УДК 517.97

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SUBSONIC PERODIC TRAVELING WAVES IN FERMI–PASTA–ULAM TYPE SYSTEMS WITH NONLOCAL INTERACTION ON 2D-LATTICE

S. M. Bak, H. M. Kovtoniuk. *Subsonic periodic traveling waves in Fermi–Pasta–Ulam type systems with nonlocal interaction on 2d-lattice*, Mat. Stud. **62** (2024), 184–191.

The paper is devoted to Fermi–Pasta–Ulam type system that describe an infinite system of nonlinearly coupled particles with nonlocal interaction on a two dimensional integer-valued lattice. It is assumed that each particle interacts nonlinearly with several neighbors horizontally and vertically on both sides. This system forms an infinite system of ordinary differential equations and is representative of a wide class of systems called lattice dynamical systems, which have been extensively studied in recent decades. Among the solutions of such systems, traveling waves deserve special attention. The main result concerns the existence of traveling waves solutions with periodic velocity profiles. Note that the profiles of such waves are not necessarily periodic. The problem of the existence of such solutions is reduced to a variational problem for the action functionals. We obtain sufficient conditions for the existence of such solutions with the aid of the critical point method and the Linking Theorem for functionals satisfying the Palais–Smale condition and possessing linking geometry. We prove that under natural assumptions there exist subsonic traveling waves. While in our previous paper [12], the existence of supersonic periodic traveling waves in this system was established using variational techniques and a corresponding version of the Mountain Pass Theorem for action functionals that satisfy the Cerami condition instead of the Palais–Smale condition.

1. Introduction. Recently, considerable attention has been paid to models that are discrete in the spatial variables. The discrete Klein–Gordon type equations and the Fermi–Pasta–Ulam type systems are examples of equations that describe such models. We note that these systems are representative of a wide class of systems called lattice dynamical systems, which extensively studied in recent decades. These systems are of interest in view of numerous applications in physics ([1, 15–17, 21]).

Among the solutions of such systems, traveling waves deserve special attention. The existence of periodic and solitary traveling waves in discrete Klein–Gordon type equations with local interaction on 1D and 2D–lattices is studied in [4–6, 14, 19, 22–24]. Periodic and solitary traveling waves in Fermi–Pasta–Ulam type systems with local interaction on 1D-lattice is studied in many works, but a comprehensive presentation of existing results on traveling waves is given by A. Pankov in [26]. While in papers [7–9, 11] traveling waves in Fermi–Pasta–Ulam type systems with local interaction on 2D–lattice are studied.

2020 *Mathematics Subject Classification:* 37K60, 34A34, 74J30.
Keywords: Fermi–Pasta–Ulam type systems; nonlocal interaction; subsonic periodic traveling waves; 2D-lattice; critical points; Linking Theorem.
doi:10.30970/ms.62.2.184-191

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However, the existence of traveling waves in such systems with nonlocal interaction is not well-studied. Sufficient conditions for the existence of supersonic periodic traveling waves in discrete equations of the Klein-Gordon type are established in [10]. The existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam type systems with nonlocal interaction on 1D–lattice is studied in [25] and [28]. G. Friesecke and K. Matthies ([20]) showed the existence of solitary traveling waves for a two-dimensional elastic lattice of particles interacting via harmonic springs between nearest and diagonal neighbors. While in the present paper we study the Fermi–Pasta–Ulam type system that describes an infinite system of nonlinearly coupled particles on a two dimensional lattice with nonlocal interaction, i.e., each particle interacts with l neighbors horizontally and vertically on each side. The equations of motion of the system considered are of the form

$$\ddot{q}_{n,m}(t) = \sum_{j=1}^{l} \big(W'_{1j}(q_{n+j,m}(t) - q_{n,m}(t)) - W'_{1j}(q_{n,m}(t) - q_{n-j,m}(t)) + \\ + W'_{2j}(q_{n,m+j}(t) - q_{n,m}(t)) - W'_{2j}(q_{n,m}(t) - q_{n,m-j}(t))\big)\,,\ (n,m) \in \mathbb{Z}^2, \tag{1}$$

where $q_{n,m}(t)$ is the coordinate of the (n,m)-th particle at time t, $W_{1j}, W_{2j} \in C^1(\mathbb{R};\mathbb{R})$ are the potentials of interaction ($j \in \{1, 2, \dots, l\}$), in particular, W_{11}, W_{21} are the potentials of the horizontal and vertical interaction, respectively, of the (n,m)-th particle with nearest neighbors, W_{12}, W_{22} with second nearest neighbors, and so on. In the case $l = 1$ we obtain the Fermi–Pasta–Ulam type system on a two dimensional lattice with local interaction. Equations (1) form an infinite system of ordinary differential equations.

We are interested in classical solutions of system (1) in the form of traveling waves

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct), \tag{2}$$

where the function $u(s)$, $s \in \mathbb{R}$, is called the *profile function*, or simply *profile*, of the wave, the vector $\overrightarrow{l}(\cos\varphi, \sin\varphi)$ defines the direction of wave propagation, and the constant $c \neq 0$ is called the speed of the wave. If $c < 0$, then the wave moves to the opposite direction corresponding $c > 0$. Therefore, we always assume that $c > 0$.

We consider the case of periodic traveling waves. The profile function of such wave satisfies the following condition

$$u'(s + 2k) = u'(s),\ s \in \mathbb{R}, \tag{3}$$

where $k > 0$ is a real number. Note that the profile of such wave is not necessarily periodic. But the *velocity profile* $u'(s)$ is periodic. Therefore, such waves are also called periodic (see [26]).

An important role is played by some quantity c_0 called the *speed of sound* in this system (see [26]). The sufficient conditions for the existence of periodic traveling waves with the speed $c > c_0$, i.e., the case of *supersonic* periodic traveling waves, were obtained in [12]. This was done using the critical point method and a suitable version of the Mountain Pass Theorem for functionals satisfying the Cerami condition instead of the Palais–Smale condition. While in the present paper we study periodic traveling waves with the speed $0 < c \leq c_0$, i.e., the case of *subsonic* periodic traveling waves with the aid of the Linking Theorem instead of the Mountain Pass Theorem. We note that in [13] sufficient conditions for the existence of supersonic solitary traveling waves in (1) are established.

2. Formulation of the main result. We assume that:

(i) $W_{ij}(r) = \frac{c_{ij}^2}{2} r^2 + f_{ij}(r)$, where $c_{ij} \in \mathbb{R}$, $f_{ij} \in C^1(\mathbb{R};\mathbb{R})$, moreover, $f_{ij}(0) = f'_{ij}(0) = 0$ and $f'_{ij}(r) = o(r)$ as $r \to 0$, $i \in \{1, 2\}$, $j \in \{1, 2, \dots, l\}$;

(ii) there exists $\mu > 2$ such that $0 \le \mu f_{ij}(r) \le r f'_{ij}(r)$ for $r \ne 0$, $i \in \{1,2\}$, $j \in \{1,2,\dots,l\}$.

Note that the last condition is called the *Ambrosetti-Rabinowitz condition.*

For example we consider the following functions $f_{ij}(r) = d_{ij}^2 |r|^\mu$, where $d_{ij} \in \mathbb{R}$, $\mu > 2$. It is easy to see that the functions $f_{ij}(r)$ satisfy the assumptions (i) and (ii).

We define c_0 by the equality

$$c_0 = c_0(\varphi) := \left(\sum_{j=1}^{l} (c_{1j}^2 \cos^2 \varphi + c_{2j}^2 \sin^2 \varphi) j^2 \right)^{1/2}.$$

The main result of this paper is the following theorem that establishes the existence of subsonic periodic waves.

Theorem 1. *Assume* (i) *and* (ii). *Then for every* $c \in (0, c_0]$ *there exists a nonconstant traveling wave solution of* (1) *with profile* $u(s)$ *satisfying* (3).

3. Auxiliary statements. To obtain the main result, we will need the Linking Theorem.

Let $I\colon H \to \mathbb{R}$ be a C^1-functional on a Hilbert space H with the scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. We say that I satisfies the *Palais-Smale condition*, if the following condition is satisfied (see [26, 27, 29]):

(PS) If the sequence $\{u_n\} \subset H$ is such that $\{I(u_n)\}$ is convergent and $I'(u_n) \to 0$, $n \to \infty$, then $\{u_n\}$ contains a convergent subsequence.

Let $H = Y \oplus Z$, $\rho > r > 0$ and $z \in Z$ be given such that $\|z\| = r$. Define

$$M_z := \{u = y + \lambda z : \; y \in Y, \; \lambda \ge 0, \; \|u\| \le \rho\}$$

and $\partial M_z = \{u = y + \lambda z : \; y \in Y, \; \lambda \ge 0$ and $\|u\|_k = \rho$, or $y \in Y$, $\lambda = 0$ and $\|u\|_k \le \rho\}$, i.e., ∂M_z is the boundary of M_z. Let $N := \{u \in Z : \; \|u\| = r\}$.

We suppose that $\beta := \inf\limits_{u \in N} I(u) > \alpha := \sup\limits_{u \in \partial M_z} I(u)$. In this situation we say that I possesses the *linking geometry.*

Next, we will need Linking Theorem, which we will present in a form convenient for us.

Proposition 1 (Linking Theorem, [26, 27, 29])**.** *Suppose that a* C^1*-functional* $I\colon H \to \mathbb{R}$ *satisfies the Palais-Smale condition and possesses the linking geometry. Then there exists a nontrivial critical point* $u \in H$ *of the functional* I.

4. Proof of the main result. Substituting (2) into (1), we obtain the following equation for the profile function

$$c^2 u''(s) = \sum_{j=1}^{l} \left[W'_{1j}(A_j^+ u(s)) - W'_{1j}(A_j^- u(s)) + W'_{2j}(B_j^+ u(s)) - W'_{2j}(B_j^- u(s)) \right], \qquad (4)$$

where $s = n \cos\varphi + m \sin\varphi - ct$,

$$A_j^+ u(s) := u(s + j\cos\varphi) - u(s), \;\; A_j^- u(s) := u(s) - u(s - j\cos\varphi),$$
$$B_j^+ u(s) := u(s + j\sin\varphi) - u(s), \;\; B_j^- u(s) := u(s) - u(s - j\sin\varphi).$$

Thus, our problem is reduced to problem (4), (3). It is easy to see that if we have a solution $u(s)$ of (4) satisfying (3), then $u(s) + C$ is also a solution of this problem. Therefore, to obtain the main result, we impose an additional condition

$$u(0) = 0. \qquad (5)$$

In what follows, a solution of (4) is understood as a function $u(s)$ from the space $C^2(\mathbb{R}; \mathbb{R})$ satisfying (4) for all $s \in \mathbb{R}$.

We denote by E_k the Hilbert space $E_k = \{u \in H^1_{\text{loc}}(\mathbb{R}) : u'(s+2k) = u'(s),\ u(0) = 0\}$ with the scalar product $(u,v)_k = \int_{-k}^{k} u'(s)v'(s)ds$ and corresponding norm $\|u\|_k = (u,u)^{\frac{1}{2}}$. The norm in the dual space E_k^* is denoted by $\|\cdot\|_{k,*}$. By the embedding theorem, $E_k \subset C(\mathbb{R})$, where $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$.

Note that the difference operators $A_j^{\pm}$ and $B_j^{\pm}$ are bounded linear operators on E_k, moreover, these operators satisfy the inequalities (see [3], Lemma 6.1)

$$\begin{gathered}\|A_j^{\pm}u\|_{L^\infty(-k,k)} \le l_1(k)j^{1/2}\|u\|_k, \quad \|A_j^{\pm}u\|_{L^2(-k,k)} \le |\cos\varphi| j\|u\|_k,\\ \|B_j^{\pm}u\|_{L^\infty(-k,k)} \le l_2(k)j^{1/2}\|u\|_k, \quad \|B_j^{\pm}u\|_{L^2(-k,k)} \le |\sin\varphi| j\|u\|_k,\\ \|A_j^{\pm}u\|^2_{H^1(-k,k)} + \|B_j^{\pm}u\|^2_{H^1(-k,k)} \le (j^2+8)\|u\|_k^2,\end{gathered} \tag{6}$$

where

$$l_1(k) = \begin{cases} |\cos\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0<2k<1;\\ |\cos\varphi|, & 2k \ge 1,\end{cases} \qquad l_2(k) = \begin{cases} |\sin\varphi|\sqrt{\left[\frac{1}{2k}\right]+1}, & 0<2k<1;\\ |\sin\varphi|, & 2k \ge 1,\end{cases}$$

and $\left[\frac{1}{2k}\right]$ is the integer part of $\frac{1}{2k}$.

On the space E_k, we consider the functional

$$\begin{aligned} J_k(u) = \int_{-k}^{k}\Big[\frac{c^2}{2}(u'(s))^2 - \sum_{j=1}^{l}\big(W_{1j}(A_j^+u(s)) + W_{2j}(B_j^+u(s))\big)\Big]ds \equiv \int_{-k}^{k}\Big[\frac{c^2}{2}(u'(s))^2 -\\ -\sum_{j=1}^{l}\left(\frac{c_{1j}^2}{2}(A_j^+u(s))^2 + \frac{c_{2j}^2}{2}(B_j^+u(s))^2\right) - \sum_{j=1}^{l}\Big(f_{1j}(A_j^+u(s)) + f_{2j}(B_j^+u(s))\Big)\Big]ds. \end{aligned} \tag{7}$$

Remark 1. It is easily verified that, under the assumptions imposed, the functional J_k is well-defined C^1-functional on E_k, and its derivative is given by the formula (see [12])

$$\begin{aligned} \langle J_k'(u), h\rangle &= \int_{-k}^{k}\Big[c^2u'(s)h'(s) - \sum_{j=1}^{l}\big(W_{1j}'(A_j^+u(s))A_j^+h(s) + W_{2j}'(B_j^+u(s))B_j^+h(s)\big)\Big]ds =\\ &= \int_{-k}^{k}\Big[c^2u'(s)h'(s) - \sum_{j=1}^{l}\big(c_{1j}^2A_j^+u(s)A_j^+h(s) + c_{2j}^2B_j^+u(s)B_j^+h(s)\big) -\\ &\qquad -\sum_{j=1}^{l}\big(f_{1j}'(A_j^+u(s))A_j^+h(s) + f_{2j}'(B_j^+u(s))B_j^+h(s)\big)\Big]ds \end{aligned}$$

for $u, h \in E_k$. Moreover, any critical point of the functional J_k is a C^2-solution of (4) satisfying (3).

Thus, to establish the existence of solutions to (4) satisfying (3), it is suffice to prove the existence of nontrivial critical points of the functional J_k.

Lemma 1. *Under the assumptions of Theorem 1 the functional J_k satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset E_k$ be a Palais-Smale sequence at some level b, i.e., $I(u_n) \to b$ and $I'(u_n) \to 0$ as $n \to \infty$.

Step 1. First we show that $\{u_n\}$ is bounded. Choose $\beta \in (\mu^{-1}, 2^{-1})$. Then for n large enough we have

$$b+1+\beta\|u_n\|_k \geq J_k(u_n) - \beta\langle J_k'(u_n), u_n\rangle =$$

$$= \left(\frac{1}{2}-\beta\right)\int_{-k}^{k}\left[c^2(u_n'(s))^2 - \sum_{j=1}^{l}\left(c_{1j}^2(A_j^+u(s))^2 + c_{2j}^2(B_j^+u(s))^2\right)\right]ds+$$

$$+\int_{-k}^{k}\sum_{j=1}^{l}\left[\beta(f_{1j}'(A_j^+u_n(s))A_j^+u_n(s) + f_{2j}'(B_j^+u_n(s))B_j^+u_n(s))-\right.$$

$$\left.-f_{1j}(A_j^+u_n(s)) - f_{2j}(B_j^+u_n(s))\right]ds = \left(\frac{1}{2}-\beta\right)\int_{-k}^{k}c^2(u_n'(s))^2ds-$$

$$-\left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l}\int_{-k}^{k}\left(c_{1j}^2(A_j^+u(s))^2 + c_{2j}^2(B_j^+u(s))^2\right)ds+$$

$$+\sum_{j=1}^{l}\int_{-k}^{k}\left[\beta(f_{1j}'(A_j^+u_n(s))A_j^+u_n(s) + f_{2j}'(B_j^+u_n(s))B_j^+u_n(s))-\right.$$

$$\left.-f_{1j}(A_j^+u_n(s)) - f_{2j}(B_j^+u_n(s))\right]ds \geq$$

$$\geq \left(\frac{1}{2}-\beta\right)c^2\|u_n\|_k^2 - \left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l}\left[c_{1j}^2\|A_j^+u_n\|_{L^2(-k,k)}^2 + c_{2j}^2\|B_j^+u_n\|_{L^2(-k,k)}^2\right]+$$

$$+(\beta\mu-1)\sum_{j=1}^{l}\int_{-k}^{k}\left[f_{1j}(A_j^+u_n(s)) + f_{2j}(B_j^+u_n(s))\right]ds \geq$$

$$\geq \left(\frac{1}{2}-\beta\right)c^2\|u_n\|_k^2 - \left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l}\left[c_{1j}^2\|A_j^+u_n\|_{L^2(-k,k)}^2 + c_{2j}^2\|B_j^+u_n\|_{L^2(-k,k)}^2\right]+$$

$$+C(\beta\mu-1)\sum_{j=1}^{l}\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^\mu + \|B_j^+u_n\|_{L^\mu(-k,k)}^\mu\right] - C_0.$$

Since $\mu > 2$, we have

$$c_{1j}^2\|A_j^+u_n\|_{L^2(-k,k)}^2 + c_{2j}^2\|B_j^+u_n\|_{L^2(-k,k)}^2 \leq C\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^2 + \|B_j^+u_n\|_{L^\mu(-k,k)}^2\right] \leq$$
$$\leq K(\varepsilon) + \varepsilon\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^\mu + \|B_j^+u_n\|_{L^\mu(-k,k)}^\mu\right],$$

where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Then

$$b+1+\beta\|u_n\|_k \geq \left(\frac{1}{2}-\beta\right)c^2\|u_n\|_k^2 - \left(\frac{1}{2}-\beta\right)lK(\varepsilon) - \left(\frac{1}{2}-\beta\right)\varepsilon\sum_{j=1}^{l}\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^\mu+\right.$$

$$\left.+\|B_j^+u_n\|_{L^\mu(-k,k)}^\mu\right] + C(\beta\mu-1)\sum_{j=1}^{l}\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^\mu + \|B_j^+u_n\|_{L^\mu(-k,k)}^\mu\right] - C_0.$$

Choosing $\varepsilon > 0$ small enough, we obtain

$$b+1+\beta\|u_n\|_k \geq \left(\frac{1}{2}-\beta\right)c^2\|u_n\|_k^2 + C(\beta\mu-1)\sum_{j=1}^{l}\left[\|A_j^+u_n\|_{L^\mu(-k,k)}^\mu + \|B_j^+u_n\|_{L^\mu(-k,k)}^\mu\right] - C_0.$$

Since $\beta\mu - 1 > 0$, $b+1+\beta\|u_n\|_k \geq \left(\frac{1}{2}-\beta\right)c^2\|u_n\|_k^2 - C_0$. The last inequality implies that $\{u_n\}$ is bounded.

Step 2. Since $\{u_n\}$ is bounded in Hilbert space E_k we have, up to a subsequence (with the same denotation), $u_n \to u$ weakly in E_k, hence, $A_j^+ u_n \to A_j^+ u$ and $B_j^+ u_n \to B_j^+ u$ ($j \in \{1, 2, \dots, l\}$) weakly in E_k, and strongly in $L^2(-k, k)$ and $C([-k, k])$ (by the compactness of Sobolev embedding, [2, 26]). A straightforward calculation shows that

$$c^2\|u_n - u\|_k^2 = \int_{-k}^{k} \left(c^2(u_n'(s) - u'(s))^2 + c^2(u_n(s) - u(s))^2\right) ds =$$

$$= \langle J_k'(u_n) - J_k'(u), u_n - u\rangle + \sum_{j=1}^{l} \left[c_{1j}^2\|A_j^+ u_n - A_j^+ u\|_{L^2(-k,k)}^2 + c_{2j}^2\|B_j^+ u_n - B_j^+ u\|_{L^2(-k,k)}^2\right] +$$

$$+ \sum_{j=1}^{l} \int_{-k}^{k} \left(f_{1j}'(A_j^+ u_n(s)) - f_{1j}'(A_j^+ u(s))\right)\left(A_j^+ u_n(s) - A_j^+ u(s)\right) ds +$$

$$+ \sum_{j=1}^{l} \int_{-k}^{k} \left(f_{1j}'(B_j^+ u_n(s)) - f_{1j}'(B_j^+ u(s))\right)\left(B_j^+ u_n(s) - B_j^+ u(s)\right) ds.$$

Obviously that all the terms on the right hand part converge to 0 (first, fourth and fifth by weak convergence, second and third terms converge to 0 by strong convergence). Thus, $\|u_n - u\|_k \to 0$ as $n \to \infty$, and proof is complete. □

Lemma 2. *Under the assumptions of Theorem 1 the functional J_k possesses the linking geometry.*

Proof. First we note that the space E_k splits into orthogonal sum of the 1-dimensional subspace generated by the function $h_0(s) = s$ and the space $H_{k,0}^1$ of all $2k$-periodic functions from E_k with zero mean value (see [26]). Consider the operator L defined by

$$(Lu)(s) := c^2 u''(s) - \sum_{j=1}^{l} \left[c_{1j}^2(A_j^+ u)(s) + c_{1j}^2(B_j^+ u)(s)\right].$$

Elementary Fourier analysis shows that L is a self-adjoint operator in $L^2(-k; k)$, bounded below and that L has discrete spectrum which accumulated at $+\infty$ (see [18, 26]). All eigenvalues, λ_j, with nonconstant eigenfunctions are double. Denote by $h_j^\pm \in H_{k,0}^1$ linearly independent pairs of eigenfunctions with the eigenvalues λ_j.

Let Z be the subspace of $H_{k,0}^1$ generated by the functions of the operator L with positive eigenvalues and Y be the subspace of E_k generated by the functions with non-positive eigenvalues and the function h_0. It is readily verified that $Y \perp Z$ and $E_k = Y \oplus Z$.

Step 1. Denote by Q_k the quadratic part of the functional J_k

$$Q_k(u) = \frac{1}{2}\int_{-k}^{k} \left[c^2(u'(s))^2 - \sum_{j=1}^{l} \left(c_{1j}^2(A_j^+ u(s))^2 + c_{1j}^2(B_j^+ u(s))^2\right)\right] ds.$$

Obviously, $Q_k(y + z) = Q_k(y) + Q_k(z)$, where $y \in Y$, $z \in Z$.

Note that the quadratic form Q_k is positive definite on Z, i.e., $Q_k(u) \geq \alpha\|u\|_k^2$, with $\alpha > 0$. Assumption (i) implies that, given $\varepsilon > 0$, there exists $r_0 > 0$ such that $|f_{ij}(r)| \leq \varepsilon r^2$ as $|r| \leq r_0$ ($i \in \{1, 2\}; j \in \{1, 2, \dots, l\}$). Then, by (6),

$$J_k(u) \geq Q_k(u) -$$

$$-\varepsilon \int_{-k}^{k} \sum_{j=1}^{l} \left[(A_j^+ u(s))^2 + (B_j^+ u(s))^2\right] ds = Q_k(u) - \varepsilon \sum_{j=1}^{l} \int_{-k}^{k} \left[(A_j^+ u(s))^2 + (B_j^+ u(s))^2\right] ds \geq$$

$$\geq Q_k(u) - \varepsilon\|u\|_k^2 \sum_{j=1}^{l} j^2 = Q_k(u) - \varepsilon\frac{l(l+1)(2l+1)}{6}\|u\|_k^2 \geq \delta\|u\|_k^2,$$

where $\delta > 0$. Hence, $J_k(u) > 0$ on $N = \{u \in Z : \ \|u\|_k = r\}$ provided $r > 0$ is small enough.
Step 2. Now we fix $z \in Z$, $\|z\|_k = 1$ and set $M_z = \{u = y + \lambda z : \ y \in Y, \ \lambda \geq 0, \ \|u\|_k \leq \rho\}$. We prove that $J_k(u) \leq 0$ on ∂M_z provided that ρ is large enough. Recall that

$$\partial M_z = \{u = y + \lambda z : \ y \in Y, \ \lambda \geq 0 \text{ and } \|u\|_k = \rho, \text{ or } y \in Y, \ \lambda = 0 \text{ and } \|u\|_k \leq \rho\}.$$

Since, by assumptions (i) and (ii), there exist constants $d > 0$, $d_0 \geq 0$ (see [3], Lemma 3.1) such that $f_{ij}(r) \geq d|r|^\mu - d_0$, $\mu > 2$ $(i \in \{1,2\}; j \in \{1,2,\ldots,l\})$, then, given that $Q_k(y) \leq 0$,

$$J_k(y+\lambda z) = Q_k(y+\lambda z) - \int_{-k}^{k}\sum_{j=1}^{l}\left[f_{1j}(A_j^+(y+\lambda z)) + f_{2j}(B_j^+(y+\lambda z))\right] ds =$$

$$\leq Q_k(y) + \lambda^2 Q_k(z) - \sum_{j=1}^{l}\int_{-k}^{k}\left[f_{1j}(A_j^+(y+\lambda z)) + f_{2j}(B_j^+(y+\lambda z))\right] ds \leq$$

$$\leq \lambda^2 Q_k(z) - \sum_{j=1}^{l}\int_{-k}^{k}\left[d|A_j^+(y+\lambda z)|^\mu + d|B_j^+(y+\lambda z)|^\mu - 2d_0\right] ds =$$

$$= \lambda^2\gamma_0 + 4kld_0 - d\sum_{j=1}^{l}\left[\|A_j^+(y+\lambda z)\|^\mu_{L^\mu(-k,k)} + \|B_j^+(y+\lambda z)|^\mu_{L^\mu(-k,k)}\right] ds,$$

where $\gamma_0 = Q_k(z)$. Since $\rho^2 = \|y+\lambda z\|_k^2 = \|y\|_k^2 + \lambda^2$, then $\lambda^2 \leq \rho^2$. Furthermore, on finite dimensional spaces all norms are equivalent. Then $\|y+\lambda z\|_{L^\mu(-k,k)} \geq c\|y+\lambda z\|_k = c\rho$. Therefore, we have that $J_k(y+\lambda z) \leq \gamma_0\rho^2 + 4kld_0 - C\rho^\mu$ with some $C > 0$. Since $\mu > 2$, the right-hand side part here is negative if ρ is large enough. Hence, $J_k(y+\lambda z) \leq 0$. If $u \in \partial M_z$, $\|u\|_k \leq \rho$ and $\lambda = 0$, then $u = y \in Y$ and, obviously, $J_k(u) \leq 0$.

Thus, J_k possesses the linking geometry, and the lemma is proved. □

Due to Lemma 1 and Lemma 2, the functional J_k satisfies all conditions of the Linking Theorem (see Proposition 1). Hence, J_k has a nontrivial critical point $u \in E_k$. By Remark 1, u is a C^2-solution of (4) that satisfy (3), and hence, (1) has a nontrivial periodic traveling wave solution. This solution is nonconstant due to the condition (5). Thus, the proof of Theorem 1 is complete.

Conclusion. In the present paper we obtain some result on the existence of non-constant subsonic traveling waves with periodic velocity profiles in Fermi-Pasta-Ulam type systems with nonlocal interaction on a two-dimensional lattice.

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Received 14.07.2024