}_{\alpha\beta}$ by condition

$$\int_{\sigma_0}^{0} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{|\sigma|^2\beta(1/|\sigma|)}d\sigma < +\infty.$$

Theorem 3. *Let $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, 0)$, $\beta \in L^0$, $\alpha(M_G^{-1}(e^x)) \in L^0$ and (5) holds. If $F_j \in \mathfrak{C}_{\alpha\beta}$ for all j then $F \in \mathfrak{C}_{\alpha\beta}$.*

Proof. We will use such a property of class L^0 [13]: *if $\gamma \in L^0$ then $\gamma(cx) = O(\gamma(x))$ for every $c \in (0, +\infty)$ as $x \to +\infty$.*

Since $\mu_F(m\sigma) \leq \sum\limits_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}$, we have

$$\alpha(M_G^{-1}(\mu_F(m\sigma))) \leq \alpha\Big(M_G^{-1}\Big(\sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}\Big)\Big) =$$

$$= \alpha\Big(M_G^{-1}\Big(\exp\Big\{\ln \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|M_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot M_{F_p}(\sigma)^{k_p}\Big\}\Big)\Big) \leq$$

$$\leq \alpha\Big(M_G^{-1}\Big(\exp\Big\{\sum_{k_1+\dots+k_p=m} (k_1 \ln M_{F_1}(\sigma) + \dots + k_p \ln M_{F_p}(\sigma)) + K_1\Big\}\Big)\Big),$$

where $K_1 = \ln p + \sum\limits_{k_1+\dots+k_p=m} \ln |c_{k_1\dots k_p}|$. From hence by condition $\alpha(M_G^{-1}(e^x)) \in L^0$ we get

$$\alpha(M_G^{-1}(\mu_F(m\sigma))) \le K_2\alpha\Big(M_G^{-1}\Big(\exp\Big\{m \sum_{k_1+\dots+k_p=m} (\ln M_{F_1}(\sigma) + \cdot + \ln M_{F_p}(\sigma))\Big\}\Big)\Big) \le$$

$$\le K_2\alpha\Big(M_G^{-1}\Big(\exp\Big\{m \sum_{k_1+\dots+k_p=m} p\cdot \max\{\ln M_{F_j}(\sigma) : 1 \le j \le p\}\Big\}\Big)\Big) =$$

$$= K_2\alpha\Big(M_G^{-1}\Big(\exp\Big\{mpK_3 \max\{\ln M_{F_j}(\sigma) : 1 \le j \le p\}\Big\}\Big)\Big) \le$$

$$\le K_4\alpha\Big(M_G^{-1}\Big(\exp\Big\{\max\{\ln M_{F_j}(\sigma) : 1 \le j \le p\}\Big\}\Big)\Big) =$$

$$= K_4 \max\{\alpha\Big(M_G^{-1}\Big(M_{F_j}(\sigma)\Big)\Big) : 1 \le j \le p\} \le K_4 \sum_{j=1}^{p} \alpha\Big(M_G^{-1}\Big(M_{F_j}(\sigma)\Big)\Big),$$

where K_j are some positive constants. Thus,

$$\int_{\sigma_0}^{0} \frac{\alpha(M_G^{-1}(\mu_F(m\sigma)))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \le K_4 \sum_{j=1}^{p} \int_{\sigma_0}^{\infty} \frac{\alpha(M_G^{-1}(M_{F_j}(\sigma)))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma.$$

In view of the condition $\beta \in L^0$ from hence it follows that if $F_j \in \mathfrak{C}_{\alpha\beta}$ for all j then

$$\int_{\sigma_0}^{0} \frac{\alpha(M_G^{-1}(\mu_F(\sigma)))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty. \tag{13}$$

Finally, by Lemma 1 in view of condition $\alpha(M_G^{-1}(e^x)) \in L^0$ and $\beta \in L^0$ we get

$$\int_{\sigma_0}^{0} \frac{\alpha(M_G^{-1}(M_F(\sigma))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma) \le \int_{\sigma_0}^{0} \frac{\alpha\Big(M_G^{-1}\Big(K(\varepsilon)\mu_F\Big(\frac{1-\varepsilon}{1+h}\sigma\Big)^{1+h+\varepsilon}\Big)\Big)}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \le$$

$$\le K_5 \int_{\sigma_0}^{0} \frac{\alpha\Big(M_G^{-1}\Big(\mu_F\Big(\frac{1-\varepsilon}{1+h}\sigma\Big)\Big)\Big)}{|\sigma|^2\beta(1/|\sigma|)} d\sigma \le K_6 \int_{\sigma_0}^{0} \frac{\alpha(M_G^{-1}(\mu_F(\sigma)))}{|\sigma|^2\beta(1/|\sigma|)} d\sigma < +\infty,$$

i.e. $F \in \mathfrak{C}_{\alpha\beta}$. □

To study the growth of functions $F \in S(\Lambda, 0)$ of the generalized (α,β)-order $\varrho = \varrho^0_{\alpha\beta}[F]$ one can use the generalized convergence class defined in [5, 6] by the condition $\int_{\sigma_0}^{0} \frac{\ln M(\sigma,F)}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))} d\sigma < +\infty$. Therefore, in addition to class $\mathfrak{C}_{\alpha\beta}$, to study the growth of functions $F \in S(\Lambda, 0)$ one can use the relative generalized convergence class $\mathfrak{C}_{\alpha\beta,\varrho}$ defined by the condition

$$\int_{\sigma_0}^{0} \frac{M_G^{-1}(M_F(\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))} d\sigma < +\infty, \quad \varrho = \varrho^0_{\alpha\beta}[F]_G. \tag{14}$$

The analogue for $\mathfrak{C}_{\alpha\beta,\varrho}$ of Theorem 3 is the following statement.

Proposition 2. *Let $F\in S(\Lambda,0)$ with $\varrho^0_{\alpha\beta}[F]_G=\varrho\in(0,+\infty)$ be the Hadamard composition of genus $m\ge 1$ of the functions $F_j\in S(\Lambda,0)$ with $\varrho^0_{\alpha\beta}[F_j]_G=\varrho$. Suppose that $M^{-1}_G(e^x)\in L^0$, $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$ and (5) holds. If $F_j\in\mathfrak{C}_{\alpha\beta,\varrho}$ for all j then $F\in\mathfrak{C}_{\alpha\beta,\varrho}$.*

Proof. As in the proof of Theorem 3 now we obtain

$$\int\limits_{\sigma_0}^{0}\frac{M^{-1}_G(\mu_F(m\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma\le K_4\sum_{j=1}^{p}\int\limits_{\sigma_0}^{\infty}\frac{M^{-1}_G(M_{F_j}(\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma,$$

whence in view of the condition $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$ instead of (13) we have

$$\int\limits_{\sigma_0}^{0}\frac{M^{-1}_G(\mu_F(\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma<+\infty. \tag{15}$$

Finally, by Lemma 1 in view of condition $M^{-1}_G(e^x)\in L^0$ and $\alpha^{-1}(\varrho\beta(x))\in L^0$ as in the proof of Theorem 3 we get

$$\int\limits_{\sigma_0}^{0}\frac{M^{-1}_G(M_F(\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma)\le K_6\int\limits_{\sigma_0}^{0}\frac{M^{-1}_G(\mu_F(\sigma))}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma<+\infty,$$

i. e. $F\in\mathfrak{C}_{\alpha\beta,\varrho}$. □

In the proof of Theorem 3 and Proposition 2, condition (5) on the coefficients was significantly used. To replace it with a condition on exponents, we need the following addition to Lemma 2.

Lemma 3. *If $F\in S(\Lambda,0)$, $G\in S(\Lambda,+\infty)$, $\alpha\in L_{si}$, $\beta\in L_{si}$ and for some $\eta>0$*

$$\alpha\big((M^{-1}_G(n))^{1+\eta}\big)=o\big(\beta\big(\lambda_n/\ln n\big)\big),\quad n\to\infty, \tag{16}$$

then for every $\varepsilon\in(0,1)$, $\delta>0$ and all $\sigma_0(\varepsilon,\delta)\le\sigma<0$

$$M_F(\sigma)\le\mu_F((1-\varepsilon)\sigma)M_G\Big((\alpha^{-1}(\delta\beta(1/|\sigma|)))^{1/(1+\eta)}\Big) \tag{17}$$

Proof. As in the proof of Lemma 2, from (16) we get instead (11)

$$\lambda_n/\ln n\ge\frac{2}{\varepsilon}\beta^{-1}\big(\alpha(M^{-1}_G(n)^{1+\eta})/\delta\big).$$

Now we put $N(\sigma)=[M_G\big((\alpha^{-1}(\delta\beta(1/|\sigma|)))^{1/(1+\eta)}\big)]+1$. Then, as above,

$$|\sigma|\ge 1/\beta^{-1}\Big(\alpha\big((M^{-1}_G(N(\sigma))^{1+\eta}\big)/\delta\Big)\ge 1/\beta^{-1}\Big(\alpha(M^{-1}_G(n)^{1+\eta}/\delta\Big)$$

for $n\ge N(\sigma)$ and, therefore, $\sum_{n=N(\sigma)}^{\infty}\exp\{-\varepsilon|\sigma|\lambda_n\}\le\sum_{n=N(\sigma)}^{\infty}\exp\{-2\ln n\}$, whence

$$\sum_{n=1}^{\infty}e^{-\varepsilon|\sigma|\lambda_n}\le N(\sigma)-1+\sum_{n=N(\sigma)}^{\infty}e^{-\varepsilon|\sigma|\lambda_n}\le M_G(\alpha^{-1}(\delta\beta(1/|\sigma|))),$$

i.e. (10) implies (17). □

Theorem 4. *Let $F\in S(\Lambda,0)$ with $\varrho^0_{\alpha\beta}[F]_G=\varrho\in(0,+\infty)$ be the Hadamard composition of genus $m\ge 1$ of the functions $F_j\in S(\Lambda,0)$ with $\varrho^0_{\alpha\beta}[F_j]_G=\varrho$, $M^{-1}_G(e^x)\in L^0$ and $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$. Suppose that for some $\eta>0$ condition (16) holds and $\int_{x_0}^{\infty}\frac{dx}{(\alpha^{-1}(\varrho\beta(x)))^{\eta/(1+\eta)}}<+\infty$. If $F_j\in\mathfrak{C}_{\alpha\beta,\varrho}$ for all j then $F\in\mathfrak{C}_{\alpha\beta,\varrho}$.*

Proof. Since $\alpha^{-1}(\varrho\beta(x))\in L^0$ and $F_j\in\mathfrak{C}_{\alpha\beta,\varrho}$ for all j then as in the proof of Theorem 3 we get (15). On the other hand, since $M_G^{-1}(e^x)\in L^0$, by Lemma 3 as above for $\delta<\varrho$ we get

$$\begin{aligned}M_G^{-1}(M_F(\sigma))\le M_G^{-1}(\exp\{\ln\,\mu_F((1-\varepsilon)\sigma)+\ln\,M_G\Big((\alpha^{-1}(\delta\beta(1/|\sigma)))^{1/(1+\eta)}\Big)\})\le\\ \le K\max\{M_G^{-1}(\mu_F((1-\varepsilon)\sigma)),(\alpha^{-1}(\delta\beta(1/|\sigma)))^{1/(1+\eta)}\}\le\\ \le K(M_G^{-1}(\mu_F((1-\varepsilon)\sigma))+(\alpha^{-1}(\varrho\beta(1/|\sigma)))^{1/(1+\eta)}),\quad K=\text{const}>0.\end{aligned}\tag{18}$$

The condition $\int_{x_0}^{\infty}\frac{dx}{(\alpha^{-1}(\varrho\beta(x))^{\eta/(1+\eta)}}<+\infty$ implies

$$\int\limits_{\sigma_0}^{0}\frac{(\alpha^{-1}(\varrho\beta(1/|\sigma|)))^{1/(1+\eta)})}{|\sigma|^2\alpha^{-1}(\varrho\beta(1/|\sigma|))}d\sigma<+\infty.\tag{19}$$

Since $\alpha^{-1}(\varrho\beta(x))\in L^0$, from (18), (15) and (19) we obtain (14). □

Finally, consider the belonging of functions to the class $\mathfrak{C}_{\alpha\beta}$ if we impose a condition on the sequence Λ. To do this we need a slightly different version of Lemma 3.

Lemma 4. *If $F\in S(\Lambda,0)$, $G\in S(\Lambda,+\infty)$, $\alpha\in L_{si}$, $\beta\in L_{si}$ and for some $\eta>0$*

$$\alpha^{1+\eta}\big(M_G^{-1}(n)\big)=o\big(\beta\big(\lambda_n/\ln\,n\big)\big),\quad n\to\infty,\tag{20}$$

then for every $\varepsilon\in(0,1)$, $\delta>0$ and all $\sigma_0(\varepsilon,\delta)\le\sigma<0$

$$M_F(\sigma)\le\mu_F((1-\varepsilon)\sigma)M_G\big(\alpha^{-1}\big(\delta\beta^{1/(1+\eta)}(1/|\sigma|)\big)\big).\tag{21}$$

Proof. Indeed, as in the proofs of Lemma 2 and 3, from (20) we obtain

$$\lambda_n/\ln\,n\ge\frac{2}{\varepsilon}\beta^{-1}\big(\alpha^{1+\eta}(M_G^{-1}(n))/\delta\big).$$

Putting $N(\sigma)=[M_G(\alpha^{-1}((\delta\beta(1/|\sigma|))^{1/(1+\eta)}))]+1$, as above, we get

$$|\sigma|\ge1/\beta^{-1}\big(\alpha^{1+\eta}(M_G^{-1}(N(\sigma)))/\delta\big)\ge1/\beta^{-1}\big(\alpha^{1+\eta}(M_G^{-1}(n))/\delta\big)\quad\text{for }n\ge N(\sigma)$$

and, therefore, $\sum_{n=N(\sigma)}^{\infty}\exp\{-\varepsilon|\sigma|\lambda_n\}\le\sum_{n=N(\sigma)}^{\infty}\exp\{-2\ln\,n\}$, whence

$$\sum_{n=1}^{\infty}\exp\{-\varepsilon|\sigma|\lambda_n\}\le M_G\big(\alpha^{-1}\big((\delta\beta(1/|\sigma|))^{1/(1+\eta)}\big)\big),$$

i.e. (10) implies (21). □

Using Lemma 4 and repeating the proof of Theorem 4, we get the following statement.

Proposition 3. *Let $F\in S(\Lambda,0)$ be the Hadamard composition of genus $m\ge1$ of the functions $F_j\in S(\Lambda,0)$, $\alpha(M_G^{-1}(e^x))\in L^0$ and $\beta\in L^0$. Suppose that for some $\eta>0$ condition (20) holds and $\int_{x_0}^{\infty}\beta^{-\eta/(1+\eta)}(x)dx<+\infty$. If $F_j\in\mathfrak{C}_{\alpha\beta}$ for all j then $F\in\mathfrak{C}_{\alpha\beta}$.*

4. Open problem. In view of recent articles [14–17] about Dirichlet series with complex exponents it is naturally to extend the results of this paper to the case of an arbitrary sequence of complex exponents.

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