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UNIFORM ESTIMATES FOR LOCAL PROPERTIES OF ANALYTIC FUNCTIONS IN A COMPLETE REINAHRDT DOMAIN

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Using recent estimates of maximum modulus for partial derivatives of the analytic functions with bounded **L**-index in joint variables we describe maximum modulus of these functions at the polydisc skeleton with given radii by the maximum modulus with lesser radii. Such a description is sufficient and necessary condition of boundedness of **L**-index in joint variables for functions which are analytic in a complete Reinhardt domain. The vector-valued function **L** is a positive and continuous function in the domain and its values at a point is greater than reciprocal of distance from the point to the boundary of the Reinhardt domain multiplied by some constant.

1. Introduction. We also need the following standard notations from the theory of holomorphic functions of bounded index in joint variables (see, for example, [2, 13–15]). In particular, $\mathbb{R}_+$ means the non-negative real semi-axis, $\mathbf{0} = (0, \dots, 0)$ $\mathbf{1} = (1, \dots, 1)$, $\mathbf{1}_j$ is the n-dimensional unit vector whose j-th component equals 1, and all other components are zeros. For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, we denote $AB = (a_1b_1, \cdots, a_nb_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1}a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = \sum_{j=1}^n a_j$. The inequality $A < B$ means that $a_j \leq b_j$ for each $j \in \{1, \dots, n\}$, and so on. For $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ we put $K! = k_1! \cdot \ldots \cdot k_n!$ Also we will use such notations: $\mathbb{D}^n(z^0, R) = \{z \in \mathbb{C}^n \colon |z_j - z_j^0| < r_j \text{ for } j \in \{1, \dots, n\}\}$. $\mathbb{T}^n(z^0, R) = \{z \in \mathbb{C}^n \colon\ |z_j - z_j^0| = r_j,\ j \in \{1, \dots, n\}\}$, $\mathbb{D}^n[z^0, R] = \{z \in \mathbb{C}^n \colon\ |z_j - z_j^0| \leq r_j,\ j = 1, \dots, n\}$. The domain $\mathbb{G} \subset \mathbb{C}^n$ is called the complete Reinhardt domain [10] if $\forall z = (z_1, \dots, z_n) \in \mathbb{G}$ and $\forall R = (r_1, \dots, r_n) \in [0, 1]^n$ the product $Rz = (r_1z_1, \dots, r_nz_n) \in \mathbb{G}$ and $\forall \Theta = (\theta_1, \dots, \theta_n) \in [0; 2\pi]^n)$ one has $(z_1e^{i\theta_1}, \dots, z_ne^{i\theta_n}) \in \mathbb{G}$. By $\mathcal{A}(\mathbb{G})$ we denote a class of functions which are analytic in the $\mathbb{G}$.

For $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$ the partial derivatives of the function H we denote $H^{(J)}(z) = \frac{\partial^{\|J\|}H}{\partial z^J}(z) = \frac{\partial^{j_1+j_2+\cdots+j_n}H}{\partial z_1^{j_1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(z_1, z_2, \dots, z_n)$. By $\overline{\mathbb{G}}$, we denote the closure of the complete Reinhardt domain $\mathbb{G}$ and $\partial\mathbb{G} = \overline{\mathbb{G}} \setminus \mathbb{G}$. We suppose that a continuous function $\mathbf{L}(z) = (l_1(z), l_2(z), \dots, l_n(z))$ satisfies the following condition in whole $\mathbb{G}$

$$l_j(z) > \frac{\beta}{\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G}, \\ r>1}} (r|z_j|) - |z_j|} \tag{1}$$

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for some real $\beta > 1$. Here, $\widehat{R}_j = (1, \dots, 1, \underbrace{r}_{j\text{-th item}}, 1, \dots, 1)$. At the same time, if the set $\{z\widehat{R}_j \colon r > 1\}$ is unbounded for a given $z \in \mathbb{G}$, then we will only require that the condition $l_j(z) > 0$ be fulfilled. We will assume $l_j(z) > 0$ in the case when $\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G},\\ r>1}} r|z_j| = +\infty$. Such a case is possible, for example, if $\mathbb{G} = \mathbb{C}\times\mathbb{D}$ (see [9,12]). We also write $\mathcal{B} = (0, \beta]$, $\mathcal{B}^n = (0, \beta]^n$, and $\boldsymbol{\beta} = (\beta, \dots, \beta)$. Below we suppose everywhere that $\mathbb{G} \subset \mathbb{C}^n$ is the complete Reinhardt domain, and we will not repeat this assumption in the following assertions and definitions.

A multivariate holomorphic function $H \in \mathcal{A}(\mathbb{G})$ is called a function with *bounded (finite)* $\mathbf{L}$*-index (in joint variables)* (see [1]) if, for some non-negative integer n_0, the following inequality holds for every order J of partial derivatives in the whole domain $\mathbb{G}$:

$$\frac{|H^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon\ K \in \mathbb{Z}_+^n,\ \|K\| \le n_0\right\}. \tag{2}$$

The least corresponding number n_0 is the $\mathbf{L}$*-index in joint variables* for the function H, and $N(H, \mathbf{L}, \mathbb{G}) = n_0$ stands for the index. If the Reinhardt domain $\mathbb{G}$ matches with n-dimensional complex space $\mathbb{C}^n$, and if the mapping $\mathbf{L}$ identically equals $\mathbf{1}$, then it is a definition of an entire multivariate function of a bounded index [14,15]. In addition, if $n = 1$ and $\mathbf{L} = l$, then it becomes the definition of univariate entire function of bounded l-index [16], and if, finally, $l \equiv 1$, then we obtain the definition of the entire function having a bounded (finite) index [11]. Entire functions of bounded L-index in direction are considered in [4,8].

In addition, we define the global characteristics λ_j for the function l_j :

$$\lambda_j(R) = \sup_{z,w\in\mathbb{G}}\left\{\frac{l_j(z)}{l_j(w)} \colon |z_k - w_k| \le \frac{r_k}{\min\{l_k(z), l_k(w)\}}, k \in \{1, \dots, n\}\right\}, \tag{3}$$

where $r_k \in (0, \beta]$. By $Q(\mathbb{G})$ we denote the class of functions $\mathbf{L}\colon \mathbb{G} \to \mathbb{R}_+^n$ satisfying (1) and $\lambda_j(R)$ from (3) is finite.

2. Auxiliary propositions.

Theorem 1 ([1])**.** *Let* $\mathbf{L} \in Q(\mathbb{G})$*,* $H \in \mathcal{A}(\mathbb{G})$*. The index* $N(H, \mathbf{L})$ *is finite if and only if* $\forall R \in \mathcal{B}^n$ $\exists n_0 \in \mathbb{Z}_+$ $\exists d_0 > 0$ *such that* $\forall z^0 \in \mathbb{G}$ $\exists K^0 \in \mathbb{Z}_+^n$ $\|K^0\| \le n_0$ *and*

$$\max\left\{\frac{|H^{(k_1,\dots,k_n)}(z)|}{k_1!\cdots k_n!(\mathbf{L}(z))^{(k_1,\dots,k_n)}} \colon k_1 + \dots + k_n \le n_0,\ z \in \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \le d_0\frac{|H^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{4}$$

Theorem 2 ([1])**.** *Let* $\mathbf{L} \in Q(\mathbb{G})$*,* $H \in \mathcal{A}(\mathbb{G})$*. If the index* $N(H, \mathbf{L})$ *is finite then* $\forall R \in \mathcal{B}^n$ $\exists n_0 \in \mathbb{Z}_+$ $\exists d \ge 1$ *such that* $\forall z^0 \in \mathbb{G}$ $\exists K^0 \in \mathbb{Z}_+^n$ *with* $\|K^0\| \le n_0$ *and*

$$\max\left\{|H^{(K^0)}(z)| \colon z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le d|H^{(K^0)}(z^0)|. \tag{5}$$

And if $\forall R \in \mathcal{B}^n$ $\exists n_0 \in \mathbb{Z}_+$*,* $\exists d \ge 1$ *and* $\forall z^0 \in \mathbb{G}$ $\forall j \in \{1, \dots, n\}$*,* $\exists K_j^0 = k_j^0 \cdot \mathbf{1}_j$ *with* $k_j^0 \le n_0$

$$\max\left\{|H^{(K_j^0)}(z)| \colon z \in \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \le d|H^{(K_j^0)}(z^0)|, \tag{6}$$

then the index $N(H, \mathbf{L})$ *is finite.*

In proof of Theorem 2 one should observe that K^0 is the same as K^0 in Theorem 1.

Let $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z), \ldots, \widetilde{l}_n(z)) \in Q(\mathbb{G})$. The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist $\Theta_1 = (\theta_{1,j}, \ldots, \theta_{1,n})$, $\Theta_2 = (\theta_{2,j}, \ldots, \theta_{2,n}) \in \mathbb{R}^n_+$ such that $\theta_{1,j}\widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j}\widetilde{l}_j(z)$ for every $j \in \{1, 2, 3, \ldots, n\}$ and for every $z \in \mathbb{G}$.

Theorem 3 ([1]). *Let $\mathbf{L} \in Q(\mathbb{G})$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta\Theta_1 > \mathbf{1}$, $H \in \mathcal{A}(\mathbb{G})$. The joint index $N(H, \widetilde{\mathbf{L}})$ is finite if and only if $N(H, \mathbf{L})$ is finite.*

3. Local behaviour of maximum modulus of analytic in ball function. For an analytic in $\mathbb{G}$ function F we put $M(r, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{T}^n(z^0, r)\}$, where $z^0 \in \mathbb{G}$, $r \in \mathbb{R}^n_+$. Then $M(R, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{D}^n[z^0, R]\}$, because the maximum modulus for an analytic function in a closed polydisc is attained on its skeleton. Similar results for another classes of holomorphic functions were deduced in the case of a unit ball [2], polydisc [5], slice holomorphicity [6]. They need to justify growth estimates [3] and describe zero distribution [7].

Theorem 4. *Let $\mathbf{L} \in Q(\mathbb{G})$, $F \in \mathcal{A}(\mathbb{G})$. If $\exists R'$, $R'' \in \mathcal{B}^n$, $R' < R''$, and $p_1 \ge 1$ such that $\forall z^0 \in \mathbb{G}$ one has*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \le p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) \tag{7}$$

then F is of bounded $\mathbf{L}$-index in joint variables.

Proof. At first, we assume that $\mathbf{0} < R' < \mathbf{1} < R''$. Let $z^0 \in \mathbb{G}$ be an arbitrary point. We expand a function F in power series

$$F(z) = \sum_{K \ge \mathbf{0}} b_K(z - z^0)^K = \sum_{k_1, \ldots, k_n \ge 0} b_{k_1, \ldots, k_n}(z_1 - z_1^0)^{k_1} \ldots (z_n - z_n^0)^{k_n}, \tag{8}$$

where $b_K = b_{k_1, \ldots, k_n} = \frac{F^{(K)}(z^0)}{K!}$. Let $\mu(R, z^0, F) = \max\{|b_K|R^K\colon\ K \ge \mathbf{0}\}$ be a maximal term of power series (8) and $\nu(R) = \nu(R, z^0, F) = (\nu_1^0(R), \ldots, \nu_n^0(R))$ be a set of indices such that $\mu(R, z^0, F) = |b_{\nu(R)}|R^{\nu(R)}$,

$$\|\nu(R)\| = \sum_{j=1}^{n} \nu_j(R) = \max\{\|K\|\colon K \ge \mathbf{0},\ |b_K|R^K = \mu(R, z^0, F)\},$$

where $R = (r_1, r_2, \ldots, r_n)$. Remind the notation $\widehat{R}_j = (1, \ldots, 1, \underbrace{r}_{j\text{-th item}}, 1, \ldots, 1)$. Since

$$|F^{(K)}(z^0)|R^K/K! \le \max\{|F(z)|\colon z \in \mathbb{T}^n(z^0, R)\}$$

we obtain that for any $r_j < \inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G},\\ r > 1}} (r|z_j^0|) - |z_j^0|$ one has $\mu(R, z^0, F) \le M(R, z^0, F)$. Then for given R' with $\mathbf{0} < R' < \mathbf{1}$ we conclude

$$M(R'R, z^0, F) \le \sum_{K \ge \mathbf{0}} |b_k|(R'R)^k \le \mu(R, z^0, F) \sum_{K \ge \mathbf{0}} (R')^k = \prod_{j=1}^{n} \frac{\mu(R, z^0, F)}{1 - r_j'}. \tag{9}$$

Besides, for given R'' with $\mathbf{1} < R'' < \boldsymbol{\beta}$

$$\ln \mu(R, z^0, F) = \ln\{|b_{\nu(R)}|R^{\nu(R)}\} = \ln\left\{|b_{\nu(R)}|(RR'')^{\nu(R)}\frac{1}{(R'')^{\nu(R)}}\right\} =$$

$$= \ln\{|b_{\nu(R)}|(RR'')^{\nu(R)}\} + \ln\left\{\frac{1}{(R'')^{\nu(R)}}\right\} \leq \ln\mu(R''R, z^0, F) - \|\nu(R)\| \ln \min_{1\leq j\leq n} r''_j.$$

From the last estimate we express the $\|\nu(R)\|$ by the difference of logarithms of maximal terms and substitute the lower estimate of $\mu(R, z^0, F)$ from (9)

$$\|\nu(R)\| \leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}(\ln\mu(R''R, z^0, F) - \ln\mu(R, z^0, F)) \leq$$

$$\leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}\Big(\ln M(R''R, z^0, F) - \ln\Big(\prod_{j=1}^{n}(1-r'_j)M(R'R, z^0, F)\Big)\Big) \leq$$

$$\leq \frac{1}{\ln\min_{1\leq j\leq n} r''_j}\left(\ln M(R''R, z^0, F) - \ln M(R'R, z^0, F)\right) - \frac{\sum_{j=1}^{n}\ln(1-r'_j)}{\ln\min_{1\leq j\leq n} r''_j} =$$

$$= \frac{1}{\min_{1\leq j\leq n} r''_j} \ln\frac{M(R''R, z^0, F)}{M(R'R, z^0, F)} - \frac{\sum_{j=1}^{n}\ln(1-r_j)}{\ln\min_{1\leq j\leq n} r''_j}. \tag{10}$$

Put $R = \frac{\mathbf{1}}{\mathbf{L}(z^0)}$. Now let $N(F, z^0, \mathbf{L})$ be the $\mathbf{L}$-index of the function F in joint variables at the point z^0 i. e. it is the least integer for which inequality (2) holds at the point z^0. Clearly that

$$N(F, z^0, \mathbf{L}) \leq \left\|\nu\left(\frac{1}{\mathbf{L}(z^0)}, z^0, F\right)\right\| = \|\nu(R, z^0, F)\|. \tag{11}$$

But $M\left(R''/\mathbf{L}(z^0), z^0, F\right) \leq p_1(R', R'')M\left(R'/\mathbf{L}(z^0), z^0, F\right)$. Consequently substituting the last inequality and (10) in (11) we obtain that for any $z^0 \in \mathbb{G}$

$$N(F, z^0, \mathbf{L}) \leq \frac{-\sum_{j=1}^{n}\ln(1-r'_j)}{\ln\min_{1\leq j\leq n} r''_j} + \frac{\ln p_1(R', R'')}{\ln\min_{1\leq j\leq n} r''_j}.$$

This means that F has bounded $\mathbf{L}$-index in joint variables, if $\mathbf{0} < R' < \mathbf{1} < R'' < \boldsymbol{\beta}$.

Now we will prove the theorem for any $\mathbf{0} < R' < R''$, where R', $R'' \in \mathcal{B}^n$. From (7) with $\mathbf{0} < R' < R''$ it follows that

$$\max\Big\{|F(z)|\colon z \in \mathbb{T}^n\Big(z^0, \frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\Big)\Big\} \leq P_1 \max\Big\{|F(z)|\colon z \in \mathbb{T}^n\Big(z^0, \frac{2R'}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\Big)\Big\},$$

where $\mathbf{0} < \frac{2R'}{R'+R''} < \mathbf{1} < \frac{2R''}{R'+R''}$, $\widetilde{\mathbf{L}}(z) = \frac{2\mathbf{L}(z)}{R'+R''}$. Taking into account the first part of the proof, we conclude that the function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables. To apply Theorem 3, we check that $\beta \cdot \frac{2\cdot\mathbf{1}}{R'+R''} > \beta \cdot \frac{2\cdot\mathbf{1}}{\boldsymbol{\beta}+\boldsymbol{\beta}} = \mathbf{1}$. Thus, the function F is of bounded $\mathbf{L}$-index in joint variables. □

Also the corresponding necessary conditions are valid.

Theorem 5. *Let $\mathbf{L} \in Q(\mathbb{G})$. If an analytic in $\mathbb{G}$ function F has bounded $\mathbf{L}$-index in joint variables then for any R', $R'' \in \mathcal{B}^n$, $R' < R''$, there exists a number $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{G}$ inequality (7) holds.*

Proof. Let $N(F, \mathbf{L}, \mathbb{G}) = N < +\infty$. Suppose that inequality (7) does not hold i.e. there exist R', $R'' \in \mathcal{B}^n$, $R' < R''$, such that for each $p_* \geq 1$ and for some $z^0 = z^0(p_*)$

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) > p_* M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right). \tag{12}$$

By Theorem 1, there exists a number $p_0 = p_0(R'') \geq 1$ such that for every $z^0 \in \mathbb{G}$ and some $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \leq N$, (i.e. $n_0 = N$, see proof of necessity of [1, Theorem 1]) one has for any J with $\|J\| \leq N$

$$\max\left\{\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}:\ z \in \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \leq p_0 \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)} \tag{13}$$

and by Theorem 2 for the same K^0

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F^{(K^0)}\right) \leq p_1 |F^{(K^0)}(z^0)|. \tag{14}$$

We put

$$B_n(r,k) = p_0\left(\prod_{j=1}^n \lambda_j^N(R'')\right)(N!)^{n-k}\left(\sum_{j=1}^N \frac{(N-j)!}{(r)^j}\right),$$

$$b_k = B_n(r''_k, k)\cdot\left(\frac{r''_k r''_{k+1}\ldots r''_n}{r'_k r'_{k+1}\ldots r'_n}\right)^N \cdot \max\left\{1, \frac{1}{(r'_1\ldots r'_{k-1})^N}\right\} \quad (2 \leq k \leq n),$$

$$b_1 = p_0\left(\prod_{j=1}^n \lambda_j^N(R'')\right)(N!)^{n-1}\left(\sum_{j=1}^N \frac{(N-j)!}{(r''_1)^j}\right)\left(\frac{r''_1 r''_2\ldots r''_n}{r'_1 r'_2\ldots r'_n}\right)^N,$$

$$p_* = (N!)^n p_0 \left(\frac{r''_1 r''_2\ldots r''_n}{r'_1 r'_2\ldots r'_n}\right)^N + \sum_{k=1}^n b_k + 1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (12) holds and K^0 be such that (14) holds and

$$M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|, \quad M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F^{(J)}\right) = |F^{(J)}(z^*_J)|$$

for every $J \in \mathbb{Z}^n_+$, $\|J\| \leq N$. We apply Cauchy's inequality

$$|F^{(J)}(z^0)| \leq J!\left(\frac{\mathbf{L}(z^0)}{R'}\right)^J |F(z^*)| \tag{15}$$

for estimate the difference

$$|F^{(J)}(z^*_J) - F^{(J)}(z^0_1, z^*_{J,2}, \ldots, z^*_{J,n})| = \Big|\int_{z^0_1}^{z^*_{J,1}} F^{(J+\mathbf{1}_1)}(\xi, z^*_{J,2}, \ldots, z^*_{J,n})d\xi\Big| \leq$$
$$\leq |F^{(J+\mathbf{1}_1)}(z^*_{J+\mathbf{1}_1})|\frac{r''_1}{l_1(z^0)}. \tag{16}$$

Such a point $(z^0_1, z^*_{J,2}, \ldots, z^*_{J,n})$ belongs to the polydisc $\mathbb{D}^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]$, because for all $k \in \{2, 3, \ldots, n\}$ $|z^*_{J,k} - z^0_k| = \frac{r''_k}{l_k(z^0)}$. Then by inequality (13) we establish

$$|F^{(J)}(z^0_1, z^*_{J,2}, \ldots, z^*_{J,n})| \leq J!\mathbf{L}^J(z^0_1, z^*_{J,2}, \ldots, z^*_{J,n}) \max\left\{\frac{|F^{(J)}(z)|}{J!L^J(Z)}: z \in \mathbb{D}^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]\right\} \leq$$
$$\leq \frac{J!\mathbf{L}^J(z^0_1, z^*_{J,2}, \ldots, z^*_{J,n})}{K^0!\mathbf{L}^{K^0}(z^0)} p_0 |F^{(K^0)}(z^0)|. \tag{17}$$

Since $\mathbf{L}\in Q(\mathbb{G})$ the estimate $l_k(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*) \le \lambda_k(R'')l_k(z^0)$ holds. Applying it and (15) with $J=K^0$ to (17) we deduce

$$|F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)| \le \frac{J!\mathbf{L}^J(z^0)\prod_{k=1}^n \lambda_k^{j_k}(R'')}{K^0!\mathbf{L}^{K^0}(z^0)} p_0 K^0! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^{K^0} |F(z^*)| = \\ = \frac{p_0 J!\mathbf{L}^J(z^0)\prod_{k=1}^n \lambda_k^{j_k}(R'')}{(R')^{K^0}}|F(z^*)|. \tag{18}$$

Rewrite inequality (16) in the reverse order and substitute estimate (18) instead of $|F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)|$

$$\left|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1+1}\partial z_2^{j_2}\ldots\partial z_n^{j_n}}(z^*_{(j_1+1,j_2,\ldots,j_n)})\right| \ge \frac{l_1(z^0)}{r_1''}\left\{|F^{(J)}(z_J^*)| - |F^{(J)}(z_1^0, z_{J,2}^*, \ldots, z_{J,n}^*)|\right\} \ge \\ \ge \frac{l_1(z^0)}{r_1''}|F^{(J)}(z_J^*)| - \frac{p_0 J!\mathbf{L}^{(j_1+1,j_2,\ldots,j_n)}(z^0)\prod_{k=1}^n \lambda_k^{j_k}(R'')}{r_1''(R')^{K^0}}|F(z^*)|. \tag{19}$$

Then we will consequently apply (19) for $J=K^0$, $J=K^0-\mathbf{1}_1$, $J=K^0-2\cdot\mathbf{1}_1$, and so on. On each stage we will apply (19) to decrease partial derivative order of the function F by one in the variable z_1

$$|F^{(K^0)}(z^*_{K^0})| \ge \frac{l_1(z^0)}{r_1''}|f^{(K^0-\mathbf{1}_1)}(z^*_{(K^0-\mathbf{1}_1)})| - \frac{p_0(k_1^0-1)!k_2^0!\ldots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=1}^n\lambda_i^{k_i^0}(R'')}{r_1''(R')^{K^0}}|F(z^*)| \ge \\ \ge \frac{l_1^2(z^0)}{(r_1'')^2}|f^{(K^0-2\cdot\mathbf{1}_1)}(z^*_{K^0-2\cdot\mathbf{1}_1})| - \frac{p_0(k_1^0-2)!k_2^0!\ldots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=1}^n\lambda_i^{k_i^0}(R'')}{(r_1'')^2(R')^{K^0}}|F(z^*)| - \\ - \frac{p_0(k_1^0-1)!k_2^0!\ldots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=1}^n\lambda_i^{k_i^0}(R'')}{r_1''(R')^{K^0}}|F(z^*)| \ge \ldots \ge \\ \ge \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\left|\frac{\partial^{\|K^0\|-k_1^0}f}{\partial z_2^{k_2^0}\ldots\partial z_n^{k_n^0}}(z^*_{(0,k_2^0,\ldots,k_n^0)})\right| - \frac{p_0\mathbf{L}^{K^0}(z^0)}{(R')^{K^0}}\prod_{i=1}^n\lambda_i^{k_i^0}(R'')k_2^0!\ldots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)|. \tag{20}$$

Estimate (19) is proved for decrease of partial derivative order in the variable z_1. Repeating similar considerations it is possible to prove similar estimate in any variable, i.e. for any $s\in\{1,\ldots,n\}$ one has

$$|F^{(J+\mathbf{1}_s)}(z^*_{J+\mathbf{1}_s})| \ge \frac{l_s(z^0)}{r_s''}|F^{(J)}(z_J^*)| - \frac{p_0 J!\mathbf{L}^{(J+\mathbf{1}_s)}(z^0)\prod_{k=1}^n\lambda_k^{j_k}(R'')}{r_s''(R')^{K^0}}|F(z^*)|. \tag{21}$$

Applying (21) to (20) in each other variable s_2, s_3, …, s_n we deduce

$$|F^{(K^0)}(z^*_{K^0})| \ge \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\frac{l_2^{k_2^0}(z^0)}{(r_2'')^{k_2^0}}\left|\frac{\partial^{\|K^0\|-k_1^0-k_2^0}f}{\partial z_3^{k_3^0}\ldots\partial z_n^{k_n^0}}(z^*_{(0,0,k_3^0,\ldots,k_n^0)})\right| - \\ - \frac{l_1^{k_1^0}(z^0)p_0L^{(0,k_2^0,\ldots,k_n^0)}(z^0)}{(r_1'')^{k_1^0}(R')^{K^0}}\left(\prod_{i=1}^n\lambda_i^{k_i^0}(R'')\right)k_3^0!\ldots k_n^0!\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}|F(z^*)| -$$

$$-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=1}^n \lambda_i^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)| \geq$$

$$\geq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*_{\mathbf{0}})| - |F(z^*)|\sum_{i=1}^b \tilde{b}_i, \tag{22}$$

where $\tilde{b}_i$ denotes multiplier near $|F(z^*)|$ of such a form

$$\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=1}^n \lambda_i^{k_i^0}(R'')\right)\frac{k_{i+1}^0!\dots k_n^0!}{(r_1'')^{k_1^0}\cdot\ldots\cdot(r_{i-1}'')^{k_n^0}}\sum_{j_i=1}^{k_i^0}\frac{(k_i^0-j_i)!}{(r_i'')^{j_i}}|F(z^*)|.$$

In view of the inequalities $\lambda_i(R'') \geq 1$ and $R'' \geq R'$, the parameters $\tilde{b}_i$ from (22) can be estimated by b_i, if we replace K^0 by $N\cdot\mathbf{1}$

$$\tilde{b}_1 = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=1}^n \lambda_i^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}} =$$

$$= \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left(\frac{R''}{R'}\right)^{K^0}p_0\left(\prod_{i=1}^n \lambda_i^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}} \leq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_1,$$

$$\tilde{b}_2 = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n \lambda_i^{k_i^0}(R'')\right)\frac{k_3^0!\dots k_n^0!}{(r_1'')^{k_1^0}}\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}} \leq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_2,$$

$$\tilde{b}_{n-1} = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\frac{\lambda_{n-1}^{k_{n-1}^0}(R'')\lambda_n^{k_n^0}(R'')k_n^0!}{(r_1'')^{k_1^0}\dots(r_{n-2}'')^{k_{n-2}^0}}\sum_{j_{n-1}=1}^{k_{n-1}^0}\frac{(k_{n-1}^0-j_{n-1})!}{(r_{n-1}'')^{j_{n-1}}} \leq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_{n-1},$$

$$\tilde{b}_n = \frac{p_0}{(R')^{K^0}}\lambda_n^{k_n^0}(R'')\mathbf{L}^{K^0}(z^0)\frac{1}{(r_1'')^{k_1^0}\dots(r_{n-1}'')^{k_{n-1}^0}}\sum_{j_n=1}^{k_n^0}\frac{(k_n^0-j_n)!}{(r_n'')^{j_n}} \leq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_n.$$

Thus, (22) implies that

$$|F^{(K^0)}(z^*_{K^0})| \geq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|} - \sum_{j=1}^n b_j\right\}.$$

But in view of (12) and a choice of p_* we have

$$\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|} \geq p_* > \sum_{j=1}^n b_j.$$

Thus, (14) and (15) imply

$$|F^{(K^0)}(z^*_{K^0})| \geq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{p_* - \sum_{j=1}^n b_j\right\} \geq$$

$$\geq \left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left\{p_* - \sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^0)|(R')^{K^0}}{K^0!\mathbf{L}^{K^0}(z^0)} \geq \left(\frac{r_1'\dots r_n'}{r_1''\dots r_n''}\right)^N\left\{p_* - \sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^*_{K^0})|}{p_0(N!)^n}.$$

Hence, we have

$$p_* \leq p_0\big(\tfrac{r_1'\dots r_n'}{r_1''\dots r_n''}\big)^N(N!)^n + \textstyle\sum_{j=1}^n b_j,$$

but this contradicts the choice of p_*. □

REFERENCES

1. A. Bandura, T. Salo, O. Skaskiv, *Analytic functions in a complete Reinhardt domain having bounded L-index in joint variables*, Symmetry, **16** (2024), №3, Article ID: 351. https://doi.org/10.3390/sym16030351
2. A.I. Bandura, O.B. Skaskiv, *Analytic functions in the unit ball of bounded L-index: asymptotic and local properties*, Mat. Stud., **48** (2017), №1, №1, 37–73. https://doi.org/10.15330/ms.48.1.37-73
3. A. Bandura, O. Skaskiv, *Asymptotic estimates of entire functions of bounded* **L***-index in joint variables*, Novi Sad J. Math., **48** (2018), №1, 103–116. https://doi.org/10.30755/NSJOM.06997
4. A. Bandura, *Composition of entire functions and bounded L-index in direction*, Mat. Stud., **47** (2017), №2, 179–184. https://doi.org/10.15330/ms.47.2.179-184
5. A. Bandura, N. Petrechko, O. Skaskiv, *Maximum modulus in a bidisc of analytic functions of bounded* **L***-index and an analogue of Hayman's theorem*, Mat. Bohemica, **143** (2018), №4, 339–354. https://doi.org/10.21136/MB.2017.0110-16
6. A. Bandura, O. Skaskiv, *Slice holomorphic functions in several variables with bounded L-index in direction*, Axioms, **8** (2019), №3, Article ID: 88. https://doi.org/10.3390/axioms8030088
7. A. Bandura, O. Skaskiv, *Entire functions of bounded* **L***-index: Its zeros and behavior of partial logarithmic derivatives*, J. Complex Analysis, **2017** (2017), 1–10. Article ID: 3253095. https://doi.org/10.1155/2017/3253095
8. I.M. Hural, *About some problem for entire functions of unbounded index in any direction*, Mat. Stud., **51** (2019), №1, 107–110. https://doi.org/10.15330/ms.51.1.107-110
9. V.P. Kostov, *A domain free of the zeros of the partial theta function*, Mat. Stud., **58** (2022), №2, 142–158. https://doi.org/10.30970/ms.58.2.142-158
10. A. Kuryliak, O. Skaskiv, *Wiman's type inequality in multiple-circular domain*, Axioms, **10** (2021), №4, Article ID: 348. https://doi.org/10.3390/axioms10040348
11. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Proc. Sympos. Pure Math., **11** (1968), 298–307.
12. T.H. Nguyen, *On the conditions for a special entire function related to the partial Theta-function and the q-Kummer functions to belong to the Laguerre-Polya class*, Comput. Methods Funct. Theory, **22** (2022), №1, 7–25. https://doi.org/10.1007/s40315-021-00361-0
13. F. Nuray, R.F. Patterson, *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*, Mat. Stud., **49** (2018), №1, 67–74. https://doi.org/10.15330/ms.49.1.67-74
14. F. Nuray, R.F. Patterson, *Entire bivariate functions of exponential type*, Bull. Math. Sci., **5** (2015), №2, 171–177. http://dx.doi.org/10.1007/s13373-015-0066-x
15. F. Nuray, R. F. Patterson, *Multivalence of bivariate functions of bounded index*, Le Matematiche (Catania), **70** (2015), №2, 225–233. https://doi.org/10.4418/2015.70.2.14
16. M.M. Sheremeta, A.D. Kuzyk, *Logarithmic derivative and zeros of an entire function of bounded l-index*, Sib. Math. J., **33** (1992), №2, 304–312. https://doi.org/10.1007/BF00971102

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