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INFINITE LOCALLY FINITE GROUPS WITH GIVEN PROPERTIES OF THE NORM OF ABELIAN NON-CYCLIC SUBGROUPS

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In the paper the properties of infinite locally finite groups with non-Dedekind locally nilpotent norms of Abelian non-cyclic subgroups are studied. It is proved that such groups are finite extensions of a quasicyclic subgroup and contain Abelian non-cyclic p-subgroups for a unique prime p. In particular, in the paper is prove the following assertions:

1) Let G be an infinite locally finite group and contain the locally nilpotent norm N_G^A with the non-Hamiltonian Sylow p-subgroup $(N_G^A)_p$. Then G is a finite extension of a quasicyclic p-subgroup, all Sylow p'-subgroups are finite and do not contain Abelian non-cyclic subgroups. In particular, Sylow q-subgroups (q is an odd prime, $q \in \pi(G)$, $q \neq p$) are cyclic, Sylow 2-subgroups ($p \neq 2$) are either cyclic or finite quaternion 2-groups (Theorem 1).

2) Let G be a locally finite non-locally nilpotent group with the infinite locally nilpotent non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. Then $G = G_p \rtimes H$, where G_p is an infinite $\overline{HA}_p$-group of one of the types (1)–(4) of Proposition 2 in present paper, which coincides with the Sylow p-subgroup of the norm N_G^A, H is a finite group, all Abelian subgroups of which are cyclic, and $(|H|, p) = 1$. Any element $h \in H$ of odd order that centralizes some Abelian non-cyclic subgroup $M \subset N_G^A$ is contained in the centralizer of the norm N_G^A. (Theorem 2).

3) Let G be an infinite locally finite non-locally nilpotent group with the finite nilpotent non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. Then $G = H \rtimes K$, where H is a finite group, all Abelian subgroups of which are cyclic, $(|H|, 2) = 1$, K is an infinite 2-group of one of the types (5)–(6) of Proposition 2 (in present paper). Moreover, the norm N_K^A of Abelian non-cyclic subgroups of the group K is finite, $K \cap N_G^A = N_K^A$ and coincides with the Sylow 2-subgroup $(N_G^A)_2$ of the norm N_G^A of a group G. Moreover, any element $h \in H$ of the centralizer of some Abelian non-cyclic subgroup $M \subset N_G^A$ is contained in the centralizer of the norm N_G^A. (Theorem 4).

1. Introduction. The year 1935, when R. Baer first introduced the norm of a group, marked the beginning of a new direction in group theory research. According to [1], the norm $N(G)$ of a group G is the intersection of the normalizers of all subgroups of a group. Since the norm $N(G)$ normalizes all subgroups of a group, all subgroups are normal in $N(G)$. If a group coincides with its norm $N(G)$, then all subgroups are normal in a group. The structure of such groups was known at that time. These are so-called Dedekind groups, which are either Abelian or Hamiltonian. The innovation of R. Baer was that he began to investigate the case when the norm $N(G)$ is a proper subgroup of a group G.

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Generalizing the problem formulated by R. Baer, let's consider a system Σ of subgroups of a group G that possess some theoretical group property. Accordingly, the Σ-norm of a group G is defined as the intersection of the normalizers of all subgroups from the system Σ in a group G. It is clear that the norm of a group is contained in all other Σ-norms, which, in turn, can be considered as its generalizations.

It is evident that in a group, which coincides with its Σ-norm, all subgroups from Σ are normal (under the condition that the system of such groups is non-empty). Groups with systems Σ of normal subgroups were studied quite actively since the end of the 19th century. Under these conditions the structure and properties of many natural systems Σ of subgroups (in particular, the system Σ of all subgroups of a group, all non-cyclic subgroups, all non-cyclic Abelian subgroups, infinite Abelian subgroups, infinite cyclic subgroups, non-Abelian subgroups, etc.) were described. In this regard, it would be natural to raise the question of studying the properties of groups with the proper Σ-norm, which satisfies some restrictions.

This research direction proved to be so fruitful that algebraists often turned to studying groups with restrictions on various Σ-norms (see, for example, [4–6], [11–23], [25]). In this article, one of these generalized norms – the norm of Abelian non-cyclic subgroups of a group – and its impact on the properties of a group under certain restrictions are considered.

The norm N_G^A of Abelian non-cyclic subgroups of a group G is defined as the intersection of the normalizers of all Abelian non-cyclic subgroups of a group, provided that the system of such subgroups is non-empty (see [21]). If the norm N_G^A contains at least one Abelian non-cyclic subgroup, then all such subgroups are normal in N_G^A. Non-Abelian groups with this property were characterized by F. Lyman [10] and called $\overline{HA}$-groups (accordingly, $\overline{HA_p}$-groups, if they are p-groups). Thus, if the norm of Abelian non-cyclic subgroups contains an Abelian non-cyclic subgroup, then it is either Dedekind or a non-Hamiltonian $\overline{HA}$-group.

The study of the properties of the norm of Abelian non-cyclic subgroups, its impact on the properties of a group and its relations with other generalized norms under given restrictions were considered in findings [12,13], [16,17], [19], [21–23]. The non-Dedekindness of the norm of a group was chosen as a defining restriction. In particular, in the findings [21–23] locally finite p-groups were studied, while in [17] torsion locally nilpotent groups that contain an Abelian non-cyclic subgroup and have the non-Dedekind norm of Abelian non-cyclic subgroups were considered. Under the additional condition of infiniteness such groups are finite extensions of a quasicyclic subgroup.

In the findings [12, 13] the relations between the norm N_G^A of Abelian non-cyclic subgroups and the norm N_G^d of decomposable subgroups of a group were studied in locally finite and non-periodic locally-by-solvable groups. It was proved that under these restrictions one of the inclusions $N_G^A \supseteq N_G^d$ or $N_G^d \supseteq N_G^A$ holds (Theorem 1.2 and 1.3 [12]).

The paper purpose is to study properties of infinite locally finite groups with the locally nilpotent non-Dedekind norm of Abelian non-cyclic subgroups.

As will be shown further, all such groups are finite extensions of a quasicyclic subgroup and contain Abelian non-cyclic p-subgroups for a unique prime p. Moreover, such subgroups are semidirect products of an infinite Sylow p-subgroup, which is either coincides with a Sylow p-subgroup of the norm N_G^A or contains it, and a finite group, all Abelian subgroups of which are cyclic.

2. Preliminary results. The intersection of normalizers of all Abelian non-cyclic subgroups of a group G under the condition that the system of such subgroups is non-empty is called *the norm* N_G^A *of Abelian non-cyclic subgroups* of a group G.

If a group G does not contain Abelian non-cyclic subgroups, then we will claim that $G = N_G^A$. But, only groups, which contain at least one Abelian non-cyclic subgroup, will be considered further. Clearly, this condition holds for infinite locally finite groups because such groups contain an infinite (so non-cyclic) Abelian subgroup by the Kargapolov-Hall-Kulatilaka theorem (see [7, 8]).

In a group G that contains an Abelian non-cyclic subgroup and coincides with the norm N_G^A of Abelian non-cyclic subgroups all Abelian non-cyclic subgroups are normal, so it is either Dedekind or a non-Hamiltonian $\overline{HA}$-group. Properties and the structure of torsion non-primary locally nilpotent $\overline{HA}$-groups are described in the following proposition.

Proposition 1 ([10]). *A torsion locally nilpotent non-Hamiltonian group G is a $\overline{HA}$-group if and only if $G = G_p \times B$, where G_p is a Sylow p-subgroup of a group G, which is non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind p'-group, all Abelian subgroups of which are cyclic.*

Thus, if the norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, contains an Abelian non-cyclic subgroup and is a locally nilpotent $\overline{HA}$-group, then it is of the structure mentioned in Proposition 1. But, the subgroup N_G^A can contain no Abelian non-cyclic subgroups. The example of such a group is the infinite torsion Frobenius group mentioned in [2] (Example 3.4), where $N_G^A = E$.

Let's prove that the norm N_G^A is a non-Hamiltonian $\overline{HA}$-group mentioned in Proposition 1 by its non-Dedekindness and locally nilpotency in the class of infinite locally finite groups.

Lemma 1. *If the norm N_G^A of Abelian non-cyclic subgroups of an infinite locally finite group G is non-Dedekind and locally nilpotent, then it is a non-Hamiltonian $\overline{HA}$-group.*

Proof. Let group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy lemma conditions. If N_G^A contains Abelian non-cyclic subgroups, then they are normal in it and in this case N_G^A is a non-Hamiltonian $\overline{HA}$-group.

Suppose that N_G^A does not contain Abelian non-cyclic subgroups. Since N_G^A is locally nilpotent, by Proposition 1.4 [3] it is the direct product of its Sylow p-subgroups. By the non-Dedekindness of N_G^A, Lemma 3 [17] and the assumption, N_G^A is the direct product of a finite quaternion 2-group Q of order greater than 8 and a cyclic Sylow $2'$-subgroup $\langle h\rangle$:

$$N_G^A = Q \times \langle h\rangle,$$

where $Q = \langle a\rangle\langle b\rangle$, $|a| = 2^n$, $n \geq 3$, $|b| = 4$, $a^{2^{n-1}} = b^2$, $b^{-1}ab = a^{-1}$, $|h| = m$, $(m, 2) = 1$.

Since G is infinite, it contains an infinite Abelian subgroup A (see for instance [9] p. 499). Let's prove that A satisfies the minimal condition for Abelian subgroups. Indeed, it contains the direct product M of infinitely many subgroups of prime order. Hence, $|N_G^A \cap M| < \infty$.

Let $N_G^A \cap M = M_1$. Then $M = M_1 \times M_2$, where $|M_2| = \infty$. Since $N_G^A \cap M_2 = E$, by Lemma 2 [17] the norm N_G^A must be Dedekind, which contradicts the condition. Therefore, A is a group satisfying the minimal condition for Abelian subgroups (and by [24] for all subgroups), hence, it is a finite extension of the direct product of a finitely many quasicyclic subgroups.

Let the divisible part of the subgroup A denote by P. Then P is the direct product of quasicyclic subgroups. By $\left|N_G^A\right| < \infty$, $N_G^A \lhd G$, it follows that $\left|G : C_G(N_G^A)\right| < \infty$ and $P \subset C_G(N_G^A)$. Therefore, P is contained in the center of the group $G_1 = P \cdot N_G^A$. By Lemma 1 [17] $G_1 = N_{G_1}^A$ and G_1 is a non-Hamiltonian $\overline{HA}$-group. By the description of non-Hamiltonian $\overline{HA}$-groups (see [10]), we conclude that P is a quasicyclic p-group.

Since by Lemma 1 [17] $N_G^A \cap P \neq E$ and N_G^A contains a generalized quaternion group, contrary to the description of $\overline{HA}$-groups [10]. Therefore, a group does not contain infinite Abelian subgroups, which is impossible. Thus, this case does not occur and the norm N_G^A contains Abelian non-cyclic subgroups, i.e. it is a non-Hamiltonian locally nilpotent $\overline{HA}$-group. The lemma is proved. □

According to Lemma 1, the condition of the existence of Abelian non-cyclic subgroups in infinite locally finite group is equivalent to the condition of the existence of such subgroups in its norm N_G^A with the additional restriction of the non-Dedekindness and locally nilpotency of the norm.

Corollary 1. *An infinite locally finite group G with the non-Dedekind locally nilpotent norm N_G^A of Abelian non-cyclic subgroups contains Abelian non-cyclic subgroups if and only if its norm N_G^A contains such subgroups.*

Proof. The sufficiency of the corollary conditions is evident. The necessity follows from Lemma 1 because the norm N_G^A is a non-Hamiltonian $\overline{HA}$-group and contains Abelian non-cyclic subgroups. □

Note that the similar statement was proved earlier for torsion locally nilpotent groups with the non-Dedekind norm N_G^A in [17].

Taking into account Lemma 1 and the definition of the norm N_G^A of Abelian non-cyclic subgroups of a group, the study of the properties of infinite locally finite groups with the non-Dedekind locally nilpotent norm N_G^A will be provided under the condition that N_G^A is a non-Hamiltonian locally nilpotent $\overline{HA}$-group.

The following statement are needed for the sequel. It follows from Proposition 1 and 3 [17].

Proposition 2. *Any infinite locally finite p-group (p is a prime) G has the non-Dedekind norm N_G^A if and only if it is a group of one of the following types:*

1) *$G = (A \times \langle b\rangle) \leftthreetimes \langle c\rangle$, A is a quasi-cyclic p-group, $|b| = |c| = p$, $[A, \langle c\rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = p$;*

2) *$G = A\langle b\rangle$, A is a quasicyclic 2-group, $|b| = 4$, $b^2 \in A$, $b^{-1}ab = a^{-1}$ for any element $a \in A$;*

3) *$G = A\langle b\rangle$, A is a quasicyclic 2-group, $|b| = 8$, $b^4 \in A$, $b^{-1}ab = a^{-1}$ for any element $a \in A$;*

4) *$G = A \times H$, A is a quasicyclic 2-group, $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2$, $[h_1, h_2] = h_1^2$;*

5) *$G = (A \times \langle b\rangle) \leftthreetimes \langle c\rangle \leftthreetimes \langle d\rangle$, where A is a quasicyclic 2-group, $|b| = |c| = |d| = 2$, $[A, \langle c\rangle] = E$, $[b, c] = [b, d] = [c, d] = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ for any element $a \in A$; $N_G^A = (\langle a_2\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$, $a_2 \in A$, $|a_2| = 4$;*

6) *$G = (A\langle y\rangle)Q$, where A is a quasicyclic 2-group, $[A, Q] = E$, $Q = \langle q_1, q_2\rangle$, $|q_1| = 4$, $q_1^2 = q_2^2 = [q_1, q_2]$, $|y| = 4$, $y^2 = a_1 \in A$, $y^{-1}ay = a^{-1}$ for any element $a \in A$, $[\langle y\rangle, Q] \subseteq \langle a_1\rangle \times \langle q_1^2\rangle$; $N_G^A = \langle a_2\rangle \times Q$, $a_2 \in A$, $|a_2| = 4$.*

Proposition 2 implies the following corollaries.

Corollary 2. *If the norm N_G^A of Abelian non-cyclic subgroups of a locally finite p-group (p is a prime) G is infinite and non-Dedekind, then it coincides with a group, i.e. $N_G^A = G$, and is a group of one of the types (1)–(4) of Proposition 2.*

Corollary 3. *If an infinite locally finite p-group (p is a prime) G has the non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and a central quasicyclic subgroup, then $N_G^A = G$ and G is a group of one of the types (1) or (4) of Proposition 2.*

The following theorem describes the properties of infinite locally finite groups with the non-Dedekind locally nilpotent norm N_G^A.

Theorem 1. *Let G be an infinite locally finite group and contain the locally nilpotent norm N_G^A with the non-Hamiltonian Sylow p-subgroup $(N_G^A)_p$. Then G is a finite extension of a quasicyclic p-subgroup, all Sylow p'-subgroups are finite and do not contain Abelian non-cyclic subgroups. In particular, Sylow q-subgroups (q is an odd prime, $q \in \pi(G)$, $q \neq p$) are cyclic, Sylow 2-subgroups ($p \neq 2$) are either cyclic or finite quaternion 2-groups.*

Proof. Let a group G and its norm N_G^A satisfy the theorem conditions. Then by Lemma 1 N_G^A is a non-Dedekind locally nilpotent $\overline{HA}$-group. By Proposition 1, $N_G^A = (N_G^A)_p \times B$, where $(N_G^A)_p$ is Sylow p-subgroup of the norm, which is a non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind group, all Abelian subgroups of which are cyclic and $(|B|, p) = 1$. By the description of non-Hamiltonian $\overline{HA}_p$-groups (see for instance Proposition 1 [17]), N_G^A is either finite or a finite extension of a quasicyclic p-subgroup for a prime $p \in \pi(G)$.

Let $G_{p'}$ be an arbitrary Sylow p'-subgroup of a group G. Let's prove that all Abelian subgroups of the group $G_{p'}$ are cyclic. Indeed, let $A \leq G_{p'}$ be an Abelian non-cyclic subgroup. Since $(N_G^A)_p$ is characteristic and A is N_G^A-admissible, $[\langle x \rangle, A] \subseteq (N_G^A)_p \cap A = E$ for any element $x \in (N_G^A)_p$. Taking into account that $\langle x, A \rangle = \langle x \rangle \times A$ is Abelian non-cyclic and N_G^A-admissible, we get $\langle x, A \rangle \cap (N_G^A)_p = \langle x \rangle \lhd (N_G^A)_p$. But in this case $(N_G^A)_p$ is Dedekind, which contradicts the condition. Therefore, all Abelian p'-subgroups of a group G are cyclic.

Since $G_{p'}$ does not contain infinite Abelian subgroups, by the Kargapolov-Hall-Kulatilaka theorem (see [7], [8]) $G_{p'}$ is a finite group and by the proved above all its Abelian subgroups are cyclic. By these it follows that all Sylow q-subgroups of a group G ($q \in \pi(G)$, $q \neq p$) are cyclic for odd primes, Sylow 2-subgroups ($p \neq 2$) are either cyclic or finite quaternion 2-groups.

Let's prove that G satisfies the minimal condition for Abelian subgroups. Suppose the converse. Then G contains an Abelian subgroup A, which is the direct product if infinitely many subgroups of prime order. Let $A_1 = N_G^A \cap A$. Then $|A_1| < \infty$ and $A = A_1 \times A_2$, where $|A_2| = \infty$ and $N_G^A \cap A_2 = E$. By Lemma 2 [17] the norm N_G^A must be Dedekind, which contradicts the condition. Therefore, G is a group with the minimal condition for Abelian subgroups, moreover, by [24] for all subgroup, and is Chernikov group. But then G is a finite extension of divisible Abelian subgroup P.

Since all Sylow q-subgroup of a group G ($q \neq p$) are either cyclic or quaternion 2-groups by the proved above, P is the direct product of finitely many quasicyclic p-subgroups.

Let $P \supseteq (A_1 \times A_2)$, where A_1 and A_2 are quasicyclic p-subgroups. Since

$$N_G^A \lhd G_1 = (A_1 \times A_2) \cdot N_G^A$$

and G_1/N_G^A is a divisible Abelian subgroup, by Theorem 1.16 [3] the center of G_1 contains a divisible Abelian subgroup A such that $|A \cap N_G^A| < \infty$ and $G_1 = A \cdot N_G^A$. Thus, G_1 is a locally nilpotent group with the infinite center. By Lemma 1 [17] G_1 is a $\overline{HA}$-group, so by the description of such groups (see [10]), we conclude that $P = A$ is a quasicyclic p-subgroup, which is the maximal divisible subgroup of a group G. The theorem is proved. □

Corollary 4. *If the norm of Abelian non-cyclic subgroups of a non-primary locally finite group G is locally nilpotent non-Dedekind and $2 \notin \pi(G)$, then G has non-cyclic Sylow p-subgroups for a unique prime $p \in \pi(G)$.*

Corollary 5. *Any infinite locally finite group G with the infinite locally nilpotent non-Dedekind norm N_G^A is a finite extension of this norm.*

3. Infinite torsion non-locally nilpotent groups with the locally nilpotent non-Dedekind norm N_G^A. In this section we will consider infinite locally finite non-locally nilpotent groups with locally nilpotent non-Dedekind norm of Abelian non-cyclic subgroups.

Theorem 2. *Let G be a locally finite non-locally nilpotent group with the infinite locally nilpotent non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. Then $G = G_p \leftthreetimes H$, where G_p is an infinite $\overline{HA}_p$-group of one of the types (1)–(4) of Proposition 2, which coincides with the Sylow p-subgroup of the norm N_G^A, H is a finite group, all Abelian subgroups of which are cyclic, and $(|H|, p) = 1$. Any element $h \in H$ of odd order that centralizes some Abelian non-cyclic subgroup $M \subset N_G^A$ is contained in the centralizer of the norm N_G^A.*

Proof. Let a group G and its norm N_G^A satisfy the theorem conditions. Then by Proposition 1 N_G^A is a finite extension of the Sylow p-subgroup $(N_G^A)_p$.

Let G_p be an arbitrary Sylow p-subgroup of a group G. Since $(N_G^A)_p$ is a characteristic subgroup of the norm N_G^A, $(N_G^A)_p \lhd G$ and $(N_G^A)_p \subseteq G_p$. Thus, $(N_G^A)_p$ is contained in the norm $N_{G_p}^A$ of Abelian non-cyclic subgroups of a group G_p and then G_p is a locally finite p-group with the infinite non-Dedekind norm of Abelian non-cyclic subgroups. By Corollary 2

$$(N_G^A)_p = N_{G_p}^A = G_p.$$

In other words, G_p is an infinite $\overline{HA}_p$-group, which is normal in G. By the description of such groups (Proposition 1 [17] and Proposition 2), we conclude that G_p is a group of one of the types (1)–(4) of Proposition 2.

Taking into account the proved above and Theorem 1, we get $[G : G_p] < \infty$. Then by the generalized Shur theorem (see for instance [3] p. 214) the subgroup G_p is complemented in G and $G = G_p \leftthreetimes H$, where H is a finite group and $(|H|, p) = 1$. By Theorem 1 all Abelian subgroups of a group H are cyclic and the first assertion of the theorem is proved.

Let h be an arbitrary element of odd order of the subgroup H that centralizes some Abelian non-cyclic subgroup $M \subset N_G^A$. Without loss of generality, we can consider that $M \subset G_p$. Since the subgroup $(M \times \langle h \rangle)$ is N_G^A-admissible, its characteristic subgroup $\langle h \rangle$ is also N_G^A-admissible. Taking into account that the norm N_G^A is locally nilpotent and all its Sylow q-subgroups $(q, 2p) = 1$ are cyclic, we conclude that $\langle h \rangle \subset C_G(N_G^A)$. The theorem is proved. □

Note that the subgroup H mentioned in Theorem 2 can be non-nilpotent. Besides, a group of the structure mentioned in the theorem can contain the non-locally nilpotent norm of Abelian non-cyclic subgroups. So the conditions of Theorem 2 are necessary but not sufficient. The examples of such group are below.

Example 1. $G = ((A \times \langle b \rangle) \leftthreetimes \langle c \rangle) \times H$, where A is the quasicyclic 5-subgroup, $|b| = |c| = 5$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = 5$, $H = \langle d \rangle \leftthreetimes \langle h \rangle$, $|d| = 3$, $|h| = 4$, $h^{-1}dh = d^{-1}$.

It is easy to prove that in this group the norm of Abelian non-cyclic subgroups is a group of the type

$$N_G^A = ((A \times \langle b \rangle) \leftthreetimes \langle c \rangle) \times \langle h^2 \rangle .$$

The group G is nilpotent and its norm N_G^A of Abelian non-cyclic subgroups is nilpotent.

Example 2. $G = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle \leftthreetimes \langle h \rangle$, where A is a quasicyclic 7-subgroup, $|b| = |c| = 7$, $|h| = 3$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = 7$, $h^{-1}a_1h = a_1^4$, $h^{-1}a_mh = a_m^{\alpha_m}$, $\alpha_m^3 \equiv 1 \pmod{7^m}$, $\alpha_m \not\equiv 1 \pmod{7^m}$ for any element $a_m \in A$, $|a_m| = 7^m$, $m > 1$, $h^{-1}bh = b^2$, $h^{-1}ch = c^2$.

It is evident that G is a group of the type $G = G_7 \leftthreetimes H$, where $G_7 = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle$ is an infinite $\overline{HA}_7$-group and $H = \langle h \rangle$.

All Abelian non-cyclic subgroups of the group G are contained in the Sylow 7-subgroup G_7 and normal in the group G. So $N_G^A = G$ and the norm N_G^A of Abelian non-cyclic subgroups is a non-locally nilpotent group.

Note that for some additional restrictions conditions of Theorem 2 can become sufficient. In particular, the following statement holds.

Theorem 3. *Let G be an infinite locally finite group. The norm N_G^A of Abelian non-cyclic subgroups is a locally nilpotent non-Dedekind group with the infinite Sylow 2-subgroup if and only if $G = G_2 \times H$, where G_2 is an infinite $\overline{HA}_2$-group of either the type (1), when $p = 2$, or types (3)–(4) of Proposition 2, which coincides with the Sylow 2-subgroup of the norm N_G^A, H is a finite group, all Abelian subgroups of which are cyclic, and $(|H|, 2) = 1$. Moreover,*

$$N_G^A = G_2 \times Z(H).$$

Proof. Necessity. By Theorem 2, $G = G_2 \leftthreetimes H$, where G_2 is an infinite $\overline{HA}_2$-group, which coincides with the Sylow 2-subgroup of the norm N_G^A by the theorem condition and is a group of either the type (1), when $p = 2$, or types (2)–(4) of Proposition 2, H is a finite group, all Abelian subgroups of which are cyclic, $(|H|, 2) = 1$. In all cases the subgroup G_2 is a finite extension of the quasicyclic 2-group A.

Let denote an arbitrary element of the subgroup H by h. Then by Proposition 1.11 [3] we get $h \in C_G(A)$. Therefore, both the subgroup $(\langle h \rangle \times A)$ and its characteristic subgroup $\langle h \rangle$ are N_G^A-admissible. Thus, H is also N_G^A-admissible. Since $G = G_2 \leftthreetimes H$ and G_2 is a subgroup of N_G^A, $H \lhd G$ and $G = G_2 \times H$.

Let us prove that $H \cap N_G^A = Z(H)$. Let $h \in (H \cap N_G^A)$. By the proved above $\langle h \rangle$ is N_G^A-admissible, $\langle h \rangle \lhd N_G^A$. Taking into account that all $2'$-subgroups of the norm N_G^A are cyclic and normal in G, we conclude that $\langle h \rangle \lhd G$ as a characteristic subgroup of the norm N_G^A.

Since for an arbitrary element $y \in H$ of order coprime to $|h|$ the subgroup $A \times \langle y \rangle$ is Abelian non-cyclic and N_G^A-admissible, $[h, y] \subseteq \langle h \rangle \cap (A \times \langle y \rangle) = 1$. Taking into account that all Sylow q-subgroups $(q, 2) = 1$ are cyclic, we conclude that $h \in Z(H)$ and $N_G^A = G_2 \times Z(H)$.

Sufficiency. Let G be a group of the structure mentioned in the theorem. Then all its Abelian non-cyclic subgroups can be presented as $M \times \langle y \rangle$, where $M \subseteq G_2$ is an Abelian non-cyclic 2-group and $(|y|, 2) = 1$. Since G_2 is an infinite $\overline{HA}_2$-group, it normalizes all Abelian non-cyclic 2-subgroups. Besides, G_2 normalizes all subgroups of a group H because $[G_2, H] = E$. Thus, $G_2 \subseteq N_G^A$.

Let $h, y \in H \cap N_G^A$ be elements such that $(|h|, |y|) = 1$. Then subgroups $A \times \langle y \rangle$ and $A \times \langle h \rangle$ are N_G^A-admissible as Abelian non-cyclic. Since subgroups $\langle h \rangle$ and $\langle y \rangle$ are characteristic, they are also N_G^A-admissible. Thus, $[h, y] \in \langle h \rangle \cap \langle y \rangle = E$ and N_G^A is locally nilpotent.

Analysis similar to the proof of the necessity shows that $N_G^A = G_2 \times Z(H)$. □

Let's consider infinite locally finite non-locally nilpotent groups with the finite nilpotent non-Dedekind norm of Abelian non-cyclic subgroups.

Theorem 4. *Let G be an infinite locally finite non-locally nilpotent group with the finite nilpotent non-Dedekind norm N_G^A of Abelian non-cyclic subgroups. Then $G = H \leftthreetimes K$, where H is a finite group, all Abelian subgroups of which are cyclic, $(|H|, 2) = 1$, K is an infinite*

2-group of one of the types (5)–(6) of Proposition 2. Moreover, the norm N_K^A of Abelian non-cyclic subgroups of the group K is finite, $K \cap N_G^A = N_K^A$ and coincides with the Sylow 2-subgroup $(N_G^A)_2$ of the norm N_G^A of a group G.

Moreover, any element $h \in H$ of the centralizer of some Abelian non-cyclic subgroup $M \subset N_G^A$ is contained in the centralizer of the norm N_G^A.

Proof. Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the theorem condition. Then by Theorem 1 G is a finite extension of a quasicyclic subgroup A. Since the subgroup N_G^A is finite and normal in a group G, $[G : C_G(N_G^A)] < \infty$, we obtain $A \subseteq C_G(N_G^A)$.

By the theorem condition and Proposition 1, $N_G^A = (N_G^A)_p \times B$, where $(N_G^A)_p$ is a Sylow p-subgroup of the norm, which is a finite non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind group, all Abelian subgroups of which are cyclic, and $(|B|, p) = 1$.

If $p \neq 2$, then A is contained in the center of any Sylow p-subgroup G_p of a group G. Then the norm $N_{G_p}^A$ of Abelian non-cyclic subgroups of the group G_p contains the product $A \cdot (N_G^A)_p$ and is an infinite non-Dedekind p-group. By Corollary 2, $N_{G_p}^A = G_p$ and G_p is a $\overline{HA}_p$-group.

Taking into account that the subgroup $(N_G^A)_p$ is non-Dedekind and the description of non-Hamiltonian $\overline{HA}_p$-groups [10], we conclude that $G_p = A \cdot (N_G^A)_p$. Therefore, $G_p \lhd G$ as a product of normal subgroups. By the generalized Shur theorem ([3], p. 214), the subgroup G_p is complemented in G and $G = G_p \leftthreetimes H$, where H is a finite group, all Abelian subgroups of which are cyclic, and $(|H|, p) = 1$.

If all Abelian non-cyclic subgroups of a group G are p-groups, then $G_p \subseteq N_G^A$, which contradicts the finiteness of the norm N_G^A. Therefore, G contains a non-primary Abelian non-cyclic subgroup $M = M_p \times M_q$, where M_p is an Abelian non-cyclic p-group, M_q is a cyclic q-group ($p \neq q$). Taking into account the structure of the subgroup G_p, we conclude that $M_p \bigcap A \neq E$. Thus, $M_q \subseteq C_G(a_1)$, where $a_1 \in A$, $|a_1| = p$. By Proposition 1.11 [3] $M_q \subseteq C_G(A)$, so $A \subseteq N_G^A$, which contradicts the finiteness of the norm N_G^A. Thus, this case is impossible.

Let now $p = 2$. Since the quasicyclic 2-group A is normal in G and non-central (because in this case $|N_G^A| = \infty$), $[G : C_G(A)] = 2$ and $G = C_G(A) \langle x \rangle$, where $x^2 \in C_G(A)$.

By the condition $A \subseteq C_G(N_G^A)$, we get $N_G^A \subseteq C_G(A)$. Thus, $C_G(A)$ is a group, which has the infinite locally nilpotent norm of Abelian non-cyclic subgroups. Using Theorem 3 to $C_G(A)$, we get $C_G(A) = C_2 \times H$, where C_2 is an infinite $\overline{HA}_2$-group, H is a finite group, all Abelian subgroups of which are cyclic, and $(|H|, 2) = 1$.

Since the subgroup H is characteristic in $C_G(A)$, $H \lhd G$. It follows also that H contains all $2'$-elements of a group G, further, a Sylow $2'$-subgroup of a group G. Taking into account that a group G is countable, contains a normal solvable locally normal Sylow $2'$-subgroup H and, by [9] (p. 508), we conclude that H is complemented in G. Therefore, $G = H \leftthreetimes K$, where K is an infinite 2-subgroup, $A \subseteq K$.

Since the Sylow 2-subgroup $(N_G^A)_2$ of the norm N_G^A is finite, is contained in the norm N_K^A of Abelian non-cyclic subgroups of the group K and $A \nsubseteq Z(K)$, N_K^A is a finite non-Hamiltonian $\overline{HA}_2$-group. Accordingly, K is an infinite locally finite 2-group with the finite non-Dedekind norm of Abelian non-cyclic subgroups, $K \cap N_G^A = N_K^A = (N_G^A)_2$. By the description of such groups, K is a group of one of the types (5)–(6) of Proposition 2.

The proof of last theorem assertion is similar to the proof of Theorem 2. □

The existence of infinite non-locally nilpotent groups with the finite nilpotent norm of Abelian non-cyclic subgroups confirms the following example.

Example 3. $G = \langle h\rangle \leftthreetimes (A \times \langle b\rangle) \leftthreetimes \langle c\rangle) \leftthreetimes \langle d\rangle$, where A is the quasicyclic 2-group, $|b| = |c| = |d| = 2$, $[A, \langle c\rangle] = E$, $[b, c] = [b, d] = [c, d] = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ for an arbitrary element $a \in A$, $|h| = 3$, $d^{-1}hd = h^{-1}$, $[A, \langle h\rangle] = E$, $[b, h] = [c, h] = 1$.

Since $N_G^A \subseteq N_G(\langle d, a_1\rangle) \cap N_G(\langle hd, a_1\rangle) = (\langle a_2\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle = N$, where $a_2 \in A$, $|a_2| = 4$, all Abelian non-cyclic subgroups contain the involution $a_1 \in A$ and $[G, N] \subseteq \langle a_1\rangle$,

$$N_G^A = (\langle a_2\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle \,.$$

Moreover, G is an infinite non-locally nilpotent group, the norm N_G^A is a finite nilpotent group.

REFERENCES

1. R. Baer, *Der Kern, eine charakteristische Untergruppe*, Comp. Math., **1** (1935), 254–283.
2. V.M. Bursakin, A.I. Starostin, *On splitting locally finite groups*, Sbornik: Mathematics, **62** (1963), №3, 275–294. (in Russian)
3. S.N. Chernikov, Groups with given properties of system of subgroups, M., Nauka, 1980. (in Russian)
4. M.G. Drushlyak, T.D. Lukashova, F.M. Lyman, *Generalized norms of groups*, Algebra Discr. Math., **22** (2016), №1, 48–80.
5. M. de Falco, F. de Giovanni, L.A. Kurdachenko, C. Musella, *The metanorm and its influence on the group structure*, J. Algebra, **506** (2018), 76–91.
6. M. Ferrara, M. Trombetti, *Large norms in group theory*, J. Algebra, **646** (2024), 236–267. doi:10.1016/j.jalgebra.2024.02.007.
7. P. Hall, C.R. Kulatilaka, *A property of locally finite groups*, J. London Math. Soc., **39** (1964), 235–239. doi:10.1112/jlms/s1-39.1.235.
8. M.I. Kargapolov, *On O.Yu. Shmidt's problem*, Sib. Math. J., **4** (1963), №1, 232–235. (in Russian)
9. A.G. Kurosh, Theory of groups, M., Nauka, 1967. (in Russian)
10. F.N. Liman, *Periodic groups, all Abelian noncyclic subgroups of which are invariant. Groups with restrictions for subgroups*, 1971, Kyiv, Naukova Dumka, 65–96. (in Russian)
11. F.M. Liman, T.D. Lukashova, *On infinite groups with given properties of norm of infinite subgroups*, Ukr. Math. J., **53** (2001), №5, 625–630. doi:10.1023/A:10125266221
12. F.N. Liman, T.D. Lukashova, *On the norm of decomposable subgroups in locally finite groups*, Ukr. Math. J., **67** (2015), №4, 542–551.
13. F.N. Liman, T.D. Lukashova, *On the norm of decomposable subgroups in the non-periodic groups*, Ukr. Math. J., **67** (2016), №12, 1900–1912.
14. T. Lukashova, *Locally soluble groups with the restrictions on the generalized norms*, Algebra Discr. Math., **29** (2020), №1, 85–98. doi:10.12958/adm1527.
15. T.D. Lukashova, *Infinite locally finite groups with the locally nilpotent non-Dedekind norm of decomposable subgroups*, Communications in Algebra, **48** (2020), №3, 1052–1057. doi:10.1080/00927872.2019.1677683.
16. T. Lukashova, M. Drushlyak, *Generalized norms of groups: retrospective review and current status*, Algebra Discr. Math., **34** (2022), №1, 105–131. doi:10.12958/adm1968.
17. T.D. Lukashova, M.G. Drushlyak, *Torsion locally nilpotent groups with non-Dedekind norm of Abelian non-cyclic subgroups*, Carpathian Math. Publ., **14** (2022), №1, 247–259. doi:10.15330/cmp.14.1.247-259.
18. T. Lukashova, M. Drushlyak, *Torsion locally nilpotent groups with the non-Dedekind norm of decomposable subgroups*, Adv. Group Theory Appl., **17** (2023), 51–63. doi:10.32037/agta-2023-015.
19. T.D. Lukashova, M.G. Drushlyak, F.M. Lyman, *Conditions of Dedekindness of generalized norms in non-periodic groups*, Asian-European Journal of Mathematics, **12** (2019), №1, 1950093, 11 p. doi:10.1142/S1793557119500931.
20. F.M. Lyman, T.D. Lukashova, *Non-periodic locally soluble groups with non-Dedekind locally nilpotent norm of decomposable subgroups*, Ukr. Math. J., **71** (2020), №11, 1739–1750. doi:10.1007/s11253-020-01744-7.

21. F.M. Lyman, T.D. Lukashova, *On infinite 2-groups with non-Dedekind norm of Abelian non-cyclic subroups*, Bull. Univ. Kyiv, **1** (2005), 56–64. (in Ukrainian)
22. F. Lyman, T. Lukashova, M. Drushlyak, *On finite 2-groups with non-Dedekind norm of Abelian non-cyclic subgroups*, Mat. Stud., **46** (2016), №1, 20–28. doi:10.15330/ms.46.1.20-28
23. F. Lyman, T. Lukashova, M. Drushlyak, *Finite 2-groups with a non-Dedekind non-metacyclic norm of Abelian non-cyclic subgroups,* Bul. Acad. Stiinte Repub. Mold. Mat., **57** (2019), №2, 3–19.
24. V.P. Shunkov, *On locally finite groups with a minimality condition for Abelian subgroups*, Algebra Logic, **9** (1970), 579–615. (in Russian)
25. H. Wielandt, *Über den Normalisator der Subnormalen Untergruppen*, Mat. Z. **69** (1958), №5, 463–465.

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