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GENERALIZED AND MODIFIED ORDERS OF GROWTH FOR DIRICHLET SERIES ABSOLUTELY CONVERGENT IN A HALF-PLANE

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Let $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ be a non-negative sequence increasing to $+\infty$, $\tau(\lambda)=\overline{\lim}_{n\to\infty}(\ln n/\lambda_n)$, and $\mathcal{D}_0(\lambda)$ be the class of all Dirichlet series of the form $F(s)=\sum_{n=0}^{\infty}a_n(F)e^{s\lambda_n}$ absolutely convergent in the half-plane $\operatorname{Re}s<0$ with $a_n(F)\ne 0$ for at least one integer $n\ge 0$. Also, let α be a continuous function on $[x_0,+\infty)$ increasing to $+\infty$, β be a continuous function on $[a,0)$ such that $\beta(\sigma)\to+\infty$ as $\sigma\uparrow 0$, and γ be a continuous positive function on $[b,0)$. In the article, we investigate the growth of a Dirichlet series $F\in\mathcal{D}_0(\lambda)$ depending on the behavior of the sequence $(|a_n(F)|)$ in terms of its α,β,γ-orders determined by the equalities

$$R^*_{\alpha,\beta,\gamma}(F)=\overline{\lim_{\sigma\uparrow 0}}\frac{\alpha(\max\{x_0,\gamma(\sigma)\ln\mu(\sigma)\})}{\beta(\sigma)},\quad R_{\alpha,\beta,\gamma}(F)=\overline{\lim_{\sigma\uparrow 0}}\frac{\alpha(\max\{x_0,\gamma(\sigma)\ln M(\sigma)\})}{\beta(\sigma)},$$

where $\mu(\sigma)=\max\{|a_n(F)|e^{\sigma\lambda_n}\colon n\ge 0\}$ and $M(\sigma)=\sup\{|F(s)|\colon \operatorname{Re}s=\sigma\}$ are the maximal term and the supremum modulus of the series F, respectively. In particular, if for every fixed $t>0$ we have $\alpha(tx)\sim\alpha(x)$ as $x\to+\infty$, $\beta(t\sigma)\sim t^{-\rho}\beta(\sigma)$ as $\sigma\uparrow 0$ for some fixed $\rho>0$, $0<\underline{\lim}_{\sigma\uparrow 0}\gamma(t\sigma)/\gamma(\sigma)\le\overline{\lim}_{\sigma\uparrow 0}\gamma(t\sigma)/\gamma(\sigma)<+\infty$, $\Phi(\sigma)=\alpha^{-1}(\beta(\sigma))/\gamma(\sigma)$ for all $\sigma\in[\sigma_0,0)$, $\widetilde{\Phi}(x)=\max\{x\sigma-\Phi(\sigma)\colon\sigma\in[\sigma_0,0)\}$ for all $x\in\mathbb{R}$, and $\Delta_\Phi(\lambda)=\overline{\lim}_{n\to\infty}(-\ln n/\widetilde{\Phi}(\lambda_n))$, then:
(a) for each Dirichlet series $F\in\mathcal{D}_0(\lambda)$ we have

$$R^*_{\alpha,\beta,\gamma}(F)=\overline{\lim_{n\to+\infty}}\left(\frac{\ln^+|a_n(F)|}{-\widetilde{\Phi}(\lambda_n)}\right)^{\rho};$$

(b) if $\tau(\lambda)>0$, then for each $p_0\in[0,+\infty]$ and any positive function Ψ on $[c,0)$ there exists a Dirichlet series $F\in\mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F)=p_0$ and $M(\sigma,F)\ge\Psi(\sigma)$ for all $\sigma\in[\sigma_0,0)$;
(c) if $\tau(\lambda)=0$, then $(R_{\alpha,\beta,\gamma}(F))^{1/\rho}\le(R^*_{\alpha,\beta,\gamma}(F))^{1/\rho}+\Delta_\Phi(\lambda)$ for every Dirichlet series $F\in\mathcal{D}_0(\lambda)$;
(d) if $\tau(\lambda)=0$, then for each $p_0\in[0,+\infty]$ there exists a Dirichlet series $F\in\mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F)=p_0$ and $(R_{\alpha,\beta,\gamma}(F))^{1/\rho}=(R^*_{\alpha,\beta,\gamma}(F))^{1/\rho}+\Delta_\Phi(\lambda)$.

1. Introduction. We denote by $\mathbb{N}_0$ the set of all non-negative integers, and denote by Λ the class of all non-negative sequences $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ increasing to $+\infty$.

Let $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ. Consider a Dirichlet series of the form

$$F(s)=\sum_{n=0}^{\infty}a_ne^{s\lambda_n}\tag{1}$$

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and denote by $\sigma_a(F)$ the abscissa of absolute convergence of this series. Put

$$\sigma^*(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

It is easy to see that if $\sigma < \sigma^*(F)$, then $|a_n|e^{\sigma\lambda_n} \to 0$ as $n \to \infty$. Therefore, for each such σ, we can determine the maximal term

$$\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}$$

of series (1). Note also that in the case when $\sigma > \sigma^*(F)$ we have $\varlimsup_{n\to\infty} |a_n|e^{\sigma\lambda_n} = +\infty$. If $\sigma_a(F) > -\infty$, then for all $\sigma < \sigma_a(F)$ we define the supremum modulus of series (1) by equality

$$M(\sigma, F) = \sup\{|F(s)| \colon \operatorname{Re} s = \sigma\}.$$

We denote by $\mathcal{D}_0^*(\lambda)$ the class of all Dirichlet series of the form (1), for which $\sigma^*(F) \geq 0$ and $a_n \neq 0$ at least for one value of $n \in \mathbb{N}_0$. By $\mathcal{D}_0(\lambda)$ we denote the class of all Dirichlet series of the form (1) such that $\sigma_a(F) \geq 0$ and $a_n \neq 0$ at least for one value of $n \in \mathbb{N}_0$. It is clear that $\mathcal{D}_0(\lambda) \subset \mathcal{D}_0^*(\lambda)$ and, as it is well known, $\mathcal{D}_0(\lambda) = \mathcal{D}_0^*(\lambda)$ if and only if $\tau(\lambda) = 0$, where

$$\tau(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n}.$$

Put

$$\mathcal{D}_0 = \bigcup_{\lambda\in\Lambda} \mathcal{D}_0(\lambda), \quad \mathcal{D}_0^* = \bigcup_{\lambda\in\Lambda} \mathcal{D}_0^*(\lambda).$$

For $A \in (-\infty, +\infty]$, we denote by Y_A the class of all real functions $\eta \colon D_\eta \to \mathbb{R}$ such that the domain D_η of η is an interval of the form $[a, A)$.

Let $\eta \in Y_{+\infty}$ be a positive measurable function on D_η. As in [1], we call the function η slowly varying at the point $+\infty$ if for every fixed number $c > 0$ we have $\eta(cx) \sim \eta(x)$ as $x \to +\infty$, and we call the function η regularly varying at the point $+\infty$ with index $\rho \geq 0$, if $\eta(x) = x^\rho\zeta(x)$ for all $x \geq x_0$, where $\zeta \in Y_{+\infty}$ is a slowly varying function at the point $+\infty$.

Let $\eta \in Y_0$ be a positive measurable function on D_η. We call the function η slowly varying at the point 0 if for every fixed number $c > 0$ we have $\eta(c\sigma) \sim \eta(\sigma)$ as $\sigma \uparrow 0$, and we call the function η regularly varying at the point 0 with index $\rho \geq 0$, if $\eta(\sigma) = |\sigma|^{-\rho}\zeta(\sigma)$ for all $\sigma \in [\sigma_0, 0)$, where $\zeta \in Y_0$ is a slowly varying function at the point 0.

We denote by L the sub-class of all functions $l \in Y_{+\infty}$ continuous and increasing to $+\infty$ on D_l.

We denote by C_0 the sub-class of all functions $\eta \in Y_0$ continuous on D_η, and denote by Ω_0 the sub-class of all functions $\Phi \in C_0$ such that $\Phi(\sigma) \to +\infty$ as $\sigma \uparrow 0$.

Let $\Phi \in \Omega_0$. Then, as it is well known, the function

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) \colon \sigma \in D_\Phi\}, \quad x \in \mathbb{R},$$

is called the Young-conjugate of Φ, and this function has the following properties (see, for instance, [2]): $\widetilde{\Phi}$ is convex on $\mathbb{R}$; the right-hand derivative φ of $\widetilde{\Phi}$ is a negative nondecreasing function on $\mathbb{R}$, $\varphi(x) \to 0$ as $x \to +\infty$, and

$$\varphi(x) = \max\{\sigma \in D_\Phi \colon x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}, \quad x \in \mathbb{R};$$

if $x_0 = \inf\{x > 0\colon \Phi(\varphi(x)) > 0\}$, then $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increases to 0 on $(x_0, +\infty)$. It follows from these properties that $\widetilde{\Phi}$ is a decreasing continuous function on $\mathbb{R}$. It is also easy to prove that the range of the function $\widetilde{\Phi}$ is $\mathbb{R}$. In fact, if $\sigma_0 \in D_\Phi$ is a fixed number, then for all $x \in \mathbb{R}$ we have $\widetilde{\Phi}(x) \geq x\sigma_0 - \Phi(\sigma_0)$. Letting $x \to -\infty$, we obtain $\widetilde{\Phi}(-\infty) = +\infty$. In addition, if $x > 0$, then $\widetilde{\Phi}(x) = x\varphi(x) - \Phi(\varphi(x)) < -\Phi(\varphi(x))$. Letting $x \to +\infty$, we obtain $\widetilde{\Phi}(+\infty) = -\infty$. Therefore, the function $\widetilde{\Phi}$ assumes every value in $\mathbb{R}$.

Let $\alpha \in L$, $D_\alpha = [x_0, 0)$, $\beta \in \Omega_0$, and let $\eta \in Y_0$ be a function non-decreasing on D_η. The quantity

$$R_{\alpha,\beta}[\eta] = \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\max\{x_0, \eta(\sigma)\})}{\beta(\sigma)}$$

is called the *generalized order (α, β-order)* of the function η. Note that in the definition of the quantity $R_{\alpha,\beta}[\eta]$, the constant x_0 can be replaced by any other number from D_α, and if $b \in D_\eta$, then the function η can be replaced by the restriction of η to $[b, 0)$. It is also clear that if $\zeta \in Y_0$ is a function non-decreasing on D_ζ and $\zeta(\sigma) \leq \eta(\sigma)$ for all $\sigma \in [c, 0)$, then $R_{\alpha,\beta}[\zeta] \leq R_{\alpha,\beta}[\eta]$.

For each Dirichlet series $F \in \mathcal{D}_0^*$, we set $R^*_{\alpha,\beta}(F) = R_{\alpha,\beta}[\eta]$, where $\eta(\sigma) = \ln \mu(\sigma, F)$ for all $\sigma \in [-1, 0)$. If $F \in \mathcal{D}_0$, then we set $R_{\alpha,\beta}(F) = R_{\alpha,\beta}[\eta]$, where $\eta(\sigma) = \ln M(\sigma, F)$ for all $\sigma \in [-1, 0)$; the quantity $R_{\alpha,\beta}(F)$ is called the *generalized order (α, β-order)* of the Dirichlet series F. It is clear that for each Dirichlet series $F \in \mathcal{D}_0$ we have $R^*_{\alpha,\beta}(F) \leq R_{\alpha,\beta}(F)$.

The growth of a Dirichlet series $F \in \mathcal{D}_0$ is usually identified with the growth of the function $\ln M(\sigma, F)$ as $\sigma \uparrow 0$. Important characteristics of the growth of such a series are its generalized orders $R_{\alpha,\beta}(F)$. Establishing various relations according to which the generalized order $R_{\alpha,\beta}(F)$ of a Dirichlet series $F \in \mathcal{D}_0$ of the form (1) can be expressed by the sequences of modules of its coefficients $(|a_n|)_{n\in\mathbb{N}_0}$, is a well-known classical problem. In connection with this problem, note that the generalized order $R^*_{\alpha,\beta}(F)$ of a Dirichlet series $F \in \mathcal{D}_0$ of the form (1) can be relatively simply expressed in terms of the sequence $(|a_n|)_{n\in\mathbb{N}_0}$ (see, for example, [2, 3]; see also below). In view of what has been said, the following problem arises.

Problem 1. *Let $\alpha \in L$, $\beta \in \Omega_0$, and $\lambda \in \Lambda$. Find a necessary and sufficient condition on the sequence λ under which $R_{\alpha,\beta}(F) = R^*_{\alpha,\beta}(F)$ for each Dirichlet series $F \in \mathcal{D}_0(\lambda)$.*

A similar problem for entire (absolutely convergent in $\mathbb{C}$) Dirichlet series was considered in [2, 4]. In [2], moreover, Problem 1 was completely solved in the case when $\alpha(x) = x$ for all $x \in [x_0, +\infty)$. Without going into details, we note that the results obtained in [2] also allow us to find a complete solution of Problem 1 in the case when $\alpha \in L$ is an arbitrary function regularly varying at the point $+\infty$ with index $\rho > 0$. This case is partially covered in this article.

We note also that by certain assumptions about the growth of functions $\alpha \in L$ and $\beta \in \Omega_0$, sufficient conditions on a sequence $\lambda \in \Lambda$ under which $R_{\alpha,\beta}(F) = R^*_{\alpha,\beta}(F)$ for any Dirichlet series $F \in \mathcal{D}_0(\lambda)$, were found in many works (see, for example, [5, 6, 7, 8, 9, 10, 11, 12]). In this article, Problem 1 is completely solved, in particular, in the case when $\alpha \in L$ is an arbitrary function slowly varying at the point $+\infty$ and $\beta \in \Omega_0$ is an arbitrary function regularly varying at the point 0 with index $\rho > 0$.

For every function $\Phi \in \Omega_0$ and each sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ from the class Λ, we put

$$\Delta_\Phi(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{-\widetilde{\Phi}(\lambda_n)}.$$

Theorem 1. *Let $\alpha \in L$ be a function regularly varying at the point $+\infty$ with index $\rho_1 \geq 0$, $\beta \in \Omega_0$ be a function regularly varying at the point 0 with index $\rho_2 \geq 0$, $\rho = \rho_1 + \rho_2 > 0$, and $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))$ for all $\sigma \in [a, 0)$. Then the following statements are true:*
(a) for each Dirichlet series $F \in \mathcal{D}_0^*$ of the form (1) we have

$$R^*_{\alpha,\beta}(F) = \varlimsup_{\sigma\uparrow 0} \left(\frac{\ln^+ |a_n|}{-\widetilde{\Phi}(\lambda_n)}\right)^{\rho};$$

(b) if $\lambda \in \Lambda$ and $\tau(\lambda) > 0$, then for each $p_0 \in [0, +\infty]$ and any $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta}(F) = p_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;
(c) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each Dirichlet series $F \in \mathcal{D}_0(\lambda)$ the inequality $(R_{\alpha,\beta}(F))^{1/\rho} \leq (R^*_{\alpha,\beta}(F))^{1/\rho} + \Delta_\Phi(\lambda)$ holds;
(d) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = p_0$ and $(R_{\alpha,\beta,\gamma}(F))^{1/\rho} = (R^*_{\alpha,\beta,\gamma}(F))^{1/\rho} + \Delta_\Phi(\lambda)$.

Therefore, if $\lambda \in \Lambda$, then under the assumptions of Theorem 1 the equality $R_{\alpha,\beta}(F) = R^*_{\alpha,\beta}(F)$ holds for every Dirichlet series $F \in \mathcal{D}_0(\lambda)$ if and only if $\Delta_\Phi(\lambda) = 0$.

We obtain Theorem 1 from more general results proved below for modified orders of Dirichlet series from the class $\mathcal{D}_0$.

2. Auxiliary results. The following two lemmas, which we will need later, are well known (see, for example, [2, 13]).

Lemma 1. *Let $\Phi \in \Omega_0$, and let $F \in \mathcal{D}_0^*$ be a Dirichlet series of the form (1). Then the following conditions are equivalent:*
(i) there exists a number $\sigma_0 < 0$ such that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;
(ii) there exists a number $n_0 \in \mathbb{N}_0$ such that $\ln |a_n| \leq -\widetilde{\Phi}(\lambda_n)$ for all integers $n \geq n_0$.

Lemma 2. *Let $\Phi \in \Omega_0$, $D_\Phi = [a, 0)$, and p be a positive constant. Then for the function $\Psi(\sigma) = p\Phi(\sigma/p)$, $\sigma \in [pa, 0)$, we have $\Psi \in \Omega_0$ and $\widetilde{\Psi}(x) = p\widetilde{\Phi}(x)$ for all $x \in \mathbb{R}$.*

Theorem A ([2]). *Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ with $\tau(\lambda) > 0$, and $G \in \mathcal{D}_0^*(\lambda)\backslash\mathcal{D}_0(\lambda)$ be a Dirichlet series of the form $G(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n}$ with $b_n \geq 0$ for all $n \in \mathbb{N}_0$. Then for any function $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) such that $a_n = b_n$ or $a_n = 0$ for each $n \in \mathbb{N}_0$, and $M(\sigma, F) = F(\sigma) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$.*

3. Main results. Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and $\eta \in Y_0$ be a function non-decreasing on D_η. By $S_{\Phi,l}[\eta]$ denote the set of those $p > 0$ for which there exists $\sigma_0 = \sigma_0(p) < 0$ such that

$$\eta(\sigma) \leq l(p)\Phi(\sigma/l(p)), \quad \sigma \in [\sigma_0, 0). \tag{2}$$

Note that if $\eta(0-0) < +\infty$, then $S_{\Phi,l}[\eta] = (0, +\infty)$. If $\eta(0-0) = +\infty$, $p \in S_{\Phi,l}[\eta]$, and $q > p$, then $l(q) > l(p)$ and for all $\sigma < 0$ sufficiently close to 0 we have $\eta(l(q)\sigma)/l(q) \leq \eta(l(p)\sigma)/l(p)$, and therefore $q \in S_{\Phi,l}[\eta]$. If $S_{\Phi,l}[\eta] = \varnothing$, we set $p_{\Phi,l}[\eta] = +\infty$, and let $p_{\Phi,l}[\eta] = \inf S_{\Phi,l}[\eta]$ in the opposite case. It is obvious that if $b \in D_\eta$, then in the definition of the quantity $p_{\Phi,l}[\eta]$, the function η can be replaced by the restriction of η to $[b, 0)$. It is also clear that if $\zeta \in Y_0$ is a function non-decreasing on D_ζ and $\zeta(\sigma) \leq \eta(\sigma)$ for all $\sigma \in [c, 0)$, then $p_{\Phi,l}[\zeta] \leq p_{\Phi,l}[\eta]$.

For a Dirichlet series $F \in \mathcal{D}_0^*$ of the form (1), we set $S_{\Phi,l}^*(F) = S_{\Phi,l}[\eta]$ and $p_{\Phi,l}^*(F) = p_{\Phi,l}[\eta]$, where $\eta(\sigma) = \ln \mu(\sigma, F)$ for all $\sigma \in [-1, 0)$, and let

$$k_\Phi(F) = \varlimsup_{n\to+\infty} \frac{\ln^+ |a_n|}{-\widetilde{\Phi}(\lambda_n)}.$$

For every Dirichlet series $F \in \mathcal{D}_0$, we put $S_{\Phi,l}(F) = S_{\Phi,l}[\eta]$ and $p_{\Phi,l}(F) = p_{\Phi,l}[\eta]$, where $\eta(\sigma) = \ln M(\sigma, F)$ for all $\sigma \in [-1, 0)$. Note that $p_\Phi^*(F) \le p_\Phi(F)$ for an arbitrary $\Phi \in \Omega_0$ and any Dirichlet series $F \in \mathcal{D}_0$.

Using Lemmas 1 and 2, it is easy to prove the following statement.

Proposition 1. *Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and let $F \in \mathcal{D}_0^*$ be a Dirichlet series of the form* (1). *Then $p_{\Phi,l}^*(F) = l^{-1}(k_\Phi(F))$.*

Proof. First, we prove the inequality $l(p_{\Phi,L}^*(F)) \le k_\Phi(F)$. This inequality is trivial in the case when $k_\Phi(F) = +\infty$. Suppose that $k_\Phi(F) < +\infty$, and let $k > k_\Phi(F)$ be an arbitrary fixed number. Note that $k_\Phi(F) \ge 0$, and therefore $k > 0$. Setting $p = l^{-1}(k)$, from the definition of the quantity $k_\Phi(F)$ for some $n_0 \in \mathbb{N}_0$ we obtain

$$\ln^+ |a_n| \le -l(p)\widetilde{\Phi}(\lambda_n), \quad n \ge n_0. \tag{3}$$

Then by Lemmas 1 and 2 for some $\sigma_0 < 0$ we have

$$\ln \mu(\sigma, F) \le l(p)\Phi(\sigma/l(p)), \quad \sigma \in [\sigma_0, 0), \tag{4}$$

that is, $p \in S_{\Phi,l}^*(F)$. Thus, $p_{\Phi,l}^*(F) \le p$, and hence $l(p_{\Phi,l}^*(F)) \le l(p) = k$. Since $k > k_\Phi(F)$ is arbitrary, we obtain $l(p_{\Phi,l}^*(F)) \le k_\Phi(F)$.

Now we prove the opposite inequality $l^{-1}(k_\Phi(F)) \le p_{\Phi,l}^*(F)$. This inequality is trivial in the case when $p_{\Phi,l}^*(F) = +\infty$. Suppose that $p_{\Phi,l}^*(F) < +\infty$, and let $p > p_{\Phi,l}^*(F)$ be an arbitrary fixed number. From the definition of the quantity $p_{\Phi,l}^*(F)$ for some $\sigma_0 < 0$ we have (4). Since $-\widetilde{\Phi}(x) \to +\infty$ as $x \to +\infty$, by Lemmas 1 and 2 for some $n_0 \in \mathbb{N}_0$ we obtain (3), and therefore $k_\Phi(F) \le l(p)$, i.e. $l^{-1}(k_\Phi(F)) \le p$. Since $p > p_{\Phi,l}^*(F)$ is arbitrary, we obtain $l^{-1}(k_\Phi(F)) \le p_{\Phi,l}^*(F)$. □

Proposition 2. *Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ with $\tau(\lambda) > 0$, and let $\Psi \in \Omega_0$ be an arbitrary function. Then:*

(i) there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) *such that $a_n = 1$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $M(\sigma, F) \ge \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;*

(ii) for each $k_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $k_\Phi(F) = k_0$ and $M(\sigma, F) \ge \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$.

Proof. Let $k_0 \in [0, +\infty]$. In the case when $k_0 < +\infty$ for all $n \in \mathbb{N}_0$ we set $b_n = e^{-k_0\widetilde{\Phi}(\lambda_n)}$, and in the case when $k_0 = +\infty$ for all $n \in \mathbb{N}_0$ we put $b_n = e^{-\lambda_n\delta_n}$, where $(\delta_n)_{n\in\mathbb{N}_0}$ is an arbitrary sequence increasing to 0 such that $\overline{\Phi}(\lambda_n) = o(\delta_n)$ as $n \to \infty$.

Consider the Dirichlet series $G(s) = \sum_{n=0}^\infty b_n e^{s\lambda_n}$. It is easy to verify that $\sigma^*(G) = 0$, and therefore $G \in \mathcal{D}_0^*(\lambda)$. Let's fix some $\sigma \in (-\tau(\lambda), 0)$. Then $\tau(\lambda) > -\sigma$, and hence the set $E = \{k \in \mathbb{N}_0\colon \ln k \ge -\sigma\lambda_k\}$ is infinite. For an arbitrary sufficiently large $k \in E$ we have

$$\sum_{n=[k/2]}^{k} a_n e^{\sigma\lambda_n} \ge \sum_{n=[k/2]}^{k} e^{\sigma\lambda_n} \ge \sum_{n=[k/2]}^{k} e^{\sigma\lambda_k} \ge \frac{k}{2} e^{\sigma\lambda_k} \ge \frac{k}{2} e^{-\ln k} = \frac{1}{2}.$$

Therefore, the series G is divergent at the point $s = \sigma$, and hence $G \in \mathcal{D}_0^*(\lambda)\backslash\mathcal{D}_0(\lambda)$. Then, according to Theorem A, there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) such that $a_n = b_n$ or $a_n = 0$ for each $n \in \mathbb{N}_0$, and $M(\sigma, F) = F(\sigma) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. In addition, as it is not difficult to verify, $k_\Phi(F) = k_0$. This proves (ii), and also (i) if $k_0 = 0$. □

From Proposition 2 we see that if $\tau(\lambda) > 0$, then the growth of a Dirichlet series from the class $\mathcal{D}_0(\lambda)$ can be arbitrarily fast even under the condition of boundedness its maximal term.

Theorem 2. *Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from class Λ with $\tau(\lambda) = 0$. Then:*
(i) for each Dirichlet series $F \in \mathcal{D}_0(\lambda)$ we have $l(p_{\Phi,l}(F)) \leq l(p^*_{\Phi,l}(F)) + \Delta_\Phi(\lambda)$;
(ii) for every $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $p^*_{\Phi,l}(F) = p_0$ and $l(p_{\Phi,l}(F)) = l(p^*_{\Phi,l}(F)) + \Delta_\Phi(\lambda)$.

Proof. To prove (i), we consider a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) and note that the inequality $l(p_{\Phi,l}(F)) \leq l(p^*_{\Phi,l}(F)) + \Delta_\Phi(\lambda)$ does not need proof if $p^*_{\Phi,l}(F) = +\infty$ or $\Delta_\Phi(\lambda) = +\infty$. Suppose that $p^*_{\Phi,l}(F) < +\infty$ and $\Delta_\Phi(\lambda) < +\infty$ and fix an arbitrary $b > l(p^*_\Phi(F)) + \Delta_\Phi(\lambda)$. It is clear that then there exist constants $c > l(p^*_\Phi(F))$ and $\Delta > \Delta_\Phi(\lambda)$ such that $c + \Delta < b$. From Proposition 1, the definition of the quantity $\Delta_\Phi(\lambda)$, and the obvious inequality $(b-c)/\Delta > 1$, it follows the existence of a number $n_0 \in \mathbb{N}_0$ such that for all integers $n \geq n_0$ the following inequalities $|a_n| \leq e^{-c\widetilde{\Phi}(\lambda_n)}$, $n \leq e^{-\Delta\widetilde{\Phi}(\lambda_n)}$ hold and, in addition,

$$\sum_{n\geq n_0} \frac{1}{n^{(b-c)/\Delta}} \leq \frac{1}{2}.$$

Consider the auxiliary Dirichlet series

$$G(s) = \sum_{n\geq n_0} e^{-c\widetilde{\Phi}(\lambda_n)} e^{s\lambda_n}.$$

It is easy to verify that $\sigma^*(G) = 0$, that is, $G \in \mathcal{D}_0(\lambda)$. In addition, Lemmas 1 and 2 imply the existence of a constant $\sigma_0 < 0$ such that $\ln \mu(\sigma, G) \leq c\Phi(\sigma/c)$ for all $\sigma \in [\sigma_0, 0)$. Therefore, using the above inequalities, for all $\sigma \in [b\sigma_0/c, 0)$ we obtain

$$M(\sigma, G) = \sum_{n\geq n_0} e^{-c\widetilde{\Phi}(\lambda_n)} e^{\sigma\lambda_n} = \sum_{n\geq n_0} \left(e^{-c\widetilde{\Phi}(\lambda_n)} e^{(c\sigma/b)\lambda_n}\right)^{b/c} \frac{1}{e^{-(b-c)\widetilde{\Phi}(\lambda_n)}} \leq$$
$$\leq (\mu(c\sigma/b, G))^{b/c} \sum_{n\geq n_0} \frac{1}{e^{-(b-c)\widetilde{\Phi}(\lambda_n)}} \leq e^{b\Phi(\sigma/b)} \sum_{n\geq n_0} \frac{1}{n^{(b-c)/\Delta}} \leq \frac{1}{2} e^{b\Phi(\sigma/b)}.$$

Then, by taking $q = l^{-1}(b)$, for all $\sigma < 0$ sufficiently close to 0 we have

$$M(\sigma, F) \leq \sum_{n\geq n_0} |a_n| e^{\sigma\lambda_n} + M(\sigma, G) \leq e^{b\Phi(\sigma/b)} = e^{l(q)\Phi(\sigma/l(q))}.$$

Therefore, $q \in S_{\Phi,l}(F)$, and hence $l(p_{\Phi,l}(F)) \leq l(q) = b$. The required inequality follows from the arbitrariness of $b > l(p^*_{\Phi,l}(F)) + \Delta_\Phi(\lambda)$.

Now we prove (ii). The proof is trivial if $p_0 = +\infty$ or $\Delta_\Phi(\lambda) = 0$. Suppose that $p_0 < +\infty$ and $\Delta_\Phi(\lambda) > 0$, and set $c = l(p_0)$. Let us choose arbitrary positive sequences $(c_k)_{k\in\mathbb{N}_0}$,

$(\delta_k)_{k\in\mathbb{N}_0}$ and $(\Delta_k)_{k\in\mathbb{N}_0}$ such that $c_k\to c$ for $k\to\infty$, $(\delta_k)_{k\in\mathbb{N}_0}$ decreases to 1, and $(\Delta_k)_{k\in\mathbb{N}_0}$ increases to $\Delta_\Phi(\lambda)$. It is easy to justify the existence of an increasing sequence $(n_k)_{k\in\mathbb{N}_0}$ of positive integers such that for it, as well as for the sequence $(m_k)_{k\in\mathbb{N}_0}$, where $m_k=[(n_k+1)/2]$ for all $k\in\mathbb{N}_0$, we have

$$n_k<m_{k+1},\qquad \frac{\ln(n_k-m_k+1)}{-\widetilde{\Phi}(\lambda_{n_k})}\ge\Delta_k,\qquad \frac{\ln m_k}{-\widetilde{\Phi}(\lambda_{m_k})}\le\delta_k\frac{\ln n_k}{-\widetilde{\Phi}(\lambda_{n_k})}\tag{5}$$

for each $k\in\mathbb{N}_0$.

Let $n\in\mathbb{N}_0$. Put $a_n=e^{-c_k\widetilde{\Phi}(\lambda_n)}$ if $n\in[m_k,n_k]$ for some $k\in\mathbb{N}_0$, and let $a_n=0$ in the opposite case. Consider the Dirichlet series F of the form (1) with the coefficients a_n defined in this way. It is easy to verify that $\sigma^*(F)=0$, that is, $F\in\mathcal{D}_0(\lambda)$, and $k_\Phi(F)=c$. According to Proposition 1 we have $p^*_{\Phi,l}(F)=l^{-1}(c)=p_0$. Note also that the constructed series can be written in the form

$$F(s)=\sum_{k=0}^{\infty}\sum_{n=m_k}^{n_k}e^{-c_k\widetilde{\Phi}(\lambda_n)}e^{s\lambda_n}.$$

For each $k\in\mathbb{N}_0$, we set $b_k=(n_k-m_k+1)e^{-c_k\widetilde{\Phi}(\lambda_{m_k})}$, and consider the auxiliary Dirichlet series $H(s)=\sum_{k=0}^{\infty}b_ke^{s\lambda_{n_k}}$. If $\sigma<0$ and $k\in\mathbb{N}_0$, then

$$\sum_{n=m_k}^{n_k}e^{-c_k\widetilde{\Phi}(\lambda_n)}e^{\sigma\lambda_n}\ge\sum_{n=m_k}^{n_k}e^{-c_k\widetilde{\Phi}(\lambda_{m_k})}e^{\sigma\lambda_{n_k}}=b_ke^{\sigma\lambda_{n_k}},$$

and therefore $H\in\mathcal{D}_0$. In addition, $M(\sigma,F)=F(\sigma)\ge H(\sigma)=M(\sigma,H)$ for each $\sigma<0$ and, according to (5),

$$k_\Phi(H)=\varlimsup_{k\to+\infty}\frac{\ln^+|b_k|}{-\widetilde{\Phi}(\lambda_{n_k})}=\varlimsup_{k\to+\infty}\left(\frac{\ln(n_k-m_k+1)}{-\widetilde{\Phi}(\lambda_{n_k})}+c_k\frac{\widetilde{\Phi}(\lambda_{m_k})}{\widetilde{\Phi}(\lambda_{n_k})}\right)\ge\Delta_\Phi(\lambda)+c.$$

Therefore, $l(p_{\Phi,l}(F))\ge l(p_{H,l}(F))\ge l(p^*_{H,l}(F))=k_\Phi(H)\ge l(p^*_{\Phi,l}(F))+\Delta_\Phi(\lambda)$. It remains to use statement (i) of this theorem. □

Let $\alpha\in L$, $D_\alpha=[x_0,0)$, $\beta\in\Omega_0$, $\gamma\in C_0$ be a function positive on D_γ, and $\eta\in Y_0$ be a function non-decreasing on D_η. The quantity

$$R_{\alpha,\beta,\gamma}[\eta]=\varlimsup_{\sigma\uparrow 0}\frac{\alpha(\max\{x_0,\gamma(\sigma)\eta(\sigma)\})}{\beta(\sigma)}$$

is called the *modified order (α,β,γ-order)* of the function η. Note that in the definition of the quantity $R_{\alpha,\beta}[\eta]$, the constant x_0 can be replaced by any other number from D_α, and if $b\in D_\eta$, then the function η can be replaced by the restriction of η to $[b,0)$. It is also clear that if $\zeta\in Y_0$ is a function non-decreasing on D_ζ and $\zeta(\sigma)\le\eta(\sigma)$ for all $\sigma\in[c,0)$, then $R_{\alpha,\beta,\gamma}[\zeta]\le R_{\alpha,\beta,\gamma}[\eta]$.

For any Dirichlet series $F\in\mathcal{D}^*_0$, we set $R^*_{\alpha,\beta,\gamma}(F)=R_{\alpha,\beta,\gamma}[\eta]$, where $\eta(\sigma)=\ln\mu(\sigma,F)$ for all $\sigma\in[-1,0)$. If $F\in\mathcal{D}_0$, we put $R_{\alpha,\beta,\gamma}(F)=R_{\alpha,\beta,\gamma}[\eta]$, where $\eta(\sigma)=\ln M(\sigma,F)$ for all $\sigma\in[-1,0)$; the quantity $R_{\alpha,\beta,\gamma}(F)$ is called the *modified order (α,β,γ-order)* of the Dirichlet series F. It is clear that $R^*_{\alpha,\beta,\gamma}(F)\le R_{\alpha,\beta,\gamma}(F)$ for every Dirichlet series $F\in\mathcal{D}_0$.

In connection with Problem 1, it is natural to consider the following more general problem.

Problem 2. *Let $\alpha \in L$, $\beta \in \Omega_0$, $\gamma \in C_0$ be a function positive on D_γ, and $\lambda \in \Lambda$. Find a necessary and sufficient condition on the sequence $\lambda \in \Lambda$ under which $R_{\alpha,\beta,\gamma}(F) = R^*_{\alpha,\beta,\gamma}(F)$ for each Dirichlet series $F \in \mathcal{D}_0(\lambda)$.*

Below we will obtain solutions of Problem 2 under fairly general assumptions about the behavior of the functions $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$.

Proposition 3. *Let $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$ be a function positive on D_γ. If the condition*

$$\forall c > 0\colon \quad \varliminf_{\sigma\uparrow 0} \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma) < +\infty \tag{6}$$

holds, then for any sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ from the class Λ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) such that $a_n = 1$ or $a_n = 0$ for each $n \in \mathbb{N}_0$ and $R_{\alpha,\beta,\gamma}(F) = +\infty$.

Proof. In the case when $\tau(\lambda) > 0$, it is enough to use Proosition 2.

Let $\tau(\lambda) = 0$. Consider the series $F(s) = \sum_{n=0}^{\infty} e^{s\lambda_n}$. It is clear that $F \in \mathcal{D}_0(\lambda)$, $R^*_{\alpha,\beta,\gamma}(F) = 0$ and $M(\sigma, F) \uparrow +\infty$ as $\sigma \uparrow 0$. Suppose that $R_{\alpha,\beta,\gamma}(F) < c$ for some $c > 0$. Then, from the definition of the quantity $R_{\alpha,\beta,\gamma}(F)$, we have $\ln M(\sigma, F) \le \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$, which contradicts (6). Therefore, $R_{\alpha,\beta,\gamma}(F) = +\infty$. □

Theorem 3. *Let $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$ be a function positive on D_γ. Suppose that condition (6) is not satisfied, and a function $\Phi \in C_0$ is such that*

$$\forall t > 0\colon \quad \lim_{\sigma\uparrow 0} \gamma(\sigma)\Phi(\sigma/t) = +\infty \tag{7}$$

and for every $t > 0$ there exists a finite limit

$$h(t) := \lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(\sigma)t\Phi(\sigma/t))}{\beta(\sigma)}, \tag{8}$$

and $h(t)$ is a continuously function increasing to $+\infty$ on $(0, +\infty)$ with $h(0) = h(0 + 0) = 0$. Then $\Phi \in \Omega_0$ and if $t = l(p)$ is the inverse function of the function $p = h(t)$, then:

(a) for every function $\eta \in Y_0$ non-decreasing on D_η, we have $R_{\alpha,\beta,\gamma}[\eta] = p_{\Phi,l}[\eta]$;

(b) for each Dirichlet series $F \in \mathcal{D}^*_0$ we have $R^*_{\alpha,\beta,\gamma}(F) = h(k_\Phi(F))$;

(c) if $\lambda \in \Lambda$ and $\tau(\lambda) > 0$, then for every $p_0 \in [0, +\infty]$ and any $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = p_0$ and $M(\sigma, F) \ge \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(d) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for every Dirichlet series $F \in \mathcal{D}_0(\lambda)$ the inequality $l(R_{\alpha,\beta,\gamma}(F)) \le l(R^*_{\alpha,\beta,\gamma}(F)) + \Delta_\Phi(\lambda)$ holds;

(e) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = p_0$ and $l(R_{\alpha,\beta,\gamma}(F)) = l(R^*_{\alpha,\beta,\gamma}(F)) + \Delta_\Phi(\lambda)$.

Proof. If (6) is not satisfied, then for some $c > 0$ we have $\alpha^{-1}(c\beta(\sigma))/\gamma(\sigma) \to +\infty$ as $\sigma \uparrow 0$. Let's choose the number $t > 0$ so that the inequality $h(t) > c$ holds. Then, according to (8), there exists $\sigma_0 < 0$ such that $t\Phi(\sigma/t) \ge \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Therefore, $\Phi(\sigma) \to +\infty$ as $\sigma \uparrow 0$, and hence $\Phi \in \Omega_0$.

Let's prove (a). Let $\eta \in Y_0$ be a function non-decreasing on D_η. First, we show that $R_{\alpha,\beta,\gamma}[\eta] \le p_{\Phi,l}[\eta]$. This inequality is trivial if $p_{\Phi,l}[\eta] = +\infty$. Suppose that $p_{\Phi,l}[\eta] < +\infty$, and let $p > p_{\Phi,l}[\eta]$ be an arbitrary fixed number. Then there exists a number $\sigma_0 < 0$ such that (2) holds. Therefore, using (2) and (7), we obtain

$$R_{\alpha,\beta,\gamma}[\eta] \le \overline{\lim_{\sigma\uparrow 0}} \frac{\alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p)))}{\beta(\sigma)} = h(l(p)) = p,$$

and the required inequality follows from the arbitrariness of $p > p_{\Phi,l}[\eta]$.

Now we prove that $p_{\Phi,l}[\eta] \le R_{\alpha,\beta,\gamma}[\eta]$. This inequality is trivial if $R_{\alpha,\beta,\gamma}[\eta] = +\infty$. Suppose that $R_{\alpha,\beta,\gamma}[\eta] < +\infty$, and let $p > R_{\alpha,\beta,\gamma}[\eta]$ be an arbitrary fixed number, and $q \in (R_{\alpha,\beta,\gamma}[\eta], p)$. From the definition of the quantity $R_{\alpha,\beta,\gamma}[\eta]$ for some $\sigma_1 < 0$ we have

$$\gamma(\sigma)\eta(\sigma) \le \alpha^{-1}(q\beta(\sigma)), \quad \sigma \in [\sigma_1, 0). \tag{9}$$

In addition, since

$$\lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p)))}{\beta(\sigma)} = h(l(p)) = p > q,$$

for some $\sigma_2 < 0$ we obtain

$$q\beta(\sigma) \le \alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p))), \quad \sigma \in [\sigma_2, 0). \tag{10}$$

Taking $\sigma_0 = \max\{\sigma_1, \sigma_2\}$, from (9) and (10) we see that (2) is fulfilled, i.e. $p \in S_{\Phi,l}[\eta]$. Therefore, $p_{\Phi,l}[\eta] \le p$. Since $p > R_{\alpha,\beta,\gamma}[\eta]$ is arbitrary, we have $p_{\Phi,l}[\eta] \le R_{\alpha,\beta,\gamma}[\eta]$.

Further, according to the part of the theorem that has already been proved, for each Dirichlet series $F \in \mathcal{D}_0^*$ we obtain $R^*_{\alpha,\beta,\gamma}(F) = p^*_{\Phi,l}(F)$, and for each Dirichlet series $F \in \mathcal{D}_0$ we have $R_{\alpha,\beta,\gamma}(F) = p_{\Phi,l}(F)$. Therefore, (b) follows from Proposition 1, and (c) follows from Proposition 2. In addition, for an arbitrary sequence $\lambda \in \Lambda$ with $\tau(\lambda) = 0$, according to Theorem 2, we have (d) and (e). □

4. Corollaries. Let us give some consequences from the results proved above.

Theorem 4. *Let $\alpha \in L$ be a function slowly varying at the point $+\infty$, $\beta \in \Omega_0$ be a function regularly varying at the point 0 with index $\rho > 0$, $\gamma \in C_0$ be a function positive on D_γ such that for each fixed $t > 0$ the inequalities*

$$0 < \underline{\lim_{\sigma\uparrow 0}} \frac{\gamma(t\sigma)}{\gamma(\sigma)} \le \overline{\lim_{\sigma\uparrow 0}} \frac{\gamma(t\sigma)}{\gamma(\sigma)} < +\infty \tag{11}$$

hold, and $\Phi \in C_0$ be a function such that $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Then $\Phi \in \Omega_0$ and the following statements are true:

(a) for every Dirichlet series $F \in \mathcal{D}_0^*$ we have

$$R^*_{\alpha,\beta,\gamma}(F) = \overline{\lim_{\sigma\uparrow 0}} \left(\frac{\ln^+ |a_n|}{-\widetilde{\Phi}(\lambda_n)} \right)^{\rho};$$

(b) if $\lambda \in \Lambda$ and $\tau(\lambda) > 0$, then for every $p_0 \in [0, +\infty]$ and any $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = p_0$ and $M(\sigma, F) \ge \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(c) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for every Dirichlet series $F \in \mathcal{D}_0(\lambda)$ the inequality $(R_{\alpha,\beta,\gamma}(F))^{1/\rho} \le (R^*_{\alpha,\beta,\gamma}(F))^{1/\rho} + \Delta_\Phi(\lambda)$ holds;

(d) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = p_0$ and $(R_{\alpha,\beta,\gamma}(F))^{1/\rho} = (R^*_{\alpha,\beta,\gamma}(F))^{1/\rho} + \Delta_\Phi(\lambda)$.

Proof. Let $D_\gamma = [a, 0)$. From the conditions satisfied by the function γ, for some constant $d > 1$ and all $\sigma \in [a/2, 0)$ we have $\gamma(\sigma) \le d\gamma(2\sigma)$. Let $M = \max\{\gamma(\sigma)\colon \sigma \in [a, a/2]\}$, $q = \log_2 d$ and $c = M|a|^q$. Then for all $\sigma \in [a, 0)$ we have $\gamma(\sigma) \le c|\sigma|^{-q}$. In fact, if $\sigma \in [a/2^n, a/2^{n+1}]$ for some $n \in \mathbb{N}_0$, then

$$\gamma(\sigma) \le d^n \gamma(2^n \sigma) \le 2^{qn} M \le M|a|^q |\sigma|^{-q} = c|\sigma|^{-q}.$$

Let's fix an arbitrary constant $\rho_0 \in (0, \rho)$ and choose a constant $r > 0$ so that the inequality $r\rho_0 > q$ holds. According to the well-known properties of regularly varying functions, there exist constants $\sigma_1 < 0$ and $x_0 > 0$ such that $\beta(\sigma) \ge |\sigma|^{-\rho_0}$ for all $\sigma \in [\sigma_1, 0)$ and $\alpha^{-1}(x) \ge x^r$ for all $x \in [x_0, +\infty)$. Then there exists a constant $\sigma_2 < 0$ such that $\alpha^{-1}(\beta(\sigma)) \ge (\beta(\sigma))^r \ge |\sigma|^{-r\rho_0}$ for all $\sigma \in [\sigma_2, 0)$. Therefore, using the above estimate for the γ function, we obtain $\Phi(\sigma) \to +\infty$ as $\sigma \uparrow 0$. From this, in particular, we see that $\Phi \in \Omega_0$, condition (6) is not satisfied, and (11) implies (7). In addition, using the conditions satisfied by the functions α, β, and γ, for each fixed $t > 0$ we have

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\gamma(\sigma) t \Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \uparrow 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma \uparrow 0} \frac{\beta(\sigma)}{\beta(t\sigma)} = t^\rho.$$

Therefore, all the conditions of Theorem 3 are satisfied with $h(t) = t^\rho$ for all $t > 0$, and (a), (b), (c), and (d) are consequences of the corresponding statements of Theorem 3. □

Theorem 5. *Let $\alpha \in L$ be an arbitrary function, $\beta \in \Omega_0$ be a function regularly varying at the point 0 with index $\rho > 0$, $\gamma(\sigma) = |\sigma|^{-1}$ for all $\sigma \in [-1, 0)$, and $\Phi \in C_0$ be a function such that $\Phi(\sigma) = |\sigma|\alpha^{-1}(\beta(\sigma))$ for all $\sigma \in [\sigma_0, 0)$. Then:*

(i) if $\Phi \notin \Omega_0$, then for every sequence $\lambda \in \Lambda$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = 0$, but $R_{\alpha,\beta,\gamma}(F) = +\infty$;

(ii) if $\Phi \in \Omega_0$, then statements (a), (b), (c), and (d) of Theorem 4 are true.

Proof. Noting that the condition $\Phi \notin \Omega_0$ is equivalent to condition (6), from Proposition 3 we obtain (i).

Let $\Phi \in \Omega_0$. Then, as it is easy to see, (7) holds. In addition, for every fixed $t > 0$ we have

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\gamma(\sigma) t \Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \uparrow 0} \frac{\alpha(|\sigma|^{-1} t \Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \uparrow 0} \frac{\alpha(|\sigma|^{-1}\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma \uparrow 0} \frac{\beta(\sigma)}{\beta(t\sigma)} = t^\rho.$$

Therefore, all the conditions of Theorem 3 are satisfied with $h(t) = t^\rho$ for all $t > 0$. This implies (ii). □

Theorem 6. *Let $\alpha \in L$ be an arbitrary function such that*

$$\forall q > 1\colon \quad \varliminf_{y \to +\infty} \alpha^{-1}(qy)/\alpha^{-1}(y) > 1, \tag{12}$$

$\beta \in \Omega_0$ be a function regularly varying at the point 0 with index $\rho > 0$, $\gamma \in C_0$ be a function regularly varying at the point 0 with index 1, and $\Phi \in C_0$ be a function such that $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Then statements (i) and (ii) of Theorem 5 are true.

Proof. By the assumptions of the theorem, the condition $\Phi \notin \Omega_0$ is equivalent to condition (6), and therefore from Proposition 3 we obtain statement (i) of Theorem 5.

Let $\Phi \in \Omega_0$. Then, as it is easy to see, (7) holds. Next, we note that condition (12) is satisfied if and only if for any function $\delta \in C_0$ such that $\delta(x) \to 1$ as $x \to +\infty$, we have $\alpha(x\delta(x)) \sim \alpha(x)$ as $x \to +\infty$. Using this fact, for every fixed $t > 0$ we have

$$\lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(\sigma)t\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(t\sigma)t\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma))}{t^{-\rho}\beta(\sigma)} = t^{\rho}.$$

Therefore, all the conditions of Theorem 3 are satisfied with $h(t) = t^{\rho}$ for all $t > 0$. This implies statement (ii) of Theorem 5. □

Theorem 7. *Let $\alpha \in L$ be a function regularly varying at the point $+\infty$ with index $\rho_1 > 0$, $\beta \in \Omega_0$ be a function regularly varying at the point 0 with index $\rho_2 \geq 0$, $\gamma \in C_0$ be a function regularly varying at the point 0 with index $\rho_3 \in \mathbb{R}$, $\rho = \rho_1 + \rho_2 - \rho_1\rho_3$, and $\Phi \in C_0$ be a function such that $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Then:*

(i) if $\rho_2 - \rho_1\rho_3 < 0$ or simultaneously the conditions $\rho_2 - \rho_1\rho_3 = 0$ and $\Phi \notin \Omega_0$ are satisfied, then for any sequence $\lambda \in \Lambda$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $R^*_{\alpha,\beta,\gamma}(F) = 0$ and $R_{\alpha,\beta,\gamma}(F) = +\infty$;

(ii) if simultaneously the conditions $\rho_2 - \rho_1\rho_3 = 0$ and $\Phi \in \Omega_0$ are satisfied or $\rho_2 - \rho_1\rho_3 > 0$, then statements (a), (b), (c), and (d) of Theorem 4 are true.

Proof. For every fixed $t > 0$ we have

$$\Phi(t\sigma) = \frac{\alpha^{-1}(\beta(t\sigma))}{\gamma(t\sigma)} \sim \frac{\alpha^{-1}(t^{-\rho_2}\beta(\sigma))}{t^{-\rho_3}\gamma(\sigma)} \sim \frac{t^{-\rho_2/\rho_1}}{t^{-\rho_3}}\Phi(\sigma) = t^{-(\rho_2-\rho_1\rho_3)/\rho_1}\Phi(\sigma), \quad \sigma \uparrow 0,$$

i.e. Φ is a function regularly varying at the point 0 with index $(\rho_2 - \rho_1\rho_3)/\rho_1$. We note that in the case when $\rho_2 - \rho_1\rho_3 < 0$ we have $\Phi \notin \Omega_0$, in the case when $\rho_2 - \rho_1\rho_3 > 0$ we have $\Phi \in \Omega_0$, and in the case when $\rho_2 - \rho_1\rho_3 = 0$ both the situations $\Phi \notin \Omega_0$ or $\Phi \in \Omega_0$ are possible.

Therefore, all the conditions of (i) reduce to the condition $\Phi \notin \Omega_0$, and all the conditions of (ii) reduce to the condition $\Phi \in \Omega_0$.

Noting that the condition $\Phi \notin \Omega_0$ is equivalent to condition (6), from Proposition 3 we obtain statement (i).

Let $\Phi \in \Omega_0$. Then, as it is easy to see, $\rho > 0$ and (7) holds. In addition, for every fixed $t > 0$ we have

$$\lim_{\sigma\uparrow 0} \frac{\alpha(\gamma(\sigma)t\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma\uparrow 0} \frac{t^{\rho_1}\alpha(t^{-\rho_3}\gamma(\sigma)\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma\uparrow 0} \frac{t^{\rho_1-\rho_1\rho_3}\alpha(\gamma(\sigma)\Phi(\sigma))}{t^{-\rho_2}\beta(\sigma)} = t^{\rho}.$$

Therefore, all the conditions of Theorem 3 are satisfied with $h(t) = t^{\rho}$ for all $t > 0$. This implies statement (ii). □

Finally, note that Theorem 1 is a consequence of Theorems 4 and 7 in the case when $\gamma(\sigma) = 1$ for all $\sigma \in [-1, 0)$.

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