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GENERALIZED AND MODIFIED ORDERS OF GROWTH FOR DIRICHLET SERIES ABSOLUTELY CONVERGENT IN A HALF-PLANE


Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a non-negative sequence increasing to $+\infty$, $\tau(\lambda) = \lim_{n\to\infty}(\ln n/\lambda_n)$, and $D_0(\lambda)$ be the class of all Dirichlet series of the form $F(s) = \sum_{n=0}^{\infty} a_n(F)e^{\lambda_n s}$ absolutely convergent in the half-plane $Re s < 0$ with $a_n(F) \neq 0$ for at least one integer $n \geq 0$. Also, let $\alpha$ be a continuous function on $[x_0, +\infty)$ increasing to $+\infty$, $\beta$ be a continuous function on $[0, 0)$ such that $\beta(\sigma) \to +\infty$ as $\sigma \uparrow 0$, and $\gamma$ be a continuous positive function on $[b, 0)$. In the article, we investigate the growth of a Dirichlet series $F \in D_0(\lambda)$ depending on the behavior of the sequence $(|a_n(F)|)$ in terms of its $\alpha, \beta, \gamma$-orders determined by the equalities

$$R_{\alpha, \beta, \gamma}^*(F) = \lim_{n\to+\infty} \frac{\alpha(\max\{x_0, \gamma(\sigma)\ln M(\sigma)\})}{\beta(\sigma)}, \quad R_{\alpha, \beta, \gamma}(F) = \lim_{\sigma\to0} \frac{\alpha(\max\{x_0, \gamma(\sigma)\ln M(\sigma)\})}{\beta(\sigma)},$$

where $\mu(\sigma) = \max\{|a_n(F)|e^{\sigma\lambda_n}: n \geq 0\}$ and $M(\sigma) = \sup\{|F(s)|: Re s = \sigma\}$ are the maximal term and the supremum modulus of the series $F$, respectively. In particular, if for every fixed $t > 0$ we have $\alpha(tx) \sim \alpha(x)$ as $x \to +\infty$, $\beta(t\sigma) \sim \sigma^{-\rho}\beta(\sigma)$ as $\sigma \uparrow 0$ for some fixed $\rho > 0$, $0 < \lim_{t\to0} \gamma(t\sigma)/\gamma(\sigma) \leq \lim_{t\to0+} \gamma(t\sigma)/\gamma(\sigma) < +\infty$, $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$, $\Phi(x) = \max\{\sigma x - \Phi(\sigma): \sigma \in [\sigma_0, 0)\}$ for all $x \in \mathbb{R}$, and $\Delta_{\Phi}(\lambda) = \lim_{n\to\infty}(\ln n/\Phi(\lambda_n))$, then:

(a) for each Dirichlet series $F \in D_0(\lambda)$ we have

$$R_{\alpha, \beta, \gamma}^*(F) = \lim_{n\to+\infty} \left(\frac{\ln^+ |a_n(F)|}{\Phi(\lambda_n)}\right)^\rho;$$

(b) if $\tau(\lambda) > 0$, then for each $p_0 \in [0, +\infty)$ and any positive function $\Psi$ on $[c, 0)$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha, \beta, \gamma}^*(F) = p_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(c) if $\tau(\lambda) = 0$, then $(R_{\alpha, \beta, \gamma}(F))^{1/\rho} \leq (R_{\alpha, \beta, \gamma}^*(F))^{1/\rho} + \Delta_{\Phi}(\lambda)$ for every Dirichlet series $F \in D_0(\lambda)$;

(d) if $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty)$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha, \beta, \gamma}^*(F) = p_0$ and $(R_{\alpha, \beta, \gamma}(F))^{1/\rho} = (R_{\alpha, \beta, \gamma}^*(F))^{1/\rho} + \Delta_{\Phi}(\lambda)$.

1. Introduction. We denote by $\mathbb{N}_0$ the set of all non-negative integers, and denote by $\Lambda$ the class of all non-negative sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ increasing to $+\infty$.

Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class $\Lambda$. Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s}$$

(1)

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and denote by $\sigma_a(F)$ the abscissa of absolute convergence of this series. Put

$$\sigma^*(F) = \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$ 

It is easy to see that if $\sigma < \sigma^*(F)$, then $|a_n| e^{\sigma \lambda_n} \to 0$ as $n \to \infty$. Therefore, for each such $\sigma$, we can determine the maximal term

$$\mu(\sigma, F) = \max\{|a_n| e^{\sigma \lambda_n} : n \in \mathbb{N}_0\}$$

of series (1). Note also that in the case when $\sigma > \sigma^*(F)$ we have $\lim_{n \to \infty} |a_n| e^{\sigma \lambda_n} = +\infty$. If $\sigma_a(F) > -\infty$, then for all $\sigma < \sigma_a(F)$ we define the supremum modulus of series (1) by equality

$$M(\sigma, F) = \sup\{|F(s)| : \text{Re} s = \sigma\}.$$ 

We denote by $D_0^*(\lambda)$ the class of all Dirichlet series of the form (1), for which $\sigma^*(F) \geq 0$ and $a_n \neq 0$ at least for one value of $n \in \mathbb{N}_0$. By $D_0(\lambda)$ we denote the class of all Dirichlet series of the form (1) such that $\sigma_a(F) \geq 0$ and $a_n \neq 0$ at least for one value of $n \in \mathbb{N}_0$. It is clear that $D_0(\lambda) \subset D_0^*(\lambda)$ and, as it is well known, $D_0(\lambda) = D_0^*(\lambda)$ if and only if $\tau(\lambda) = 0$, where

$$\tau(\lambda) = \lim_{n \to \infty} \ln n / \lambda_n.$$ 

Put

$$D_0 = \bigcup_{\lambda \in \Lambda} D_0(\lambda), \quad D_0^* = \bigcup_{\lambda \in \Lambda} D_0^*(\lambda).$$

For $A \in (-\infty, +\infty]$, we denote by $Y_A$ the class of all real functions $\eta: D_\eta \to \mathbb{R}$ such that the domain $D_\eta$ of $\eta$ is an interval of the form $[a, A)$.

Let $\eta \in Y_{+\infty}$ be a positive measurable function on $D_\eta$. As in [1], we call the function $\eta$ slowly varying at the point $+\infty$ if for every fixed number $c > 0$ we have $\eta(cx) \sim \eta(x)$ as $x \to +\infty$, and we call the function $\eta$ regularly varying at the point $+\infty$ with index $\rho \geq 0$, if $\eta(x) = x^\rho \zeta(x)$ for all $x \geq x_0$, where $\zeta \in Y_{+\infty}$ is a slowly varying function at the point $+\infty$.

Let $\eta \in Y_0$ be a positive measurable function on $D_\eta$. We call the function $\eta$ slowly varying at the point 0 if for every fixed number $c > 0$ we have $\eta(c x) \sim \eta(x)$ as $x \uparrow 0$, and we call the function $\eta$ regularly varying at the point 0 with index $\rho \geq 0$, if $\eta(x) = |x|^{-\rho} \zeta(x)$ for all $x \in [x_0, 0)$, where $\zeta \in Y_0$ is a slowly varying function at the point 0.

We denote by $L$ the sub-class of all functions $l \in Y_{+\infty}$ continuous and increasing to $+\infty$ on $D_l$.

We denote by $C_0$ the sub-class of all functions $\eta \in Y_0$ continuous on $D_\eta$, and denote by $\Omega_0$ the sub-class of all functions $\Phi \in C_0$ such that $\Phi(\sigma) \to +\infty$ as $\sigma \uparrow 0$.

Let $\Phi \in \Omega_0$. Then, as it is well known, the function

$$\tilde{\Phi}(x) = \max\{|x \sigma - \Phi(\sigma) : \sigma \in D_\Phi\}, \quad x \in \mathbb{R},$$

is called the Young-conjugate of $\Phi$, and this function has the following properties (see, for instance, [2]): $\tilde{\Phi}$ is convex on $\mathbb{R}$; the right-hand derivative $\varphi$ of $\tilde{\Phi}$ is a negative nondecreasing function on $\mathbb{R}$, $\varphi(x) \to 0$ as $x \to +\infty$, and

$$\varphi(x) = \max\{|\sigma \in D_\Phi : x \sigma - \Phi(\sigma) = \tilde{\Phi}(x)|, \quad x \in \mathbb{R};$$
if \( x_0 = \inf \{ x > 0 : \Phi(\varphi(x)) > 0 \} \), then \( \overline{\Phi}(x) = \overline{\Phi}(x)/x \) increases to 0 on \((x_0, +\infty)\). It follows from these properties that \( \overline{\Phi} \) is a decreasing continuous function on \( \mathbb{R} \). It is also easy to prove that the range of the function \( \overline{\Phi} \) is \( \mathbb{R} \). In fact, if \( \sigma_0 \in D_\Phi \) is a fixed number, then for all \( x \in \mathbb{R} \) we have \( \overline{\Phi}(x) \geq x \sigma_0 - \Phi(\sigma_0) \). Letting \( x \to -\infty \), we obtain \( \overline{\Phi}(-\infty) = +\infty \). In addition, if \( x > 0 \), then \( \overline{\Phi}(x) = x \varphi(x) - \Phi(\varphi(x)) < -\Phi(\varphi(x)) \). Letting \( x \to +\infty \), we obtain \( \overline{\Phi}(+\infty) = -\infty \). Therefore, the function \( \overline{\Phi} \) assumes every value in \( \mathbb{R} \).

Let \( \alpha \in L, D_\alpha = [x_0, 0), \beta \in \Omega_0 \), and let \( \eta \in Y_0 \) be a function non-decreasing on \( D_\eta \). The quantity

\[
R_{\alpha,\beta}[\eta] = \lim_{\sigma \uparrow 0} \frac{\alpha(\max\{x_0, \eta(\sigma)\})}{\beta(\sigma)}
\]

is called the generalized order \((\alpha, \beta\text{-order})\) of the function \( \eta \). Note that in the definition of the quantity \( R_{\alpha,\beta}[\eta] \), the constant \( x_0 \) can be replaced by any other number from \( D_\alpha \), and if \( b \in D_\eta \), then the function \( \eta \) can be replaced by the restriction of \( \eta \) to \([b, 0)\). It is also clear that if \( \zeta \in Y_0 \) is a function non-decreasing on \( D_\zeta \) and \( \zeta(\sigma) \leq \eta(\sigma) \) for all \( \sigma \in [c, 0) \), then \( R_{\alpha,\beta}[\zeta] \leq R_{\alpha,\beta}[\eta] \).

For each Dirichlet series \( F \in D_0 \), we set \( R_{\alpha,\beta}^*(F) = R_{\alpha,\beta}[\eta] \), where \( \eta(\sigma) = \ln \mu(\sigma, F) \) for all \( \sigma \in [-1, 0) \). If \( F \in D_0 \), then we set \( R_{\alpha,\beta}(F) = R_{\alpha,\beta}[\eta] \), where \( \eta(\sigma) = \ln M(\sigma, F) \) for all \( \sigma \in [-1, 0) \); the quantity \( R_{\alpha,\beta}(F) \) is called the generalized order \((\alpha, \beta\text{-order})\) of the Dirichlet series \( F \). It is clear that for each Dirichlet series \( F \in D_0 \) we have \( R_{\alpha,\beta}^*(F) \leq R_{\alpha,\beta}(F) \).

The growth of a Dirichlet series \( F \in D_0 \) is usually identified with the growth of the function \( \ln M(\sigma, F) \) as \( \sigma \uparrow 0 \). Important characteristics of the growth of such a series are its generalized orders \( R_{\alpha,\beta}(F) \). Establishing various relations according to which the generalized order \( R_{\alpha,\beta}(F) \) of a Dirichlet series \( F \in D_0 \) of the form \((1)\) can be expressed by the sequences of modules of its coefficients \((|a_n|)_{n \in \mathbb{N}_0}\), is a well-known classical problem. In connection with this problem, note that the generalized order \( R_{\alpha,\beta}^*(F) \) of a Dirichlet series \( F \in D_0 \) of the form \((1)\) can be relatively simply expressed in terms of the sequence \((|a_n|)_{n \in \mathbb{N}_0}\) (see, for example, [2, 3]; see also below). In view of what has been said, the following problem arises.

**Problem 1.** Let \( \alpha \in L, \beta \in \Omega_0 \), and \( \lambda \in \Lambda \). Find a necessary and sufficient condition on the sequence \( \lambda \) under which \( R_{\alpha,\beta}(F) = R_{\alpha,\beta}^*(F) \) for each Dirichlet series \( F \in D_0(\lambda) \).

A similar problem for entire (absolutely convergent in \( \mathbb{C} \)) Dirichlet series was considered in [2, 4]. In [2], moreover, Problem 1 was completely solved in the case when \( \alpha(x) = x \) for all \( x \in [x_0, +\infty) \). Without going into details, we note that the results obtained in [2] also allow us to find a complete solution of Problem 1 in the case when \( \alpha \in L \) is an arbitrary function regularly varying at the point \(+\infty\) with index \( \rho > 0 \). This case is partially covered in this article.

We note also that by certain assumptions about the growth of functions \( \alpha \in L \) and \( \beta \in \Omega_0 \), sufficient conditions on a sequence \( \lambda \in \Lambda \) under which \( R_{\alpha,\beta}(F) = R_{\alpha,\beta}^*(F) \) for any Dirichlet series \( F \in D_0(\lambda) \), were found in many works (see, for example, [5, 6, 7, 8, 9, 10, 11, 12]). In this article, Problem 1 is completely solved, in particular, in the case when \( \alpha \in L \) is an arbitrary function slowly varying at the point \(+\infty\) and \( \beta \in \Omega_0 \) is an arbitrary function regularly varying at the point 0 with index \( \rho > 0 \).

For every function \( \Phi \in \Omega_0 \) and each sequence \( \lambda = (\lambda_n)_{n \in \mathbb{N}_0} \) from the class \( \Lambda \), we put

\[
\Delta_\Phi(\lambda) = \lim_{n \to \infty} \frac{\ln n}{\Phi(\lambda_n)}.
\]
Theorem 1. Let $\alpha \in L$ be a function regularly varying at the point $+\infty$ with index $\rho_1 \geq 0$, $\beta \in \Omega_0$ be a function regularly varying at the point $0$ with index $\rho_2 \geq 0$, $\rho = \rho_1 + \rho_2 > 0$, and $\Phi(\sigma) = \alpha^{-1}(\beta(\sigma))$ for all $\sigma \in [a,0)$. Then the following statements are true:

(a) for each Dirichlet series $F \in D_0^*$ of the form (1) we have

$$R_{\alpha,\beta}^*(F) = \lim_{\sigma \uparrow 0} \left( \frac{\ln|a_n|}{-\Phi(\lambda_n)} \right)^{\rho};$$

(b) if $\lambda \in \Lambda$ and $\tau(\lambda) > 0$, then for each $p_0 \in [0, +\infty]$ and any $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha,\beta}^*(F) = p_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(c) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each Dirichlet series $F \in D_0(\lambda)$ the inequality $(R_{\alpha,\beta}^*(F))^{1/\rho} \leq (R_{\alpha,\beta}^*(F))^{1/\rho} + \Delta_F(\lambda)$ holds;

(d) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha,\beta,\gamma}^*(F) = p_0$ and $(R_{\alpha,\beta,\gamma}^*(F))^{1/\rho} = (R_{\alpha,\beta,\gamma}^*(F))^{1/\rho} + \Delta_F(\lambda)$.

Therefore, if $\lambda \in \Lambda$, then under the assumptions of Theorem 1 the equality $R_{\alpha,\beta}^*(F) = R_{\alpha,\beta}^*(F)$ holds for every Dirichlet series $F \in D_0(\lambda)$ if and only if $\Delta_F(\lambda) = 0$.

We obtain Theorem 1 from more general results proved below for modified orders of Dirichlet series from the class $D_0$.

2. Auxiliary results. The following two lemmas, which we will need later, are well known (see, for example, [2, 13]).

Lemma 1. Let $\Phi \in \Omega_0$, and let $F \in D_0^*$ be a Dirichlet series of the form (1). Then the following conditions are equivalent:

(i) there exists a number $\sigma_0 < 0$ such that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(ii) there exists a number $n_0 \in \mathbb{N}_0$ such that $\ln|a_n| \leq -\tilde{\Phi}(\lambda_n)$ for all integers $n \geq n_0$.

Lemma 2. Let $\Phi \in \Omega_0$, $D_0 = [a,0)$, and $p$ be a positive constant. Then for the function $\Psi(\sigma) = p\Phi(\sigma/p), \sigma \in [pa,0)$, we have $\Psi \in \Omega_0$ and $\Psi(x) = p\Phi(x)$ for all $x \in \mathbb{R}$.

Theorem A ([2]). Let $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ be a sequence from the class $\Lambda$ with $\tau(\lambda) > 0$, and $G \in D_0^*(\lambda) \setminus D_0(\lambda)$ be a Dirichlet series of the form $G(s) = \sum_{n=0}^{\infty} b_n e^{\lambda_n s}$ with $b_n \geq 0$ for all $n \in \mathbb{N}_0$. Then for any function $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in D_0(\lambda)$ of the form (1) such that $a_n = b_n$ or $a_n = 0$ for each $n \in \mathbb{N}_0$, and $M(\sigma, F) = F(\sigma) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$.

3. Main results. Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and $\eta \in Y_0$ be a function non-decreasing on $D_l$. By $S_{\Phi,l}[\eta]$ denote the set of those $p > 0$ for which there exists $\sigma_0 = \sigma_0(p) < 0$ such that

$$\eta(\sigma) \leq l(p)\Phi(\sigma/l(p)), \quad \sigma \in [\sigma_0, 0).$$

Note that if $\eta(0 - 0) < +\infty$, then $S_{\Phi,l}[\eta] = (0, +\infty)$. If $\eta(0 - 0) = +\infty$, $p \in S_{\Phi,l}[\eta]$, and $q > p$, then $l(q) > l(p)$ and for all $\sigma < 0$ sufficiently close to $0$ we have $\eta(l(q)\sigma)/l(q) \leq \eta(l(p)\sigma)/l(p)$, and therefore $q \in S_{\Phi,l}[\eta]$. If $S_{\Phi,l}[\eta] = \emptyset$, we set $p_{\Phi,l}[\eta] = +\infty$, and let $p_{\Phi,l}[\eta] = \inf S_{\Phi,l}[\eta]$ in the opposite case. It is obvious that if $b \in D_l$, then in the definition of the quantity $p_{\Phi,l}[\eta]$, the function $\eta$ can be replaced by the restriction of $\eta$ to $[b,0)$. It is also clear that if $\zeta \in Y_0$ is a function non-decreasing on $D_\zeta$ and $\zeta(\sigma) \leq \eta(\sigma)$ for all $\sigma \in [c,0)$, then $p_{\Phi,l}[\zeta] \leq p_{\Phi,l}[\eta]$. 

For a Dirichlet series $F \in D^*_0$ of the form (1), we set $S^*_{\Phi,l}(F) = S_{\Phi,l}[\eta]$ and $p^*_{\Phi,l}(F) = p_{\Phi,l}[\eta]$, where $\eta(\sigma) = \ln \mu(\sigma, F)$ for all $\sigma \in [-1, 0)$, and let
\[
k_{\Phi}(F) = \lim_{n \to +\infty} \frac{\ln^+ |a_n|}{-\Phi(\lambda_n)}.
\]
For every Dirichlet series $F \in D_0$, we put $S_{\Phi,l}(F) = S_{\Phi,l}[\eta]$ and $p_{\Phi,l}(F) = p_{\Phi,l}[\eta]$, where $\eta(\sigma) = \ln M(\sigma, F)$ for all $\sigma \in [-1, 0)$. Note that $p^*_{\Phi}(F) \leq p_{\Phi}(F)$ for an arbitrary $\Phi \in \Omega_0$ and any Dirichlet series $F \in D_0$.

Using Lemmas 1 and 2, it is easy to prove the following statement.

**Proposition 1.** Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and let $F \in D^*_0$ be a Dirichlet series of the form (1). Then $p^*_{\Phi,l}(F) = l^{-1}(k_{\Phi}(F))$.

**Proof.** First, we prove the inequality $l(p^*_{\Phi,l}(F)) \leq k_{\Phi}(F)$. This inequality is trivial in the case when $k_{\Phi}(F) = +\infty$. Suppose that $k_{\Phi}(F) < +\infty$, and let $k > k_{\Phi}(F)$ be an arbitrary fixed number. Note that $k_{\Phi}(F) \geq 0$, and therefore $k > 0$. Setting $p = l^{-1}(k)$, from the definition of the quantity $k_{\Phi}(F)$ for some $n_0 \in \mathbb{N}_0$ we obtain
\[
\ln^+ |a_n| \leq -l(p)\Phi(\lambda_n), \quad n \geq n_0.
\] (3)
Then by Lemmas 1 and 2 for some $\sigma_0 < 0$ we have
\[
\ln \mu(\sigma, F) \leq l(p)\Phi(\sigma/l(p)), \quad \sigma \in [\sigma_0, 0),
\] (4)
that is, $p \in S^*_{\Phi,l}(F)$. Thus, $p^*_{\Phi,l}(F) \leq p$, and hence $l(p^*_{\Phi,l}(F)) \leq l(p) = k$. Since $k > k_{\Phi}(F)$ is arbitrary, we obtain $l(p^*_{\Phi,l}(F)) \leq k_{\Phi}(F)$.

Now we prove the opposite inequality $l^{-1}(k_{\Phi}(F)) \leq p^*_{\Phi,l}(F)$. This inequality is trivial in the case when $p^*_{\Phi,l}(F) = +\infty$. Suppose that $p^*_{\Phi,l}(F) < +\infty$, and let $p > p^*_{\Phi,l}(F)$ be an arbitrary fixed number. From the definition of the quantity $p^*_{\Phi,l}(F)$ for some $\sigma_0 < 0$ we have (4). Since $-\Phi(x) \to +\infty$ as $x \to +\infty$, by Lemmas 1 and 2 for some $n_0 \in \mathbb{N}_0$ we obtain (3), and therefore $k_{\Phi}(F) \leq l(p)$, i.e. $l^{-1}(k_{\Phi}(F)) \leq p$. Since $p > p^*_{\Phi,l}(F)$ is arbitrary, we obtain $l^{-1}(k_{\Phi}(F)) \leq p^*_{\Phi,l}(F)$.  

**Proposition 2.** Let $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ be a sequence from the class $A$ with $\tau(\lambda) > 0$, and let $\Psi \in \Omega_0$ be an arbitrary function. Then:

(i) there exists a Dirichlet series $F \in D_0(\lambda)$ of the form (1) such that $a_n = 1$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0]$;

(ii) for each $k_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $k_{\Phi}(F) = k_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0]$.

**Proof.** Let $k_0 \in [0, +\infty]$. In the case when $k_0 < +\infty$ for all $n \in \mathbb{N}_0$ we set $b_n = e^{-k_0\Phi(\lambda_n)}$, and in the case when $k_0 = +\infty$ for all $n \in \mathbb{N}_0$ we put $b_n = e^{-\lambda_n\delta_n}$, where $(\delta_n)_{n \in \mathbb{N}_0}$ is an arbitrary sequence increasing to $0$ such that $\Phi(\lambda_n) = o(\delta_n)$ as $n \to \infty$.

Consider the Dirichlet series $G(s) = \sum_{n=0}^{\infty} b_n e^{\lambda_n s}$. It is easy to verify that $\sigma^*(G) = 0$, and therefore $G \in D^*_0(\lambda)$. Let’s fix some $\sigma \in (-\tau(\lambda), 0)$. Then $\tau(\lambda) > -\sigma$, and hence the set $E = \{k \in \mathbb{N}_0 : \ln k \geq -\sigma \lambda_k\}$ is infinite. For an arbitrary sufficiently large $k \in E$ we have
\[
\sum_{n=\lfloor k/2 \rfloor}^{k} a_n e^{\sigma \lambda_n} \geq \sum_{n=\lfloor k/2 \rfloor}^{k} e^{\sigma \lambda_n} \geq \sum_{n=\lfloor k/2 \rfloor}^{k} e^{\sigma \lambda_k} \geq \frac{k}{2} e^{\sigma \lambda_k} \geq \frac{k}{2} e^{-\ln k} = \frac{1}{2}.
\]
Therefore, the series $G$ is divergent at the point $s = \sigma$, and hence $G \in \mathcal{D}_0^*(\lambda) \setminus \mathcal{D}_0(\lambda)$. Then, according to Theorem A, there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) such that $a_n = b_n$ or $a_n = 0$ for each $n \in \mathbb{N}_0$, and $M(\sigma, F) = F(\sigma) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. In addition, as it is not difficult to verify, $k_{\Phi}(F) = k_0$. This proves (ii), and also (i) if $k_0 = 0$. □

From Proposition 2 we see that if $\tau(\lambda) > 0$, then the growth of a Dirichlet series from the class $\mathcal{D}_0(\lambda)$ can be arbitrarily fast even under the condition of boundedness its maximal term.

**Theorem 2.** Let $\Phi \in \Omega_0$, $l \in L$ be a function with $D_l = [0, +\infty)$ and $l(0) = 0$, and let $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ be a sequence from class $\Lambda$ with $\tau(\lambda) = 0$. Then:

(i) for each Dirichlet series $F \in \mathcal{D}_0(\lambda)$ we have $l(p_{\Phi,l}(F)) \leq l(p_{\Phi,l}^*(F)) + \Delta_{\Phi}(\lambda)$;

(ii) for every $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ such that $p_{\Phi,l}^*(F) = p_0$ and $l(p_{\Phi,l}(F)) = l(p_{\Phi,l}^*(F)) + \Delta_{\Phi}(\lambda)$.

**Proof.** To prove (i), we consider a Dirichlet series $F \in \mathcal{D}_0(\lambda)$ of the form (1) and note that the inequality $l(p_{\Phi,l}(F)) \leq l(p_{\Phi,l}^*(F)) + \Delta_{\Phi}(\lambda)$ does not need proof if $p_{\Phi,l}^*(F) = +\infty$ or $\Delta_{\Phi}(\lambda) = +\infty$. Suppose that $p_{\Phi,l}^*(F) < +\infty$ and $\Delta_{\Phi}(\lambda) < +\infty$ and fix an arbitrary $b > l(p_{\Phi,l}^*(F)) + \Delta_{\Phi}(\lambda)$. It is clear that then there exist constants $c > l(p_{\Phi,l}^*(F))$ and $\Delta > \Delta_{\Phi}(\lambda)$ such that $c + \Delta < b$. From Proposition 1, the definition of the quantity $\Delta_{\Phi}(\lambda)$, and the obvious inequality $(b - c)/\Delta > 1$, it follows the existence of a number $n_0 \in \mathbb{N}_0$ such that for all integers $n \geq n_0$ the following inequalities $|a_n| \leq e^{-c\Phi(\lambda_n)}$, $n \leq e^{-\Delta_{\Phi}(\lambda_n)}$ hold and, in addition,

$$\sum_{n \geq n_0} \frac{1}{n^{(b-c)/\Delta}} \leq \frac{1}{2}.$$

Consider the auxiliary Dirichlet series

$$G(s) = \sum_{n \geq n_0} e^{-c\Phi(\lambda_n)} e^{s\lambda_n}.$$ 

It is easy to verify that $\sigma^*(G) = 0$, that is, $G \in \mathcal{D}_0(\lambda)$. In addition, Lemmas 1 and 2 imply the existence of a constant $\sigma_0 < 0$ such that $\ln \mu(\sigma, G) \leq c\Phi(\sigma/c)$ for all $\sigma \in [\sigma_0, 0)$. Therefore, using the above inequalities, for all $\sigma \in [b\sigma_0/c, 0)$ we obtain

$$M(\sigma, G) = \sum_{n \geq n_0} e^{-c\Phi(\lambda_n)} e^{\sigma\lambda_n} = \sum_{n \geq n_0} \left(e^{-c\Phi(\lambda_n)} e^{(c/b)\lambda_n}\right)^{b/c} \frac{1}{e^{-(b-c)\Phi(\lambda_n)}} \leq \frac{1}{c} \left(\mu(c\sigma/b, G)^{b/c} \sum_{n \geq n_0} \frac{1}{n^{(b-c)\Phi(\lambda_n)}} \leq e^{b\Phi(\sigma/b)} \sum_{n \geq n_0} \frac{1}{n^{(b-c)/\Delta}} \leq \frac{1}{2} e^{b\Phi(\sigma/b)} \right).$$

Then, by taking $q = l^{-1}(b)$, for all $\sigma < 0$ sufficiently close to 0 we have

$$M(\sigma, F) \leq \sum_{n \geq n_0} |a_n| e^{\sigma\lambda_n} + M(\sigma, G) \leq e^{b\Phi(\sigma/b)} = e^{(q)\Phi(\sigma/l(q))}.$$ 

Therefore, $q \in S_{\Phi,l}(F)$, and hence $l(p_{\Phi,l}(F)) \leq l(q) = b$. The required inequality follows from the arbitrariness of $b > l(p_{\Phi,l}^*(F)) + \Delta_{\Phi}(\lambda)$.

Now we prove (ii). The proof is trivial if $p_0 = +\infty$ or $\Delta_{\Phi}(\lambda) = 0$. Suppose that $p_0 < +\infty$ and $\Delta_{\Phi}(\lambda) > 0$, and set $c = l(p_0)$. Let us choose arbitrary positive sequences $(c_k)_{k \in \mathbb{N}_0},$
In this way, it is easy to verify that according to (5),

\[ n_k < m_{k+1}, \quad \frac{\ln(n_k - m_k + 1)}{-\Phi(\lambda_{n_k})} \geq \Delta_k, \quad \frac{\ln m_k}{-\Phi(\lambda_{m_k})} \leq \delta_k \frac{\ln n_k}{-\Phi(\lambda_{n_k})} \]

for each \( k \in \mathbb{N}_0 \).

Let \( n \in \mathbb{N}_0 \). Put \( a_n = e^{-c_k \Phi(\lambda_n)} \) if \( n \in [m_k, n_k] \) for some \( k \in \mathbb{N}_0 \), and let \( a_n = 0 \) in the opposite case. Consider the Dirichlet series \( F \) of the form (1) with the coefficients \( a_n \) defined in this way. It is easy to verify that \( \sigma^*(F) = 0 \), that is, \( F \in D_0(\lambda) \), and \( k_\Phi(F) = c \). According to Proposition 1 we have \( p^*_\Phi(l)(F) = l^{-1}(c) = p_0 \). Note also that the constructed series can be written in the form

\[ F(s) = \sum_{k=0}^{\infty} \sum_{n=m_k}^{n_k} e^{-c_k \Phi(\lambda_n)} e^{s \lambda_n}. \]

For each \( k \in \mathbb{N}_0 \), we set \( b_k = (n_k - m_k + 1)e^{-c_k \Phi(\lambda_{m_k})} \), and consider the auxiliary Dirichlet series \( H(s) = \sum_{k=0}^{\infty} b_k e^{s \lambda_{m_k}} \). If \( \sigma < 0 \) and \( k \in \mathbb{N}_0 \), then

\[ \sum_{n=m_k}^{n_k} e^{-c_k \Phi(\lambda_n)} e^{s \lambda_n} \geq \sum_{n=m_k}^{n_k} e^{-c_k \Phi(\lambda_{m_k})} e^{s \lambda_{m_k}} = b_k e^{\sigma \lambda_{m_k}}, \]

and therefore \( H \in D_0 \). In addition, \( M(\sigma, F) = F(\sigma) \geq H(\sigma) = M(\sigma, H) \) for each \( \sigma < 0 \) and, according to (5),

\[ k_\Phi(H) = \lim_{k \to +\infty} \frac{\ln^{+} [b_k]}{-\Phi(\lambda_{n_k})} = \lim_{k \to +\infty} \left( \frac{\ln(n_k - m_k + 1)}{-\Phi(\lambda_{n_k})} + c_k \frac{\Phi(\lambda_{m_k})}{\Phi(\lambda_{n_k})} \right) \geq \Delta_\Phi(\lambda) + c. \]

Therefore, \( l(p^*_\Phi(l)(F)) \geq l(p^*_H(l)(F)) \geq l(p^*_{\Phi,F}(F)) = k_\Phi(H) \geq l(p^*_{\Phi,F}(F)) + \Delta_\Phi(\lambda) \). It remains to use statement (i) of this theorem. \( \square \)

Let \( \alpha \in L, \ D_\alpha = [x_0, 0), \ \beta \in \Omega_0, \ \gamma \in C_0 \) be a function positive on \( D_\gamma \), and \( \eta \in Y_0 \) be a function non-decreasing on \( D_\eta \). The quantity

\[ R_{\alpha,\beta,\gamma}[\eta] = \lim_{\sigma \to 0} \frac{\alpha(\max\{x_0, \gamma(\sigma) \eta(\sigma)\})}{\beta(\sigma)} \]

is called the modified order \( (\alpha, \beta, \gamma\text{-order}) \) of the function \( \eta \). Note that in the definition of the quantity \( R_{\alpha,\beta}[\eta] \), the constant \( x_0 \) can be replaced by any other number from \( D_\alpha \), and if \( b \in D_\eta \), then the function \( \eta \) can be replaced by the restriction of \( \eta \) to \( [b, 0] \). It is also clear that if \( \zeta \in Y_0 \) is a function non-decreasing on \( D_\zeta \) and \( \zeta(\sigma) \leq \eta(\sigma) \) for all \( \sigma \in [c, 0] \), then \( R_{\alpha,\beta,\gamma}[\zeta] \leq R_{\alpha,\beta,\gamma}[\eta] \).

For any Dirichlet series \( F \in D_0^\alpha \), we set \( R_{\alpha,\beta,\gamma}^*(F) = R_{\alpha,\beta,\gamma}[\eta] \), where \( \eta(\sigma) = \ln \mu(\sigma, F) \) for all \( \sigma \in [-1, 0] \). If \( F \in D_0^\alpha \), we put \( R_{\alpha,\beta,\gamma}(F) = R_{\alpha,\beta,\gamma}[\eta] \), where \( \eta(\sigma) = \ln M(\sigma, F) \) for all \( \sigma \in [-1, 0] \); the quantity \( R_{\alpha,\beta,\gamma}(F) \) is called the modified order \( (\alpha, \beta, \gamma\text{-order}) \) of the Dirichlet series \( F \). It is clear that \( R_{\alpha,\beta,\gamma}^*(F) \leq R_{\alpha,\beta,\gamma}(F) \) for every Dirichlet series \( F \in D_0^\alpha \).

In connection with Problem 1, it is natural to consider the following more general problem.
Problem 2. Let $\alpha \in L$, $\beta \in \Omega_0$, $\gamma \in C_0$ be a function positive on $D_\gamma$, and $\lambda \in \Lambda$. Find a necessary and sufficient condition on the sequence $\lambda \in \Lambda$ under which $R_{\alpha,\beta,\gamma}(F) = R_{\alpha,\beta,\gamma}^*(F)$ for each Dirichlet series $F \in D_0(\lambda)$.

Below we will obtain solutions of Problem 2 under fairly general assumptions about the behavior of the functions $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$.

Proposition 3. Let $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$ be a function positive on $D_\gamma$. If the condition

$$\forall c > 0: \lim_{\sigma \to 0} \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma) < +\infty$$

(6)

holds, then for any sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ from the class $\Lambda$ there exists a Dirichlet series $F \in D_0(\lambda)$ of the form (1) such that $a_n = 1$ or $a_n = 0$ for each $n \in \mathbb{N}_0$ and $R_{\alpha,\beta,\gamma}(F) = +\infty$.

Proof. In the case when $\tau(\lambda) > 0$, it is enough to use Proposition 2.

Let $\tau(\lambda) = 0$. Consider the series $F(s) = \sum_{n=0}^{\infty} e^{\lambda_n}$. It is clear that $F \in D_0(\lambda)$, $R_{\alpha,\beta,\gamma}^*(F) = 0$ and $M(\sigma, F) \uparrow +\infty$ as $\sigma \uparrow 0$. Suppose that $R_{\alpha,\beta,\gamma}(F) < c$ for some $c > 0$. Then, from the definition of the quantity $R_{\alpha,\beta,\gamma}(F)$, we have $\ln M(\sigma, F) \leq \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$, which contradicts (6). Therefore, $R_{\alpha,\beta,\gamma}(F) = +\infty$. \qed

Theorem 3. Let $\alpha \in L$, $\beta \in \Omega_0$, and $\gamma \in C_0$ be a function positive on $D_\gamma$. Suppose that condition (6) is not satisfied, and a function $\Phi \in C_0$ is such that

$$\forall t > 0: \lim_{\sigma \to 0} \gamma(\sigma)\Phi(\sigma/t) = +\infty$$

(7)

and for every $t > 0$ there exists a finite limit

$$h(t) := \lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma/t))}{\beta(\sigma)},$$

(8)

and $h(t)$ is a continuously function increasing to $+\infty$ on $(0, +\infty)$ with $h(0) = h(0 + 0) = 0$. Then $\Phi \in \Omega_0$ and if $t = l(p)$ is the inverse function of the function $p = h(t)$, then:

(a) for every function $\eta \in Y_0$ non-decreasing on $D_\eta$, we have $R_{\alpha,\beta,\gamma}[\eta] = p_{\Phi,\lambda}[\eta]$;

(b) for each Dirichlet series $F \in D_0(\lambda)$ we have $R_{\alpha,\beta,\gamma}^*(F) = h(k_\Phi(F))$;

(c) if $\lambda \in \Lambda$ and $\tau(\lambda) > 0$, then for every $p_0 \in [0, +\infty]$ and any $\Psi \in \Omega_0$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha,\beta,\gamma}(F) = p_0$ and $M(\sigma, F) \geq \Psi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$;

(d) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for every Dirichlet series $F \in D_0(\lambda)$ the inequality $l(R_{\alpha,\beta,\gamma}(F)) \leq l(R_{\alpha,\beta,\gamma}^*(F)) + \Delta_\Phi(\lambda)$ holds;

(e) if $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then for each $p_0 \in [0, +\infty]$ there exists a Dirichlet series $F \in D_0(\lambda)$ such that $R_{\alpha,\beta,\gamma}^*(F) = p_0$ and $l(R_{\alpha,\beta,\gamma}(F)) = l(R_{\alpha,\beta,\gamma}^*(F)) + \Delta_\Phi(\lambda)$.

Proof. If (6) is not satisfied, then for some $c > 0$ we have $\alpha^{-1}(c\beta(\sigma))/\gamma(\sigma) \to +\infty$ as $\sigma \uparrow 0$. Let’s choose the number $t > 0$ so that the inequality $h(t) > c$ holds. Then, according to (8), there exists $\sigma_0 < 0$ such that $t\Phi(\sigma/t) \geq \alpha^{-1}(c\beta(\sigma))/\gamma(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Therefore, $\Phi(\sigma) \to +\infty$ as $\sigma \uparrow 0$, and hence $\Phi \in \Omega_0$.

Let’s prove (a). Let $\eta \in Y_0$ be a function non-decreasing on $D_\eta$. First, we show that $R_{\alpha,\beta,\gamma}[\eta] \leq p_{\Phi,\lambda}[\eta]$. This inequality is trivial if $p_{\Phi,\lambda}[\eta] = +\infty$. Suppose that $p_{\Phi,\lambda}[\eta] < +\infty,$
and let \( p > p_{\Phi, l}[\eta] \) be an arbitrary fixed number. Then there exists a number \( \sigma_0 < 0 \) such that (2) holds. Therefore, using (2) and (7), we obtain
\[
R_{\alpha, \beta, \gamma}[\eta] \leq \lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p)))}{\beta(\sigma)} = h(l(p)) = p,
\]
and the required inequality follows from the arbitrariness of \( p > p_{\Phi, l}[\eta] \).

Now we prove that \( p_{\Phi, l}[\eta] \leq R_{\alpha, \beta, \gamma}[\eta] \). This inequality is trivial if \( R_{\alpha, \beta, \gamma}[\eta] = +\infty \). Suppose that \( R_{\alpha, \beta, \gamma}[\eta] < +\infty \), and let \( p > R_{\alpha, \beta, \gamma}[\eta] \) be an arbitrary fixed number, and \( q \in (R_{\alpha, \beta, \gamma}[\eta], p) \). From the definition of the quantity \( R_{\alpha, \beta, \gamma}[\eta] \) for some \( \sigma_1 < 0 \) we have
\[
\gamma(\sigma)\eta(\sigma) \leq \alpha^{-1}(q\beta(\sigma)), \quad \sigma \in [\sigma_1, 0). \tag{9}
\]
In addition, since
\[
\lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p)))}{\beta(\sigma)} = h(l(p)) = p > q,
\]
for some \( \sigma_2 < 0 \) we obtain
\[
q\beta(\sigma) \leq \alpha(\gamma(\sigma)l(p)\Phi(\sigma/l(p))), \quad \sigma \in [\sigma_2, 0). \tag{10}
\]
Taking \( \sigma_0 = \max\{\sigma_1, \sigma_2\} \), from (9) and (10) we see that (2) is fulfilled, i.e. \( p \in S_{\Phi, l}[\eta] \). Therefore, \( p_{\Phi, l}[\eta] \leq p \). Since \( p > R_{\alpha, \beta, \gamma}[\eta] \) is arbitrary, we have \( p_{\Phi, l}[\eta] \leq R_{\alpha, \beta, \gamma}[\eta] \).

Further, according to the part of the theorem that has already been proved, for each Dirichlet series \( F \in D_0^* \) we obtain \( R_{\alpha, \beta, \gamma}(F) = p_{\Phi, l}(F) \), and for each Dirichlet series \( F \in D_0 \) we have \( R_{\alpha, \beta, \gamma}(F) = p_{\Phi, l}(F) \). Therefore, (b) follows from Proposition 1, and (c) follows from Proposition 2. In addition, for an arbitrary sequence \( \lambda \in \Lambda \) with \( \tau(\lambda) = 0 \), according to Theorem 2, we have (d) and (e).

\[\square\]

4. Corollaries. Let us give some consequences from the results proved above.

**Theorem 4.** Let \( \alpha \in L \) be a function slowly varying at the point \( +\infty \), \( \beta \in \Omega_0 \) be a function regularly varying at the point 0 with index \( \rho > 0 \), and \( \gamma \in C_0 \) be a function positive on \( D_\gamma \) such that for each fixed \( t > 0 \) the inequalities
\[
0 < \lim_{\sigma \to 0} \frac{\gamma(t\sigma)}{\gamma(\sigma)} \leq \lim_{\sigma \to 0} \frac{\gamma(t\sigma)}{\gamma(\sigma)} < +\infty \tag{11}
\]
hold, and \( \Phi \in C_0 \) be a function such that \( \Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma) \) for all \( \sigma \in [\sigma_0, 0) \). Then \( \Phi \in \Omega_0 \) and the following statements are true:

(a) for every Dirichlet series \( F \in D_0^* \) we have
\[
R_{\alpha, \beta, \gamma}(F) = \lim_{\sigma \to 0} \left( \frac{\ln |a_n|}{-\Phi(\lambda_n)} \right)^\rho;
\]
(b) if \( \lambda \in \Lambda \) and \( \tau(\lambda) > 0 \), then for every \( p_0 \in [0, +\infty] \) and any \( \Psi \in \Omega_0 \) there exists a Dirichlet series \( F \in D_0(\lambda) \) such that \( R_{\alpha, \beta, \gamma}(F) = p_0 \) and \( M(\sigma, F) \geq \Psi(\sigma) \) for all \( \sigma \in [\sigma_0, 0) \);
(c) if \( \lambda \in \Lambda \) and \( \tau(\lambda) = 0 \), then for every Dirichlet series \( F \in D_0(\lambda) \) the inequality \( (R_{\alpha, \beta, \gamma}(F))^{1/\rho} \leq (R_{\alpha, \beta, \gamma}(F))^{1/\rho} + \Delta_\Phi(\lambda) \) holds;
(d) if \( \lambda \in \Lambda \) and \( \tau(\lambda) = 0 \), then for each \( p_0 \in [0, +\infty] \) there exists a Dirichlet series \( F \in D_0(\lambda) \) such that \( R_{\alpha, \beta, \gamma}(F) = p_0 \) and \( (R_{\alpha, \beta, \gamma}(F))^{1/\rho} = (R_{\alpha, \beta, \gamma}(F))^{1/\rho} + \Delta_\Phi(\lambda) \).
Proof. Let \( D_\gamma = [a,0) \). From the conditions satisfied by the function \( \gamma \), for some constant \( d > 1 \) and all \( \sigma \in [a/2,0) \) we have \( \gamma(\sigma) \leq d\gamma(2\sigma) \). Let \( M = \max\{\gamma(\sigma) : \sigma \in [a,a/2]\} \), \( q = \log_2 d \) and \( c = M|a|^{\rho} \). Then for all \( \sigma \in [a,0) \) we have \( \gamma(\sigma) \leq c|\sigma|^{-q} \). In fact, if \( \sigma \in [a/2^n, a/2^{n+1}] \) for some \( n \in \mathbb{N}_0 \), then
\[
\gamma(\sigma) \leq d^{n+1} \gamma(2^n \sigma) \leq 2^{\rho n} M \leq M|a|^{\rho}|\sigma|^{-q} = c|\sigma|^{-q}.
\]

Let’s fix an arbitrary constant \( \rho_0 \in (0, \rho) \) and choose a constant \( r > 0 \) so that the inequality \( r\rho_0 > q \) holds. According to the well-known properties of regularly varying functions, there exist constants \( \sigma_1 < 0 \) and \( x_0 > 0 \) such that \( \beta(\sigma) \geq |\sigma|^{-\rho} \) for all \( \sigma \in [\sigma_1,0) \) and \( \alpha^{-1}(x) \geq x^r \) for all \( x \in [x_0, +\infty) \). Then there exists a constant \( \sigma_2 < 0 \) such that \( \alpha^{-1}(\beta(\sigma)) \geq (\beta(\sigma))^r \geq |\sigma|^{-\rho_0} \) for all \( \sigma \in [\sigma_2, 0) \). Therefore, using the above estimate for the \( \gamma \) function, we obtain \( \Phi(\sigma) \to +\infty \) as \( \sigma \to 0 \). From this, in particular, we see that \( \Phi \in \Omega_0 \), condition (6) is not satisfied, and (11) implies (7). In addition, using the conditions satisfied by the functions \( \alpha \), \( \beta \), and \( \gamma \), for each fixed \( t > 0 \) we have
\[
\lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma))}{\beta(\sigma t)} = \lim_{\sigma \to 0} \frac{\beta(\sigma)}{\beta(t\sigma)} = t^\rho.
\]
Therefore, all the conditions of Theorem 3 are satisfied with \( h(t) = t^\rho \) for all \( t > 0 \), and (a), (b), (c), and (d) are consequences of the corresponding statements of Theorem 3.

Theorem 5. Let \( \alpha \in L \) be an arbitrary function, \( \beta \in \Omega_0 \) be a function regularly varying at the point 0 with index \( \rho > 0 \), \( \gamma(\sigma) = |\sigma|^{-1} \) for all \( \sigma \in [-1,0) \), and \( \Phi \in C_0 \) be a function such that \( \Phi(\sigma) = |\sigma|^{-1}(\beta(\sigma)) \) for all \( \sigma \in [\sigma_0, 0) \). Then:

(i) if \( \Phi \not\in \Omega_0 \), then for every sequence \( \lambda \in \Lambda \) there exists a Dirichlet series \( F \in D_0(\lambda) \) such that \( R_{\alpha,\beta,\gamma}^*(F) = 0 \), but \( R_{\alpha,\beta,\gamma}^{\ast}(F) = +\infty \);

(ii) if \( \Phi \in \Omega_0 \), then statements (a), (b), (c), and (d) of Theorem 4 are true.

Proof. Noting that the condition \( \Phi \not\in \Omega_0 \) is equivalent to condition (6), from Proposition 3 we obtain (i).

Let \( \Phi \in \Omega_0 \). Then, as it is easy to see, (7) holds. In addition, for every fixed \( t > 0 \) we have
\[
\lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \to 0} \frac{\alpha(|\sigma|^{-1}\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \to 0} \frac{\alpha(|\sigma|^{-1}\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma \to 0} \frac{\beta(\sigma)}{\beta(t\sigma)} = t^\rho.
\]
Therefore, all the conditions of Theorem 3 are satisfied with \( h(t) = t^\rho \) for all \( t > 0 \). This implies (ii).

Theorem 6. Let \( \alpha \in L \) be an arbitrary function such that
\[
\forall q > 1: \lim_{y \to +\infty} \frac{\gamma^{-1}(qy)}{\gamma^{-1}(y)} > 1, \tag{12}
\]
\( \beta \in \Omega_0 \) be a function regularly varying at the point 0 with index \( \rho > 0 \), \( \gamma \in C_0 \) be a function regularly varying at the point 0 with index 1, and \( \Phi \in C_0 \) be a function such that \( \Phi(\sigma) = \alpha^{-1}(\beta(\sigma)) \) for all \( \sigma \in [\sigma_0, 0) \). Then statements (i) and (ii) of Theorem 5 are true.
Proof. By the assumptions of the theorem, the condition \( \Phi \not\in \Omega_0 \) is equivalent to condition (6), and therefore from Proposition 3 we obtain statement (i) of Theorem 5.

Let \( \Phi \in \Omega_0 \). Then, as it is easy to see, (7) holds. Next, we note that condition (12) is satisfied if and only if for any function \( \delta \in C_0 \) such that \( \delta(x) \to 1 \) as \( x \to +\infty \), we have \( \alpha(x\delta(x)) \sim \alpha(x) \) as \( x \to +\infty \). Using this fact, for every fixed \( t > 0 \) we have

\[
\lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)t\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \to 0} \frac{\alpha(\gamma(t\sigma)t\Phi(\sigma))}{\beta(t\sigma)} = \lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)\Phi(\sigma))}{t^{-\rho_3}\beta(\sigma)} = t^\rho.
\]

Therefore, all the conditions of Theorem 3 are satisfied with \( h(t) = t^\rho \) for all \( t > 0 \). This implies statement (ii) of Theorem 5.

\[ \square \]

**Theorem 7.** Let \( \alpha \in L \) be a function regularly varying at the point \( +\infty \) with index \( \rho_1 > 0 \), \( \beta \in \Omega_0 \) be a function regularly varying at the point \( 0 \) with index \( \rho_2 \geq 0 \), \( \gamma \in C_0 \) be a function regularly varying at the point \( 0 \) with index \( \rho_3 \in \mathbb{R} \), \( \rho = \rho_1 + \rho_2 - \rho_1\rho_3 \), and \( \Phi \in C_0 \) be a function such that \( \Phi(\sigma) = \alpha^{-1}(\beta(\sigma))/\gamma(\sigma) \) for all \( \sigma \in [\sigma_0, 0) \). Then:

(i) if \( \rho_2 - \rho_1\rho_3 < 0 \) or simultaneously the conditions \( \rho_2 - \rho_1\rho_3 = 0 \) and \( \Phi \not\in \Omega_0 \) are satisfied, then for any sequence \( \lambda \in \Lambda \) there exists a Dirichlet series \( F \in \mathcal{D}_0(\lambda) \) such that \( R_{\alpha,\beta,\gamma}(F) = 0 \) and \( R_{\alpha,\beta,\gamma}(F) = +\infty \);

(ii) if simultaneously the conditions \( \rho_2 - \rho_1\rho_3 = 0 \) and \( \Phi \in \Omega_0 \) are satisfied or \( \rho_2 - \rho_1\rho_3 > 0 \), then statements (a), (b), (c), and (d) of Theorem 4 are true.

**Proof.** For every fixed \( t > 0 \) we have

\[
\Phi(t\sigma) = \frac{\alpha^{-1}(\beta(t\sigma))}{\gamma(t\sigma)} \sim \frac{\alpha^{-1}(t^{-\rho_2}\beta(\sigma))}{t^{-\rho_3}\gamma(\sigma)} \sim \frac{t^{-\rho_2/\rho_1}}{t^{-\rho_3}} \Phi(\sigma) = t^{-(\rho_2-\rho_1\rho_3)/\rho_1}\Phi(\sigma), \quad \sigma \uparrow 0,
\]

i.e. \( \Phi \) is a function regularly varying at the point \( 0 \) with index \( (\rho_2 - \rho_1\rho_3)/\rho_1 \). We note that in the case when \( \rho_2 - \rho_1\rho_3 < 0 \) we have \( \Phi \not\in \Omega_0 \), in the case when \( \rho_2 - \rho_1\rho_3 > 0 \) we have \( \Phi \in \Omega_0 \), and in the case when \( \rho_2 - \rho_1\rho_3 = 0 \) both the situations \( \Phi \not\in \Omega_0 \) or \( \Phi \in \Omega_0 \) are possible.

Therefore, all the conditions of (i) reduce to the condition \( \Phi \not\in \Omega_0 \), and all the conditions of (ii) reduce to the condition \( \Phi \in \Omega_0 \).

Noting that the condition \( \Phi \not\in \Omega_0 \) is equivalent to condition (6), from Proposition 3 we obtain statement (i).

Let \( \Phi \in \Omega_0 \). Then, as it is easy to see, \( \rho > 0 \) and (7) holds. In addition, for every fixed \( t > 0 \) we have

\[
\lim_{\sigma \to 0} \frac{\alpha(\gamma(\sigma)t\Phi(\sigma/t))}{\beta(\sigma)} = \lim_{\sigma \to 0} \frac{t^{\rho_1}\alpha(t^{-\rho_3}\gamma(\sigma)\Phi(\sigma))}{\beta(t)} = \lim_{\sigma \to 0} \frac{t^{\rho_1-\rho_1\rho_3}\alpha(\gamma(\sigma)\Phi(\sigma))}{t^{-\rho_2}\beta(\sigma)} = t^\rho.
\]

Therefore, all the conditions of Theorem 3 are satisfied with \( h(t) = t^\rho \) for all \( t > 0 \). This implies statement (ii).

\[ \square \]

Finally, note that Theorem 1 is a consequence of Theorems 4 and 7 in the case when \( \gamma(\sigma) = 1 \) for all \( \sigma \in [-1, 0) \).
REFERENCES


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