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THE NORMING SETS OF MULTILINEAR FORMS ON A CERTAIN NORMED SPACE $\mathbb{R}^n$

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Let $n,m \in \mathbb{N}, n,m \geq 2$ and E a Banach space. An element $(x_1,\dots,x_n) \in E^n$ is called a norming point of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \dots = \|x_n\| = 1$ and $|T(x_1,\dots,x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^nE)$, we define $\mathrm{Norm}(T)$ as the set of all $(x_1,\dots,x_n) \in E^n$ which are the norming points of T.

Let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm satisfying that $\{W_1,\dots,W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is $\{\pm W_1,\dots,\pm W_n\}$.

In the paper we characterize $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ as follows:
Let $T = (T(W_{i_1},\dots W_{i_m}))_{\substack{1\leq i_k\leq n,\\ 1\leq k\leq m}} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$, $\|T\| = 1$, $S_T = (b_{i_1\cdots i_m})_{\substack{1\leq i_k\leq n,\\ 1\leq k\leq m}} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ such that

$b_{i_1\cdots i_m} = T\big(W_{i_1},\dots W_{i_m}\big)$ if $|T\big(W_{i_1},\dots W_{i_m}\big)| = 1$ and $b_{i_1\cdots i_m} = 1$ if $|T\big(W_{i_1},\dots W_{i_m}\big)| < 1$,
and A is the Cartesian product of the set $\{1,\dots,n\}$, M is the set of indices $(i_1,\dots,i_m) \in A$ such that $|T\big(W_{i_1},\dots W_{i_m}\big)| < 1$. Then,

$$\mathrm{Norm}(T) = \bigcap_{(i_1,\dots,i_m)\in M} \bigcup_{j=1}^{m} \Big\{\Big(\sum_{1\leq i\leq n} s_i^{(1)}W_i,\dots,\sum_{1\leq i\leq n} s_i^{(j-1)}W_i, \sum_{1\leq i\leq n} s_i^{(j)}W_i - s_{i_j}W_{i_j},$$
$$\sum_{1\leq i\leq n} s_i^{(j+1)}W_i,\dots,\sum_{1\leq i\leq n} s_i^{(m)}W_i\Big) \colon \Big(\sum_{1\leq i\leq n} s_i^{(1)}W_i,\dots,\sum_{1\leq i\leq n} s_i^{(m)}W_i\Big) \in \mathrm{Norm}(S_T)\Big\}.$$

1. Introduction. In 1961 Bishop and Phelps [3] showed that the set of norm-attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. Lindenstrauss [14] studied norm-attaining operators. The problem of denseness of norm-attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm-attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [2], where they showed that the Radon-Nikodym Property is sufficient for the denseness of norm-attaining multilinear forms. Choi and Kim [4] showed that the Radon-Nikodym Property is also sufficient for the denseness of norm-attaining polynomials. Acosta [1] studied norm attaining multilinear mappings. Jiménez-Sevilla and Payá [7] studied the denseness of norm-attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Payá and Saleh [15] presented new sufficient conditions for the denseness of norm-attaining multilinear forms. Note that

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the norm denseness problem of the set of norm-attaining forms in the spaces of all continuous multilinear forms is closely related to sets with the Radon-Nikodym Property. It is also linked to the broader topic of optimization on infinite dimensional normed spaces and variational principles (see Stegall [16], Finet and Georgiev [6]).

Let $n \in \mathbb{N},\ n \geq 2$ and E a Banach space. We write S_E for the unit sphere of E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1,\cdots,x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1,\ldots,x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1,\ldots,x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\mathrm{Norm}(T) = \Big\{(x_1,\ldots,x_n) \in E^n \colon (x_1,\ldots,x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1,\ldots,x_n) \in \mathrm{Norm}(T)$ if and only if $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$. Notice that the norming sets can be empty, finite or infinite (see examples of [12]). If $\mathrm{Norm}(T) \neq \varnothing$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form (see [4]). For more details about the theory of multilinear mappings on a Banach space, we refer to [5].

For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = \ell_1^m$ or ℓ_∞^2 and $T \in \mathcal{L}(^nE)$, $\mathrm{Norm}(T) \neq \varnothing$ since S_E is compact. Kim [8, 9, 10, 12] classified Norm(T) for every $T \in \mathcal{L}_s(^2\ell_\infty^2), \mathcal{L}(^2\ell_\infty^2), \mathcal{L}(^2\ell_1^2), \mathcal{L}_s(^2\ell_1^3)$ or $\mathcal{L}_s(^3\ell_1^2)$. Kim [13] classified Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x,y)\|_{h(w)} = \max\{|y|, |x| + (1-w)|y|\}$.

Let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm satisfying that $\{W_1,\ldots,W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is $\{\pm W_1,\ldots,\pm W_n\}$.

In this paper, we characterize Norm(T) for every $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$.

2. Results. In this paper, we assume $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with the norm that $\{W_1,\ldots,W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is the set $\{\pm W_1,\ldots,\pm W_n\}$.

Example 1. $\mathbb{R}^n_{\|\cdot\|} = \ell_1^n$ with $W_j = e_j$ for $j \in \{1,\ldots,n\}$.

Example 2. Let $0 \leq \theta \leq \frac{\pi}{4}$ and $\mathbb{R}^2_{\|\cdot\|_\theta} := \ell^2_{\infty,\theta} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x,y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}$$

with $W_1 = (\cos\theta - \sin\theta,\ \cos\theta + \sin\theta),\ W_2 = (\cos\theta + \sin\theta,\ -\cos\theta + \sin\theta)$ (see [11]).

Note that $\mathbb{R}^2_{\|\cdot\|_\theta}$ and ℓ_1^2 are isometric isomorphic with $\|(x,y)\|_{(\infty,0)} = \|(x,y)\|_\infty$ and $\|(x,y)\|_{(\infty,\pi/4)} = \frac{1}{\sqrt{2}}\|(x,y)\|_1$.

For $m,n \geq 2$, let $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$. By m-linearity of T

$$T\Bigg(\sum_{1\leq i_1\leq n} x^{(1)}_{i_1} W_{i_1},\ldots,\sum_{1\leq i_m\leq n} x^{(m)}_{i_m} W_{i_m}\Bigg) = \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} T\big(W_{i_1},\ldots W_{i_m}\big)\, x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}$$

for some $x^{(k)}_{i_k} \in \mathbb{R}$ $(k \in \{1,\ldots,m\}, i_k \in \{1,\ldots,n\})$.

Recall that the Krein-Milman Theorem states that a compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points. Using the Krein-Milman Theorem, we presents an explicit formula of $\|T\|$ for every $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$.

Theorem A. *Let $m, n \geq$. If $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$, then*
$$\|T\| = \max\{\big|T\big(W_{i_1}, \dots W_{i_m}\big)\big| \colon 1 \leq i_k \leq n,\ 1 \leq k \leq m\}.$$

Proof. Let $M := \max\{\big|T\big(W_{i_1}, \dots W_{i_m}\big)\big| \colon 1 \leq i_k \leq n,\ 1 \leq k \leq m\}$. Let $(X_1, \dots, X_m) \in S_{\mathbb{R}^n_{\|\cdot\|}}$. By the Krein-Milman Theorem, the closed unit ball of $\mathbb{R}^n_{\|\cdot\|}$ is the closed convex hull of $\{\pm W_1, \dots, \pm W_n\}$. For $k \in \{1, \dots, m\}$, there are $\delta_i^{(k)} \in \{-1, 1\}$ and $t_i^{(k)} \geq 0$ with $\sum_{i=1}^n t_i^{(k)} \leq\ 1$ such that
$$X_k = \sum_{1 \leq i \leq n} \delta_i^{(k)} t_i^{(k)} W_i.$$
It follows that
$$\begin{aligned}
|T(X_1, \dots, X_m)| &\leq \Big|T\Big(\sum_{1\leq i_1\leq n} \delta_{i_1}^{(1)} t_{i_1}^{(1)} W_{i_1}, \dots, \sum_{1\leq i_m\leq n} \delta_{i_m}^{(m)} t_{i_m}^{(m)} W_{i_m}\Big)\Big| \leq \\
&\leq \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} \big|T\big(W_{i_1}, \dots W_{i_m}\big)\big|\ |\delta_{i_1}^{(1)} t_{i_1}^{(1)}| \cdots |\delta_{i_m}^{(m)} t_{i_m}^{(m)}| \leq \\
&\leq M \sum_{1\leq i_k\leq n,\ 1\leq k\leq m} t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} = M\Big(\sum_{1\leq j\leq n} t_j^{(1)}\Big) \cdots \Big(\sum_{1\leq j\leq n} t_j^{(m)}\Big) \leq M \\
&= \max\Big\{\big|T\big(W_{i_1}, \dots W_{i_m}\big)\big| \colon 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\} \leq \|T\|,
\end{aligned}$$
which shows that
$$\|T\| = \sup_{(X_1,\dots,X_m)\in S_{\mathbb{R}^n_{\|\cdot\|}}} |T(X_1, \dots, X_m)| \leq M \leq \|T\|.$$
Therefore, $\|T\| = M$. □

By simplicity we denote $T = (T(W_{i_1}, \dots W_{i_m}))_{\substack{1\leq i_k\leq n,\\ 1\leq k\leq m}}$. We call $T\big(W_{i_1}, \dots W_{i_m}\big)'$s the *coefficients* of T. Note that if $\|T\| = 1$, then $\big|T\big(W_{i_1}, \dots W_{i_m}\big)\big| \leq 1$ for all $1 \leq i_k \leq n$, $1 \leq k \leq m$.

We introduce some notations to make it easier to read as follows:

Let $W := (W_1, \dots, W_n) \in (\mathbb{R}^n_{\|\cdot\|})^n$, $x^{(k)} := (x_1^{(k)}, \dots, x_n^{(k)}) \in \mathbb{R}^n$ $(1 \leq k \leq m)$ and $x := (x^{(1)}, \dots, x^{(m)}) \in (\mathbb{R}^n)^m$. We define for $1 \leq k \leq m$, $(x^{(k)}, W) := \sum_{1\leq i_k\leq n} x_{i_k}^{(k)} W_{i_k} \in \mathbb{R}^n_{\|\cdot\|}$ and $(\langle x, W\rangle) := ((x^{(1)}, W), \dots, (x^{(m)}, W)) \in (\mathbb{R}^n_{\|\cdot\|})^m$.

Theorem B. *Let $n, m \geq 2$, $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$, $s^{(k)} = (s_1^{(k)}, \dots, s_n^{(k)}) \in \mathbb{R}^n$ $(1 \leq k \leq m)$ and $s = (s^{(1)}, \dots, s^{(m)}) \in (\mathbb{R}^n)^m$. Suppose that $(\langle s, W\rangle) \in \mathrm{Norm}(T)$. If $\big|T\big(W_{i_1'}, \dots W_{i_m'}\big)\big| < \|T\|$ for some $1 \leq i_k' \leq n,\ 1 \leq k \leq m$, then $s_{i_1'}^{(1)} = 0$ or $\cdots$ or $s_{i_m'}^{(m)} = 0$.*

Proof. Let $\delta > 0$ be such that $\big|T\big(W_{i_1'}, \dots W_{i_m'}\big)\big| + \delta < \|T\|$. We define $T_\pm \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ be such that $T_\pm(\langle x, W\rangle) = T(\langle x, W\rangle) \pm \delta x_{i_1'}^{(1)} \cdots x_{i_m'}^{(m)}$. By Theorem A, $\|T_\pm\| = \|T\|$. It follows that
$$\begin{aligned}
\|T\| \geq \max\Big\{\Big|T_+(\langle s, W\rangle)\Big|,\ \Big|T_-(\langle s, W\rangle)\Big|\Big\} &= \Big|T(\langle s, W\rangle)\Big| + \delta\Big|s_{i_1'}^{(1)} \cdots s_{i_m'}^{(m)}\Big| = \\
&= \|T\| + \delta\Big|s_{i_1'}^{(1)} \cdots s_{i_m'}^{(m)}\Big|,
\end{aligned}$$
which implies that $s_{i_1'}^{(1)} \cdots s_{i_m'}^{(m)} = 0$. □

Theorem C. *Let* $n, m \geq 2$, $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$, $\delta_{i_1\cdots i_m} = 1$ *if* $\left|T\big(W_{i_1}, \dots W_{i_m}\big)\right| = 1$ *and* $\delta_{i_1\cdots i_m} = 0$ *if* $\left|T\big(W_{i_1}, \dots W_{i_m}\big)\right| < 1$. *If* $T_\delta = (\delta_{i_1\cdots i_m}\ T\big(W_{i_1}, \dots W_{i_m}\big))_{1\leq i_k\leq n, 1\leq k\leq m} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$, *then,* $\text{Norm}(T) = \text{Norm}(T_\delta)$.

Proof. Note that $\|T_\delta\| = \|T\| = 1$ by Theorem A.

($\subseteq$). Let $(\langle s, W\rangle) \in \text{Norm}(T)$. Then

$$\|T_\delta\| = 1 = \Big|T(\langle s, W\rangle)\Big| = \Bigg| \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} +$$

$$+ \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|=1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg| =$$

$$= \Bigg| \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|=1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg| \overset{\text{(by Theorem B)}}{=}$$

$$= \Bigg| \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} \delta_{i_1\cdots i_m}\ T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} +$$

$$+ \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|=1} \delta_{i_1\cdots i_m}\ T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg| = |T_\delta(\langle s, W\rangle)|.$$

Thus, $(\langle s, W\rangle) \in \text{Norm}(T_\delta)$.

($\supseteq$). Let $(\langle s, W\rangle) \in \text{Norm}(T_\delta)$. Write

$$T(\langle x, W\rangle) = T_\delta(\langle x, W\rangle) + \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}.$$

Let $T_- \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ be such that

$$T_-(\langle x, W\rangle) = T_\delta(\langle x, W\rangle) - \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}.$$

By Theorem A, $\|T_-\| = 1$. It follows that

$$1 \geq \Big|T(\langle s, W\rangle)\Big| = \Bigg|T_\delta(\langle s, W\rangle) + \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg|,$$

$$1 \geq \Big|T_-(\langle s, W\rangle)\Big| = \Bigg|T_\delta(\langle s, W\rangle) - \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg|,$$

which implies that

$$1 \geq \Big|T_\delta(\langle s, W\rangle)\Big| + \Bigg| \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1}, \dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m}\Bigg| =$$

$$= 1 + \left| \sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1},\dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} \right|.$$

Thus,

$$\sum_{\left|T\left(W_{i_1},\dots W_{i_m}\right)\right|<1} T\big(W_{i_1},\dots W_{i_m}\big)\ s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} = 0$$

and so $1 = \big|T_\delta(\langle s, W\rangle)\big| = \big|T(\langle s, W\rangle)\big|$. Thus, $(\langle s, W\rangle) \in \mathrm{Norm}(T)$. □

The following shows that we can classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ with $\|T\| = 1$ if we have known $\mathrm{Norm}(S)$ for every $S = (b_{i_1\cdots i_m})_{1\le i_k\le n, 1\le k\le m} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ such that $\|S\| = 1 = |b_{i_1\cdots i_m}|$ for every $1 \le i_k \le n, 1 \le k \le m$.

Theorem D. *Let* $m, n \ge 2$, $T = (T\big(W_{i_1},\dots W_{i_m}\big))_{1\le i_k\le n,1\le k\le m} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ *with* $\|T\| = 1$, $S_T = (b_{i_1\cdots i_m})_{1\le i_k\le n,1\le k\le m} \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ *by* $b_{i_1\cdots i_m} = T\big(W_{i_1},\dots W_{i_m}\big)$ *if* $|T\big(W_{i_1},\dots W_{i_m}\big)| = 1$ *and* $b_{i_1\cdots i_m} = 1$ *if* $|T\big(W_{i_1},\dots W_{i_m}\big)| < 1$, $A = \{1,\dots,n\}\times\cdots\times\{1,\dots,n\}$ *and* $M = \{(i_1,\dots,i_m) \in A\colon |T\big(W_{i_1},\dots W_{i_m}\big)| < 1\}$, $s^{(k)} := (s^{(k)}_1,\dots,s^{(k)}_n) \in \mathbb{R}^n$ $(1 \le k \le m)$ *and* $s := (s^{(1)},\dots,s^{(m)}) \in (\mathbb{R}^n)^m$. *Then,*

$$\begin{aligned}&\mathrm{Norm}(T) = \bigcap_{(i_1,\dots,i_m)\in M} \Big\{\Big((s^{(1)}, W) - s_{i_1}W_{i_1}, (s^{(2)}, W),\dots,(s^{(m)}, W)\Big),\\ &\Big((s^{(1)}, W), (s^{(2)}, W) - s_{i_2}W_{i_2}, (s^{(3)}, W),\dots,(s^{(m)}, W)\Big),\cdots,\\ &\Big((s^{(1)}, W), (s^{(1)}, W),\dots,(s^{(m-1)}, W), (s^{(m)}, W) - s_{i_m}W_{i_m}\Big)\colon (\langle s, W\rangle) \in \mathrm{Norm}(S_T)\Big\}.\end{aligned}$$

Proof. Let

$$\begin{aligned}&\mathcal{F} = \bigcap_{(i_1,\dots,i_m)\in M} \Big\{\Big((s^{(1)}, W) - s_{i_1}W_{i_1}, (s^{(2)}, W),\dots,(s^{(m)}, W)\Big),\\ &\Big((s^{(1)}, W), (s^{(2)}, W) - s_{i_2}W_{i_2}, (s^{(3)}, W),\dots,(s^{(m)}, W)\Big),\cdots,\\ &\Big((s^{(1)}, W), (s^{(1)}, W),\dots,(s^{(m-1)}, W), (s^{(m)}, W) - s_{i_m}W_{i_m}\Big)\colon (\langle s, W\rangle) \in \mathrm{Norm}(S_T)\Big\}.\end{aligned}$$

We will show that $\mathrm{Norm}(T) = \mathcal{F}$. Note that by Theorem A, $\|S_T\| = 1$.

($\subseteq$). Let $(\langle s, W\rangle) \in \mathrm{Norm}(T)$. Let $(i_1,\dots,i_m) \in M$ be fixed.
Note that $\|(s^{(k)}, W) - s_{i_k}W_{i_k}\| \le 1$ for $k \in \{1,\dots,m\}$. It follows that

$$\begin{aligned}\Big|S_T(\langle s, W\rangle)\Big| &= \left| \sum_{(i_1\cdots i_m)\in M} s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} + \sum_{(i_1\cdots i_m)\notin M} T\big(W_{i_1},\dots W_{i_m}\big) s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} \right| =\\ &= \left| \sum_{(i_1\cdots i_m)\notin M} T\big(W_{i_1},\dots W_{i_m}\big) s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} \right| \overset{\text{(by Theorem B)}}{=}\\ &= \left| \sum_{(i_1\cdots i_m)\in M} T\big(W_{i_1},\dots W_{i_m}\big) s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} + \sum_{(i_1\cdots i_m)\notin M} T\big(W_{i_1},\dots W_{i_m}\big) s^{(1)}_{i_1}\cdots s^{(m)}_{i_m} \right| =\end{aligned}$$

$$\overset{\text{(by Theorem B)}}{=} \Big|T(\langle s, W\rangle)\Big| = \|T\| = 1,$$

so $(\langle s, W\rangle) \in \text{Norm}(S_T)$ satisfying $s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} = 0$ if $|a_{i_1\cdots i_m}| < 1$. Thus, for $k \in \{1, \ldots, m\}$,

$$\Big((s^{(1)}, W), \cdots, (s^{(k-1)}, W), (s^{(k)}, W) - s_{i_k} W_{i_k}, (s^{(k+1)}, W), \ldots, (s^{(m)}, W)\Big) \in \mathcal{F}.$$

$(\supseteq)$. Let $((s^{(1)}, W), \cdots, (s^{(k-1)}, W), (s^{(k)}, W) - s_{i_k} W_{i_k}, (s^{(k+1)}, W), \ldots, (s^{(m)}, W)) \in \mathcal{F}$ for some $k \in \{1, \ldots, m\}$. Note that

$$1 = \|T\| = \Big|T\Big((s^{(1)}, W), \cdots, (s^{(k-1)}, W), (s^{(k)}, W) - s_{i_k} W_{i_k}, (s^{(k+1)}, W), \ldots, (s^{(m)}, W)\Big)\Big| =$$
$$= \Big|S_T\Big((s^{(1)}, W), \cdots, (s^{(k-1)}, W), (s^{(k)}, W) - s_{i_k} W_{i_k}, (s^{(k+1)}, W), \ldots, (s^{(m)}, W)\Big)\Big| = 1,$$

which implies that

$$\Big((s^{(1)}, W), \cdots, (s^{(k-1)}, W), (s^{(k)}, W) - s_{i_k} W_{i_k}, (s^{(k+1)}, W), \ldots, (s^{(m)}, W)\Big) \in \text{Norm}(T).$$

□

Theorem E. *Let $m, n \geq 2$, $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|})$ with $\|T\| = 1$, $e = (e_1, \ldots, e_n)$ and $L_T \in \mathcal{L}(^m l_1^n)$ be such that*

$$L_T(\langle x, e\rangle) = \sum_{1\leq i_k\leq n, 1\leq k\leq m} T\big(W_{i_1}, \ldots W_{i_m}\big)\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}.$$

Then,

$$\text{Norm}(T) = \Big\{(\langle s, W\rangle) \in (S_{\mathbb{R}^n_{\|\cdot\|}})^m \colon (\langle s, e\rangle) \in \text{Norm}(L_T)\Big\}.$$

Proof. Let $\mathcal{M} = \{(\langle s, W\rangle) \in (S_{\mathbb{R}^n_{\|\cdot\|}})^m \colon (\langle s, e\rangle) \in \text{Norm}(L_T)\}$. We will show that $\text{Norm}(T) = \mathcal{M}$. Note that $\|L_T\| = \|T\| = 1$ by Theorem A.

$(\subseteq)$. Let $(\langle s, W\rangle) \in \text{Norm}(T)$. Then $\sum_{1\leq i\leq n} |s_i^{(k)}| \leq 1$ for $k \in \{1, \ldots, m\}$. It follows that

$$1 = \Big|T(\langle s, W\rangle)\Big| = \Bigg|\sum_{1\leq i_k\leq n, 1\leq k\leq m} T\big(W_{i_1}, \ldots W_{i_m}\big)\ s_{i_1}^{(1)} \cdots s_{i_m}^{(m)}\Bigg| = \Big|L_T(\langle s, e\rangle)\Big|,$$

which shows that $(\langle s, e\rangle) \in \text{Norm}(L_T)$. Thus, $(\langle s, W\rangle) \in \mathcal{M}$.

$(\supseteq)$. Let $(\langle s, W\rangle) \in \mathcal{M}$. It follows that

$$\Big|T(\langle s, W\rangle)\Big| = \Bigg|\sum_{1\leq i_k\leq n, 1\leq k\leq m} T\big(W_{i_1}, \ldots W_{i_m}\big)\ s_{i_1}^{(1)} \cdots s_{i_m}^{(m)}\Bigg| = \Big|L_T(\langle s, e\rangle)\Big| = \|L_T\| = 1.$$

Hence, $(\langle s, W\rangle) \in \text{Norm}(T)$. □

Theorem F. *Let $T \in \mathcal{L}(^n\mathbb{R}^2_{\|\cdot\|})$. Define $S_T \in \mathcal{L}(^n l_1^2)$ by*

$$S_T\Big(t_1^{(1)} e_1 + t_2^{(1)} e_2, \ldots, t_1^{(n)} e_1 + t_2^{(n)} e_2\Big) = T\Big(t_1^{(1)} W_1 + t_2^{(1)} W_2, \ldots, t_1^{(n)} W_1 + t_2^{(n)} W_2\Big),$$

where $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$. *The following assertions hold:*

(*a*) $\|T\|_{\mathcal{L}(^n\mathbb{R}^2_{\|\cdot\|})} = \|S_T\|_{\mathcal{L}(^n l_1^2)}$;

(*b*) $$\mathrm{Norm}(T) = \Big\{ \Big(t_1^{(1)} W_1 + t_2^{(1)} W_2, \ldots, t_1^{(n)} W_1 + t_2^{(n)} W_2 \Big) : \Big(t_1^{(1)} e_1 + t_2^{(1)} e_2, \ldots, t_1^{(m)} e_1 + t_2^{(m)} e_2 \Big) \in \mathrm{Norm}(S_T) \Big\}.$$

Proof. It is immediate. □

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