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AN OPERATOR RICCATI EQUATION AND REFLECTIONLESS SCHRÖDINGER OPERATORS

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In this paper, we study a connection between the operator Riccati equation

$$S'(x) = KS(x) + S(x)K - 2S(x)KS(x), \quad x \in \mathbb{R},$$

and the set of reflectionless Schrödinger operators with operator-valued potentials. Here $K \in \mathcal{B}(H)$, $K > 0$ and $S : \mathbb{R} \to \mathcal{B}(H)$, where $\mathcal{B}(H)$ is the Banach algebra of all linear continuous operators acting in a separable Hilbert space H. Let $\mathscr{S}^+(K)$ be the set of all solutions S of the Riccati equation satisfying the conditions $0 < S(0) < I$ and $S'(0) \geq 0$, with I being the identity operator in H. We show that every solution $S \in \mathscr{S}^+(K)$ generates a reflectionless Schrödinger operator with some potential q that is an analytic function in the strip

$$\Pi_K := \left\{ z = x + iy \mid x, y \in \mathbb{R},\ |y| < \tfrac{\pi}{2\|K\|} \right\};$$

moreover,

$$\|q(x+iy)\| \leq 2\|K\|^2 \cos^{-2}(y\|K\|), \quad (x+iy) \in \Pi_K.$$

1. Introduction. In this paper, we show that there is a deep connection between a special operator Riccati equation and reflectionless Schrödinger operators with operator-valued potentials. This connection has many interesting aspects; here, we present the basic results and will discuss further subtle issues elsewhere.

Let us start with notations and basic terminology. Let H be a separable Hilbert space, and $\mathcal{B}(H)$ be the Banach algebra of all everywhere-defined linear continuous operators $A\colon H \to H$. Let $\mathcal{B}_{\mathrm{inv}}(H)$ be the group of all invertible operators in $\mathcal{B}(H)$, and $\mathcal{B}_+(H)$ be the cone of nonnegative operators $A \in \mathcal{B}(H)$. The domain, range, kernel, and the spectrum of a linear operator will be denoted by $\mathrm{dom}(\cdot)$, $\mathrm{ran}(\cdot)$, $\ker(\cdot)$, and $\sigma(\cdot)$, respectively. For arbitrary operators $A, B \in \mathcal{B}(H)$, we write $A < B$ if $A \leq B$ and $\ker(B - A) = \{0\}$.

1.1. A special operator Riccati equation. We consider the Riccati equation

$$S'(x) = KS(x) + S(x)K - 2S(x)KS(x), \quad x \in \mathbb{R}, \tag{1}$$

where $K \in \mathcal{B}_+(H)$, $K > 0$ and $S\colon \mathbb{R} \to \mathcal{B}(H)$. Denote by $\mathscr{S}(K)$ the set of all solutions S of the equation (1) such that $0 < S(0) < I$, with I being the identity operator in H.

Every function $S \in \mathscr{S}(K)$ is given by an explicit formula, namely

$$S(x) = e^{xK}(S^{-1}(0) - I + e^{2xK})^{-1}e^{xK}, \quad x \in \mathbb{R}. \tag{2}$$

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It follows from (2) that for all function $S \in \mathscr{S}(K)$

$$0 < S(x) < I, \quad x \in \mathbb{R}.$$

The equation (1) is nonlinear, but it can be reduced to a linear equation under certain conditions. Indeed, let $S \in \mathscr{S}(K)$ and $S(0) \in \mathcal{B}_{\mathrm{inv}}(H)$. Then (see (2)) $S(x) \in \mathcal{B}_{\mathrm{inv}}(H)$ for all $x \in \mathbb{R}$, and the function $Y(x) := S^{-1}(x) - I$ is a solution of the Lyapunov equation

$$Y'(x) = -KY(x) - Y(x)K, \quad x \in \mathbb{R}.$$

The functions $S \in \mathscr{S}(K)$ have an analytic continuation to a strip that depends only on the norm of the operator K.

Let $S \in \mathscr{S}(K)$. Denote by $\Omega(S)$ the set of all $z \in \mathbb{C}$, for which the operator $S^{-1}(0) - I + e^{2zK}$ has an inverse operator in $\mathcal{B}(H)$.

Proposition 1. *Let $S \in \mathscr{S}(K)$. Then*

(1) the set $\Omega(S)$ is open and symmetric with respect to the real axis, and the formula

$$S(z) = e^{zK}(S^{-1}(0) - I + e^{2zK})^{-1}e^{zK}, \quad z \in \Omega(S), \tag{3}$$

is an analytic continuation of the function S;

(2) the set $\Omega(S)$ contains the strip

$$\Pi_K := \left\{ z = x + iy \mid x, y \in \mathbb{R}, \ |y| < \tfrac{\pi}{2\|K\|} \right\},$$

moreover,

$$\|S(z)\| \le [\cos(y\|K\|)]^{-1}, \quad z \in \Pi_K, \quad y = \operatorname{Im} z.$$

Let C_b be the linear space of all bounded continuous functions $f\colon \mathbb{R} \to \mathcal{B}(H)$ equipped with the locally convex topology generated by seminorms

$$\rho_{h,E}(f) := \sup_{x \in E} \|f(x)h\|, \quad f \in C_b,$$

where $h \in H$ and the set E is compact in $\mathbb{R}$. If $\dim H < \infty$, the topology in C_b is a topology of uniform convergence on compact subsets of $\mathbb{R}$.

The set $\mathscr{S}(K)$ is considered as a topological subspace in C_b, i.e., it is equipped with the topology induced by C_b. We will show that every solution $S \in \mathscr{S}(K)$ is naturally related with some Schödinger operator; moreover, the functions S from the subset

$$\mathscr{S}^+(K) := \{S \in \mathscr{S}(K) \mid S'(0) \ge 0\} \tag{4}$$

correspond to reflectionless Schrödinger operators.

We observe that the shift by $a \in \mathbb{R}$ of the function S,

$$S_{\{a\}}(x) := S(x + a), \quad x \in \mathbb{R}, \tag{5}$$

and the “mirror” reflection,

$$S^\circ(x) := I - S(-x), \quad x \in \mathbb{R},$$

are continuous automorphisms both of the set $\mathscr{S}(K)$ and its subset $\mathscr{S}^+(K)$.

It turns out that every function $S \in \mathscr{S}^+(K)$ has a nonnegative derivative ($S'(x) \ge 0$ for all $x \in \mathbb{R}$), and therefore is nondecreasing on $\mathbb{R}$, i.e., $S(x_1) \le S(x_2),\ x_1 \le x_2$.

1.2. Reflectionless potentials of Schrödinger operators. The authors are unaware of previous work on reflectionless Schrödinger operators with operator-valued potentials. However, there are many papers on reflectionless Schrödinger operators in the scalar case. In the context of current research, the work of Marchenko [1] plays a pivotal role, and [1]–[6] mark further important progress in the field.

Let $\mathcal{H} := L_2(\mathbb{R}, H)$ be the Hilbert space of square integrable functions $f : \mathbb{R} \to H$ with the inner product

$$(f \mid g)_{\mathcal{H}} := \int_{\mathbb{R}} (f(x) \mid g(x))\, dx, \quad f, g \in \mathcal{H},$$

where $(\cdot \mid \cdot)$ is the inner product in H, which is linear in the first argument.

We associate every potential $q \in C_b$ with the Schrödinger operator $T_q \colon \mathcal{H} \to \mathcal{H}$ that is defined by the formula

$$T_q f = -f'' + qf \tag{6}$$

on the domain $\operatorname{dom} T_q := W_2^2(\mathbb{R}, H)$, where $W_2^2(\mathbb{R}, H)$ is the Sobolev space. If the potential q belongs to the set

$$C_{b,s} := \{q \in C_b \mid \forall x \in \mathbb{R} \quad q^*(x) = q(x)\},$$

the operator T_q is self-adjoint. Here $q^*(x) = (q(x))^*$.

Let $q \in C_{b,s}$ and $z \in \mathbb{C}$. Let us consider the equation

$$-y'' + qy = zy. \tag{7}$$

As shown in [7], for every $z \in \mathbb{C} \setminus \mathbb{R}$ there exist the Weyl–Titchmarsh $\mathcal{B}(H)$-valued right $f_+(z, \cdot)$ and left $f_-(z, \cdot)$ normalized solutions of the equation (7), i.e., the solutions that satisfy the condition

$$f_+(z, 0) = f_-(z, 0) = I,$$

and for every $h \in H$

$$\int_{\mathbb{R}_\pm} \|f_\pm(z, x)h\|^2\, dx < \infty.$$

The functions

$$m_\pm(z) := f'_\pm(z, 0), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

are called the Weyl–Titchmarsh m-functions of the equation (7) on the half-lines $\mathbb{R}_\pm$.

Let $q \in C_{b,s}$ and $m_\pm$ be the Weyl–Titchmarsh m-functions of the equation (7). A potential q (an operator T_q) is called *reflectionless* if the function

$$g(\lambda) := \begin{cases} m_+(\lambda^2), & \operatorname{Im} \lambda > 0,\ \operatorname{Re} \lambda \neq 0; \\ m_-(\lambda^2), & \operatorname{Im} \lambda < 0,\ \operatorname{Re} \lambda \neq 0 \end{cases}$$

has an analytic continuation to the domain $\mathbb{C} \setminus i\mathbb{R}$.

Denote by $\mathcal{Q}$ the set of all reflectionless potentials $q \in C_{b,s}$ and equip $\mathcal{Q}$ with the topology induced by C_b.

In the scalar case, these definitions are equivalent to the definitions given in the works [1] and [2].

1.3. The formulation of the main result. The main aim of this paper is to construct a natural mapping

$$\mathscr{S}^+(K) \ni S \mapsto \Upsilon(S) \in \mathcal{Q}$$

from the solutions $S \in \mathscr{S}^+(K)$ of the Riccati equation to reflectionless Schrödinger operators. Namely, starting from $S \in \mathscr{S}^+(K)$, we construct the following analytic functions on $\Omega(S)$:

$$L(z) := e^{zK}(I - S(z)) + e^{-zK}S(z), \quad \Psi(z) := (S'(0))^{1/2}L(z),$$
$$q(z) := -4\Psi(z)K\Psi^*(\bar{z}) =: \Upsilon(S), \tag{8}$$

with $\Psi^*(z) := (\Psi(z))^*$.

The main result of this paper is the following theorem.

Theorem 1. *Let $S \in \mathscr{S}^+(K)$ and $q = \Upsilon(S)$. Then*

(1) the operator T_q is reflectionless, i.e., $q \in \mathcal{Q}$;

(2) the function q is analytic in Π_K and

$$\|q(z)\| \leq \frac{2\|K\|^2}{\cos^2(y\|K\|)}, \quad z \in \Pi_K, \quad y = \operatorname{Im} z.$$

The structure of the paper is as follows. In Section 2, we investigate the function $S \in \mathscr{S}^+(K)$ and discuss its main properties. In Section 3, we establish some properties of the functions Ψ and q, and prove part (2) of Theorem 1. Finally, in Section 4, we complete the proof of the main result of the paper. Some auxiliary results that are used in Sections 2 and 3 are collected in Appendix.

2. The main properties of the function S**.** We start this section by establishing properties of the function S.

Proof of Proposition 1. The fact that the function $S(z)$ is an analytic continuation of (2) whenever the operator in the parenthesis is invertible (i.e., whenever $z \in \Omega(S)$) is straightforward, as is the fact that the set $\Omega(S)$ is open and symmetric with respect to the real line.

We will next justify the second part. Let $S \in \mathscr{S}(K)$ and $0 < y < \frac{\pi}{2\|K\|}$. Let us consider the operators

$$\Gamma := S^{-1}(0) - I, \quad W := \Gamma + e^{2iyK}.$$

Since the operator Γ is self-adjoint and positive, $yK \in \mathcal{B}_+(H)$ and $\|yK\| < \pi/2$, then in view of Lemma 8, the operator $W = S^{-1}(0) - I + e^{2iyK}$ has an inverse operator W^{-1} in the algebra $\mathcal{B}(H)$ and $\|W^{-1}\| \leq [\cos(y\|K\|)]^{-1}$. This implies (see definition of $\Omega(S)$) that $iy \in \Omega(S)$ and $\|S(iy)\| \leq [\cos(y\|K\|)]^{-1}$. Note that for an arbitrary $x \in \mathbb{R}$ the shift $S_{\{x\}}$ belongs to $\mathscr{S}(K)$, and hence $\|S(x+iy)\| = \|S_{\{x\}}(iy)\| \leq [\cos(y\|K\|)]^{-1}$. Since the set $\Omega(S)$ is symmetrical about the real axis and $S(\bar{z}) = (S(z))^*$, we obtain the second part of the proposition. □

A function $S \in \mathscr{S}^+(K)$ is called *regular* if the operators $S(0)$ and $I - S(0)$ belong to $\mathcal{B}_{\mathrm{inv}}(H)$. Denote by $\mathscr{S}^+_{\mathrm{reg}}(K)$ the set of all regular functions $S \in \mathscr{S}^+(K)$.

Lemma 1. *Let $S \in \mathscr{S}^+(K)$, $B := S(0)$ and $\varepsilon \in (0, 1/2)$. Put by definition*

$$S_\varepsilon(x) = e^{xK}(B_\varepsilon^{-1} - I + e^{2xK})^{-1}e^{xK}, \quad x \in \mathbb{R}, \tag{9}$$

where $B_\varepsilon := \varepsilon I + (1-2\varepsilon)B$. Then $S_\varepsilon \in \mathscr{S}^+_{\mathrm{reg}}(K)$ for all $\varepsilon \in (0, 1/2)$.

Proof. It is obvious that the function S_ε is a solution of the equation (1), moreover,

$$S_\varepsilon(0) = B_\varepsilon, \quad S'_\varepsilon(0) = KB_\varepsilon + B_\varepsilon K - 2B_\varepsilon K B_\varepsilon.$$

Since $0 < B < I$, we get that $\varepsilon I \leq B_\varepsilon \leq (1-\varepsilon)I$. Thus the operators $S_\varepsilon(0)$ and $I - S_\varepsilon(0)$ belong to $\mathcal{B}_{\mathrm{inv}}(H)$. Using straightforward calculations, we obtain

$$KB_\varepsilon + B_\varepsilon K - 2B_\varepsilon K B_\varepsilon = (1-2\varepsilon)^2(KB + BK - 2BKB) + 2(\varepsilon - \varepsilon^2)K \geq 0.$$

Here we took into account that $KB + BK - 2BKB = S'(0) \geq 0$. Therefore, $S'_\varepsilon(0) \geq 0$, which also means that $S_\varepsilon \in \mathscr{S}^+_{\mathrm{reg}}(K)$ for all $\varepsilon \in (0, 1/2)$. □

Proposition 2. *Let $S \in \mathscr{S}^+(K)$ and S_ε be defined by the formula* (9). *Then*

$$\|S(z) - S_\varepsilon(z)\| = o(1), \quad \varepsilon \to +0,$$

uniformly on compact sets in Π_K.

Proof. Let us put $S_0 := S$. Obviously, it suffices to prove the existence of a continuous function $C\colon \Pi_K \to \mathbb{R}_+$, for which

$$\|S_0(z) - S_\varepsilon(z)\| \le C(z)\varepsilon, \quad z \in \Pi_K, \quad \varepsilon \in [0, 1/2). \tag{10}$$

Fix an arbitrary $z \in \Pi_K$ and let $c := [\cos(y\|K\|)]^{-1} + e^{2\|zK\|}$, $y = \operatorname{Im} z$. Let us consider the operators $N_\varepsilon(z) := (B_\varepsilon^{-1} - I + e^{2zK})^{-1}$, $\varepsilon \in [0, 1/2)$. By definitions $S_\varepsilon(z) = e^{zK}N_\varepsilon(z)e^{zK}$, $N_\varepsilon(z) = e^{-zK}S_\varepsilon(z)e^{-zK}$. Thus

$$\|S_0(z) - S_\varepsilon(z)\| \le c\|N_0(z) - N_\varepsilon(z)\|. \tag{11}$$

According to Proposition 1, $\|S_\varepsilon(z)\| \le c$, and hence

$$\|N_\varepsilon(z)\| \le c\|S_\varepsilon(z)\| \le c^2, \quad \varepsilon \in [0, 1/2). \tag{12}$$

It follows from the definitions of the operators N_ε that

$$N_\varepsilon(z)B_\varepsilon^{-1} = I + N_\varepsilon(z)(I - e^{2zK}), \quad B^{-1}N_0(z) = I + (I - e^{2zK})N_0(z).$$

Therefore, taking into account (12), we obtain the estimates

$$\|N_\varepsilon(z)B_\varepsilon^{-1}\| \le 1 + c\|N_\varepsilon(z)\| \le 2c^3, \quad \|B^{-1}N_0(z)\| \le 1 + c\|N_0(z)\| \le 2c^3. \tag{13}$$

It is easy to check that

$$N_0(z) - N_\varepsilon(z) = N_\varepsilon(z)(B_\varepsilon^{-1} - B^{-1})N_0(z), \quad B_\varepsilon^{-1} - B^{-1} = \varepsilon B_\varepsilon^{-1}(2B - I)B^{-1}.$$

Thus $N_0(z) - N_\varepsilon(z) = \varepsilon N_\varepsilon(z)B_\varepsilon^{-1}(2B - I)B^{-1}N_0(z)$. Using the estimates (13) and the inequality $\|2B - I\| \le 1$, we get

$$\|N_0(z) - N_\varepsilon(z)\| = \varepsilon\|N_\varepsilon(z)B_\varepsilon^{-1}\| \cdot \|2B - I\| \cdot \|B^{-1}N_0(z)\| \le 4c^6.$$

Taking into account (11), we have

$$\|S_0(z) - S_\varepsilon(z)\| \le c\|N_0(z) - N_\varepsilon(z)\| \le 4c^7.$$

Therefore, the inequality (10) holds if the function C is defined by the formula

$$C(z) := 4([\cos(y\|K\|)]^{-1} + e^{2\|zK\|})^7.$$

□

Lemma 2. *Let $S \in \mathscr{S}(K)$. Then for all $x \in \mathbb{R}$*

$$S(0)L(x) = e^{-xK}S(x). \tag{14}$$

Proof. Let $S \in \mathscr{S}(K)$ and (see (2)) $X(x) := e^{-xK}S(x) = (S^{-1}(0) - I + e^{2xK})^{-1}e^{xK}$, $x \in \mathbb{R}$. Then $S(0)(S^{-1}(0) - I + e^{2xK})X(x) = S(0)e^{xK}$, and hence

$$S(0)(e^{2xK} - I)X(x) = S(0)e^{xK} - X(x).$$

Using this equality, we obtain $S(0)L(x) = S(0)[e^{xK} - (e^{xK} - e^{-xK})S(x)] = S(0)e^{xK} - -S(0)(e^{2xK} - I)X(x) = X(x) = e^{-xK}S(x)$. □

Lemma 3. *Let $S \in \mathscr{S}^+(K)$. Then for an arbitrary $\varepsilon > 0$*

$$\|(S'(0))^{1/2}(K + \varepsilon I)^{-1}(S'(0))^{1/2}\| \le 1/2.$$

Proof. Let $\varepsilon > 0$ and $A = \sqrt{2}(K + \varepsilon I)^{-1/2}$. In view of (1), we have

$$S'(0) = S(0)K + KS(0) - 2S(0)KS(0). \tag{15}$$

Multiplying (15) on the right and left by the operator A, we get

$$AS'(0)A = AS(0)KA + AKS(0)A - 2AS(0)KS(0)A =$$
$$= I - (I - AS(0)KA)(I - AKS(0)A) - AS(0)(2K - K^2A^2)S(0)A.$$

Since $2K - K^2A^2 = 2K[I - K(K + \varepsilon I)^{-1}] \ge 0$, $(I - AS(0)KA)(I - AKS(0)A) \ge 0$, we deduce that $AS'(0)A \le I$. This means that $2(S'(0))^{1/2}(K + \varepsilon I)^{-1}(S'(0))^{1/2} \le I$, $\varepsilon > 0$. □

Lemma 4. *Let $S \in \mathscr{S}^+(K)$. Then $S'(x) \ge 0$ for all $x \in \mathbb{R}$, moreover,*

$$S'(x) = L^*(x)S'(0)L(x) = \Psi^*(x)\Psi(x), \quad x \in \mathbb{R}. \tag{16}$$

Proof. Multiplying the equality (15) on the left by $L^*(x)$ and on the right by $L(x)$, we get (see (14))

$$L^*(x)S'(0)L(x) = L^*(x)KS(0)L(x) + L^*(x)S(0)KL(x) - 2L^*(x)S(0)KS(0)L(x) =$$
$$= L^*(x)Ke^{-xK}S(x) + S(x)e^{-xK}KL(x) - 2S(x)e^{-2xK}KS(x).$$

In view of (8), we have $e^{-xK}L(x) = I - S(x) + e^{-2xK}S(x)$, $L^*(x)e^{-xK} = I - S(x) + S(x)e^{-2xK}$. Thus the equalities

$$L^*(x)Ke^{-xK}S(x) = KS(x) - S(x)KS(x) + S(x)e^{-2xK}KS(x),$$
$$S(x)e^{-xK}KL(x) = S(x)K - S(x)KS(x) + S(x)e^{-2xK}KS(x)$$

hold. Therefore, taking into account (1), we obtain

$$L^*(x)S'(0)L(x) = S(x)K + KS(x) - 2S(x)KS(x) = S'(x).$$

Since $S'(0) \ge 0$, we have $S'(x) = L^*(x)S'(0)L(x) \ge 0$, $x \in \mathbb{R}$. In view of the definitions (see (8)), $\Psi^*(x)\Psi(x) = L^*(x)S'(0)L(x) = S'(x)$. □

3. The main properties of the functions Ψ and q. Let $S \in \mathscr{S}^+(K)$ and S_ε ($\varepsilon \in (0, 1/2)$) be defined by the formula (9). Let us agree to denote by Ψ_ε and q_ε the functions Ψ and q, respectively, which are associated with the function S_ε.

Proposition 2 implies the following corollary.

Corollary 1. *Let $S \in \mathscr{S}^+(K)$. Then*

$$\|\Psi(z) - \Psi_\varepsilon(z)\| = o(1), \quad \|q(z) - q_\varepsilon(z)\| = o(1), \quad \varepsilon \to +0,$$

uniformly on compact sets in Π_K.

Let $S \in \mathscr{S}^+_{\mathrm{reg}}(K)$. Then the operators Γ and $R := (S'(0))^{1/2}S^{-1}(0)$ belong to $\mathcal{B}(H)$, moreover, $\Gamma > 0$. It follows from (2) that

$$S(z) = (I + e^{-zK}\Gamma e^{-zK})^{-1}, \quad z \in \Pi_K. \tag{17}$$

By the formulas (14), we have that $L(z) = S^{-1}(0)e^{-zK}S(z)$, and hence (see (8)),

$$\Psi(z) = Re^{-zK}S(z), \quad z \in \Pi_K. \tag{18}$$

Multiplying the equality (15) on the right and left by the operator $S^{-1}(0)$, we get

$$K\Gamma + \Gamma K = R^*R. \tag{19}$$

Lemma 5. *Let $S \in \mathscr{S}^+(K)$ and $\xi \in \mathbb{R}$. Denote by $\Psi_{\{\xi\}}, q_{\{\xi\}}, R_{\{\xi\}}, \Gamma_{\{\xi\}}$ the functions and the operators, which are associated with the function $S_{\{\xi\}}$. Then*

$$\|\Psi_{\{\xi\}}(z)\| = \|\Psi(z+\xi)\|, \quad \|q_{\{\xi\}}(z)\| = \|q(z+\xi)\|, \quad z \in \Pi_K.$$

Proof. In view of Corollary 1, we can assume that $S \in \mathscr{S}^+_{\mathrm{reg}}(K)$. Taking into account (17), we have $\Gamma_{\{\xi\}} = S^{-1}_{\{\xi\}}(0) - I = e^{-\xi K}\Gamma e^{-\xi K}$. Thus (see (19))

$$R^*_{\{\xi\}}R_{\{\xi\}} = K\Gamma_{\{\xi\}} + \Gamma_{\{\xi\}}K = e^{-\xi K}(K\Gamma + \Gamma K)e^{-\xi K} = e^{-\xi K}R^*Re^{-\xi K}.$$

Using the polar decomposition, it can be easily shown that $R_{\{\xi\}} = WRe^{-\xi K}$, where the operator W is a partial isometry of the subspaces $\overline{\operatorname{ran} R}$ and $\overline{\operatorname{ran} R_{\{\xi\}}}$. Taking into account (18), we obtain

$$\Psi_{\{\xi\}}(z) := R_{\{\xi\}}e^{-zK}S_{\{\xi\}}(z) = WRe^{-(z+\xi)K}S(z+\xi) = W\Psi(z+\xi), \quad z \in \Pi_K.$$

thus (see (8)) $q_{\{\xi\}}(z) = -4\Psi_{\{\xi\}}(z)K\Psi^*_{\{\xi\}}(\overline{z}) = Wq(z+\xi)W^*$, $z \in \Pi_K$. It follows from the above that $\|\Psi_{\{\xi\}}(z)\| = \|\Psi(z+\xi)\|$, $\|q_{\{\xi\}}(z)\| = \|q(z+\xi)\|$, $z \in \Pi_K$. □

Proposition 3. *Let $S \in \mathscr{S}^+(K)$. Then for the functions Ψ and q the inequalities*

$$\|\Psi(z)\| \le \frac{\pi\|K\|^{1/2}}{2\cos(y\|K\|)}, \quad \|q(z)\| \le \frac{2\|K\|^2}{\cos^2(y\|K\|)}, \quad y = \operatorname{Im} z, \tag{20}$$

hold in the strip Π_K.

Proof. In view of Corollary 1, we can assume that $S \in \mathscr{S}^+_{\mathrm{reg}}(K)$. Lemma 5 implies that it is sufficient to prove the estimates (20) for $z \in \Pi_K$ that lie on the imaginary axis. Denote by Ψ° and q° the functions, which are associated with the function S°. Since $S^\circ \in \mathscr{S}^+_{\mathrm{reg}}(K)$ and $\Psi^\circ(z) = \Psi(-z)$ and $q^\circ(z) = q(-z)$, it is sufficient to prove the estimates for $z \in i\mathbb{R}_+$.

Let $y \in \mathbb{R}_+$ and $\|yK\| < \pi/2$. Then (see (3), (18) and (8))

$$\Psi(iy) = R(e^{2iyK} + \Gamma)^{-1}e^{iyK}, \quad q(iy) = -4\Psi(iy)K\Psi^*(-iy).$$

Therefore, it is sufficient to prove the estimates

$$\|\Psi(iy)\| \le \frac{\pi\|K\|^{1/2}}{2\cos(y\|K\|)}, \quad \|\Psi(iy)K^{1/2}\| \le \frac{\|K\|}{\sqrt{2}\cos(y\|K\|)}. \tag{21}$$

Put $\widetilde{K} := yK$, $\widetilde{R} := \sqrt{y}R$, $\widetilde{F} := \widetilde{R}(e^{2i\widetilde{K}} + \Gamma)^{-1}$. Since $K\Gamma + \Gamma K = R^*R$, then $\widetilde{K}\Gamma + \Gamma\widetilde{K} = \widetilde{R}^*\widetilde{R}$. It is easy to check that the inequalities

$$\|\Psi(iy)\|^2 \le \|\Psi(iy)\Psi^*(iy)\| = y^{-1}\|\widetilde{F}\widetilde{F}^*\|, \quad \|\Psi(iy)K^{1/2}\|^2 \le \|\Psi(iy)K\Psi^*(iy)\| = y^{-2}\|\widetilde{F}\widetilde{K}\widetilde{F}^*\|$$

hold, and for the operators $\widetilde{K}, \widetilde{R}, \widetilde{F}$ and $\widetilde{\Gamma} := \Gamma$ the conditions of Lemma 9 are satisfied. Thus (see Lemma 9)

$$\|\widetilde{F}\widetilde{F}^*\| \leq \frac{\pi^2\|\widetilde{K}\|}{4\cos^2(\|\widetilde{K}\|)} = \frac{\pi^2 y\|K\|}{4\cos^2(\|yK\|)}, \quad \|\widetilde{F}\widetilde{K}\widetilde{F}^*\| \leq \frac{\|\widetilde{K}\|^2}{2\cos^2(\|\widetilde{K}\|)} = \frac{y^2\|K\|^2}{2\cos^2(\|yK\|)}.$$

Taking into account these estimates, we get (21). □

Lemma 6. *Let $S \in \mathscr{S}(K)$, and Ψ and q be defined by the formula* (8)*. Then*

$$-\Psi''(x) + q(x)\Psi(x) = -\Psi(x)K^2, \quad x \in \mathbb{R}. \tag{22}$$

Proof. Let $X(x) := e^{-xK}S(x)$, $x \in \mathbb{R}$. Taking into account (1), we obtain that

$$X'(x) = e^{-xK}S(x)(K - 2KS(x)) = X(x)(K - 2KS(x)),$$

and hence $X''(x) = X(x)[(K - 2KS(x))^2 - 2KS'(x)]$. Using (1), we get that

$$(K - 2KS(x))^2 = K^2 - 2K[KS(x) + S(x)K - 2S(x)KS(x)] = K^2 - 2KS'(x).$$

Thus

$$X''(x) = X(x)K^2 - 4X(x)KS'(x). \tag{23}$$

In view of (14), we have $X(x) = e^{-xK}S(x) = S(0)L(x)$. And hence the equality (23) can be rewritten as $S(0)[L''(x) - L(x)K^2 + 4L(x)KS'(x)] = 0$. Since the operator $S(0)$ has the trivial kernel, we conclude that

$$L''(x) - L(z)K^2 + 4L(x)KS'(x) = 0. \tag{24}$$

Multiplying the equality (24) on the left by the operator $(S'(0))^{1/2}$, we get

$$\Psi''(x) + 4\Psi(x)KS'(x) - \Psi(x)K^2 = 0.$$

Therefore, taking into account (8) and (16), we have that

$$-\Psi''(x) + q(x)\Psi(x) + \Psi(x)K^2 = -\Psi''(x) - 4\Psi(x)K\Psi^*(x)\Psi(x) + \Psi(x)K^2 =$$
$$= -\Psi''(x) - 4\Psi(x)KS'(x) + \Psi(x)K^2 = 0.$$

□

Lemma 7. *Let $S \in \mathscr{S}^+(K)$, and L be defined by the formula* (8)*. Then the function L satisfies the equality*

$$L'(x)\operatorname{sh}(xK) + L(x)K\operatorname{ch}(xK) = L(x)KL^*(x), \quad x \in \mathbb{R}. \tag{25}$$

Proof. Let us introduce the notations $A := \operatorname{sh}(xK)$, $B := \operatorname{ch}(xK)$. Then $A + B = e^{xK}$. It follows from the definitions that $L = A + B - 2AS$, $L' = K(A+B) - 2KBS - 2AS'$. Using the equality (1), we have $L' = K(A+B) - 2KBS - 2A[KS + SK - 2SKS]$. Thus

$$L'A + LKB = K(A+B)A - 2KBSA-$$
$$-2A[KS + SK - 2SKS]A + (A+B)KB - 2ASKB =$$
$$= K(A+B)^2 - 2(A+B)KSA - 2ASK(A+B) + 4ASKSA. \tag{26}$$

On the other hand,

$$LKL^* = (A + B - 2AS)K(A + B - 2SA) =$$
$$= K(A+B)^2 - 2ASK(A+B) - 2(A+B)KSA + 4ASKSA. \tag{27}$$

Since the right-hand sides of (26) and (27) are equal, we get $L'A + LKB = LKL^*$. □

Lemma 7 implies the following corollary.

Corollary 2. *The equality*

$$-4[\Psi'(x)\operatorname{sh}(xK)+\Psi(x)K\operatorname{ch}(xK)]\Psi^*(0)=-4\Psi(x)K\Psi^*(x)=q(x),\quad x\in\mathbb{R},\tag{28}$$

holds.

Proof. Let us multiply the equality (25) on the left by the operator $\Psi(0)$ and on the right by $\Psi^*(0)$. Since $\Psi(0)=(S'(0))^{1/2}$, we get

$$[\Psi'(x)\operatorname{sh}(xK)+\Psi(x)K\operatorname{ch}(xK)]\Psi^*(0)=\Psi(x)K\Psi^*(x)=-\tfrac{1}{4}q(x).$$

□

4. Proof of Theorem 1. Note that part (2) of Theorem 1 was proved in Proposition 3. It remains to prove part (1). The proof is divided into three steps.

1°. Let $S\in\mathscr{S}^+(K)$ and $q=\Upsilon(S)$ (see (8)). We consider the function

$$h(\lambda,x):=e^{i\lambda x}[I-\Psi(x)D(\lambda,x)\Psi^*(0)],\quad x\in\mathbb{R},\quad \lambda\in\mathcal{O}(K),$$

where $D(\lambda,x):=K_\lambda e^{-xK}+K_{-\lambda}e^{xK}$, $K_\lambda:=(K-i\lambda I)^{-1}$, $\mathcal{O}(K):=\{\lambda\in\mathbb{C}\mid \pm i\lambda\notin\sigma(K)\}$. Let us show that

$$-h''(\lambda,x)-q(x)h(\lambda,x)=\lambda^2h(\lambda,x),\quad x\in\mathbb{R},\quad \lambda\in\mathcal{O}(K).\tag{29}$$

Note that

$$D'+i\lambda D=2\operatorname{sh}(xK),\quad D''=K^2D=DK^2,\quad (D'+i\lambda D)'=2K\operatorname{ch}(xK).\tag{30}$$

Using straightforward calculations, we obtain that

$$e^{-i\lambda x}[h''-qh+\lambda^2h]=-2i\lambda[\Psi D\Psi^*(0)]'-[\Psi D\Psi^*(0)]''-q+q\Psi D\Psi^*(0)=$$
$$=-2i\lambda\Psi'D\Psi^*(0)-2i\lambda\Psi D'\Psi^*(0)-(\Psi''-q\Psi)D\Psi^*(0)-2\Psi'D'\Psi^*(0)-\Psi D''\Psi^*(0)-q.$$

Thus, taking into account (30) and (see (22)) $\Psi''-q\Psi=\Psi K^2$, we get that

$$e^{-i\lambda x}[h''-qh+\lambda^2h]=-2\Psi'(D'+i\lambda D)\Psi^*(0)-2\Psi(D'+i\lambda D)'\Psi^*(0)-q.$$

Using the equalities (30) again, we have

$$e^{-i\lambda x}[h''-qh+\lambda^2h]=-4[\Psi'\operatorname{sh}(xK)+\Psi K\operatorname{ch}(xK)]\Psi^*(0)-q.$$

Thus, in view of (28), we get that $e^{-i\lambda x}[h''-qh+\lambda^2h]=0$. Therefore, the equality (29) is proved.

2°. Taking into account that $D(\lambda,0)=2K(K^2+\lambda^2I)^{-1}$, we consider the function

$$M(\lambda):=h(\lambda,0)=I-2\Psi(0)K(K^2+\lambda^2I)^{-1}\Psi^*(0),\quad \lambda\in\mathcal{O}(K).$$

The function $M(\lambda)$ is analytic in $\mathcal{O}(K)$. Denote by $\widetilde{\mathcal{O}}(K)$ the set of all $\lambda\in\mathcal{O}(K)$, for which the operator $M(\lambda)$ is invertible in the algebra $\mathcal{B}(H)$. It is obvious that the set $\widetilde{\mathcal{O}}(K)$ is open in $\mathbb{C}$.

Let us show that the set $\mathbb{C}\setminus\widetilde{\mathcal{O}}(K)$ is a compact subset of the imaginary axis. Since

$$\|M(\lambda)-I\|=O(\lambda^{-2}),\quad \lambda\to\infty,$$

the set $\mathbb{C}\setminus\widetilde{\mathcal{O}}(K)$ is compact in $\mathbb{C}$. It suffices to prove that

$$\mathbb{C}\setminus\widetilde{\mathcal{O}}(K)\subset i\mathbb{R}.\tag{31}$$

Let us assume the contrary. Then there exists $\lambda\in\mathbb{C}\setminus i\mathbb{R}$ such that $M(\lambda)$ is not invertible. Since $(M(\lambda))^*=M(\bar\lambda)$, one of the operators $M(\lambda),M(\bar\lambda)$ is unbounded below. Without loss of generality, we may assume that $M(\lambda)$ is unbounded below. Then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in H such that

$$(\forall n\in\mathbb{N})\colon\quad \|h_n\|=1 \quad\text{and}\quad \lim_{n\to\infty}(M(\lambda)h_n\mid h_n)=0.$$

Let E be a resolution of the identity for the operator K. Let us consider nonnegative Borel measures μ_n, $n\in\mathbb{N}$, on $\mathbb{R}_+$, which are defined by the formula

$$d\mu_n(t)=\frac{2}{t}(dE(t)\Psi(0)^*h_n\mid\Psi(0)^*h_n),\quad t\in\mathbb{R}_+.$$

Since $\Psi(0)=(S'(0))^{1/2}$, it follows from Lemm 3 that for an arbitrary $\varepsilon>0$

$$\int_{\mathbb{R}_+}\frac{t\,d\mu_n(t)}{t+\varepsilon}=2(S'(0))^{1/2}(K+\varepsilon I)^{-1}(S'(0))^{1/2}h_n\mid h_n)\le\|h_n\|^2=1.$$

Thus $\int_{\mathbb{R}_+}d\mu_n(t)\le 1$, $\quad n\in\mathbb{N}$. Since the measures μ_n is concentrated on the interval $(0,\|K\|]$, by Helly's theorem, from the sequence $(\mu_n)_{n\in\mathbb{N}}$ one can choose a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, which converges weakly to some nonnegative Borel measure μ that is concentrated on the interval $(0,\|K\|]$ and $\mu(\mathbb{R}_+)\le 1$. The definitions imply that

$$(M(\lambda)h_n\mid h_n)=1-\int_{\mathbb{R}_+}\frac{t^2\,d\mu_n(t)}{t^2+\lambda^2}.$$

Thus

$$1-\int_{\mathbb{R}_+}\frac{t^2\,d\mu(t)}{t^2+\lambda^2}=\lim_{k\to\infty}\Big(1-\int_{\mathbb{R}_+}\frac{t^2\,d\mu_{n_k}(t)}{t^2+\lambda^2}\Big)=\lim_{k\to\infty}(M(\lambda)h_{n_k}\mid h_{n_k})=0.$$

As a result, the point $z=\lambda^2$ is the zero of the function

$$g(z):=1-\int_{\mathbb{R}_+}\frac{t^2\,d\mu(t)}{t^2+z},\quad z\in\mathbb{C}_+.$$

On the other hand, g is a Herglotz function, and hence it does not vanish outside the real axis. Therefore, $\lambda^2\in\mathbb{R}$. Since $\lambda\in\mathbb{C}\setminus i\mathbb{R}$, then $\lambda^2\in\mathbb{R}_+$. Thus

$$g(\lambda^2)=1-\int_{\mathbb{R}_+}\frac{t^2\,d\mu(t)}{t^2+\lambda^2}>1-\mu(\mathbb{R}_+)\ge 0,$$

meaning that $g(\lambda^2)\ne 0$, which is a contradiction. Therefore, the inclusion (31) is proved.

3°. Put by definition,

$$f(\lambda,x):=h(\lambda,x)M^{-1}(\lambda)=e^{i\lambda x}[I-\Psi(x)D(\lambda,x)\Psi^*(0)]M^{-1}(\lambda),\quad \lambda\in\widetilde{\mathcal{O}}(K). \tag{32}$$

Let $f_\pm(z,\cdot)$ be the normalized right and left Weyl–Titchmarsh solutions of the equation $-y''+qy=zy$, $z\in\mathbb{C}\setminus\mathbb{R}$. Let us show that

$$f(\lambda,\cdot)=\begin{cases}f_+(\lambda^2,\cdot), & \lambda\in\mathbb{C}_+\setminus i\mathbb{R};\\ f_-(\lambda^2,\cdot), & \lambda\in\mathbb{C}_-\setminus i\mathbb{R}.\end{cases} \tag{33}$$

Let $\Lambda:=\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>2\|K\|\}$. The formula (32) implies that

$$\|f(\lambda,x)\|\le e^{-2\|K\|x}(1+\|\Psi(x)D(\lambda,x)\Psi^*(0)M^{-1}(\lambda)\|),\quad x\ge 0,\quad \lambda\in\Lambda.$$

It is obvious that the function $\lambda\mapsto M^{-1}(\lambda)$ is bounded in Λ. According to the second estimate in (20), we have $\|\Psi(x)\|\cdot\|\Psi^*(0)\|\le 4\|K\|$, $x\in\mathbb{R}$.

It is easy to see that $\|K_{\pm\lambda}\|\le\|K\|^{-1}$, $\lambda\in\Lambda$. Thus

$$\|D(\lambda,x)\|\le 2\|K\|^{-1}e^{\|xK\|},\quad x\in\mathbb{R},\quad \lambda\in\Lambda.$$

It follows from the above that there exists a constant $C>0$ such that $\|f(\lambda,x)\|\le Ce^{-x\|K\|}$, $x\in\mathbb{R}_+$, $\lambda\in\Lambda$, and hence $\int_{\mathbb{R}_+}\|f(\lambda,x)\|^2\,dx<\infty$, i.e., $f(\lambda,\cdot)$ is the Weyl–Titchmarsh solution of the equation $-y'' + qy = \lambda^2 y$. Uniqueness of the Weyl–Titchmarsh solutions implies that $f(\lambda, x) = f_+(\lambda^2, x)$, $x \in \mathbb{R}$, $\lambda \in \Lambda$. Since at fixed x the right and left parts of the equality are analytic in $\mathbb{C}_+ \setminus i\mathbb{R}$, then

$$f(\lambda, \cdot) = f_+(\lambda^2, \cdot), \quad \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}.$$

Similarly, we prove that

$$f(\lambda, \cdot) = f_-(\lambda^2, \cdot), \quad \lambda \in \mathbb{C}_- \setminus i\mathbb{R}.$$

Put by definition $g(\lambda) := f'(\lambda, 0)$.

Obviously the function g is analytic in $\widetilde{\mathcal{O}}(K)$. The formula (33) implies that

$$g(\lambda) = \begin{cases} m_+(\lambda^2), & \lambda \in \mathbb{C}_+ \setminus i\mathbb{R}; \\ m_-(\lambda^2), & \lambda \in \mathbb{C}_- \setminus i\mathbb{R}, \end{cases}$$

and shows that the potential q is reflectionless. Theorem 1 is proved.

5. Appendix. Some auxiliary results. In Appendix, we will give auxiliary lemmas (see [8]).

Lemma 8. *Let Γ be a self-adjoint and positive operator in a Hilbert space H, $K \in \mathcal{B}_+(H)$ and $\|K\| < \pi/2$. Then $(e^{2iK} + \Gamma)^{-1} \in \mathcal{B}(H)$, moreover, $\|(e^{2iK} + \Gamma)^{-1}\| \le [\cos(\|K\|)]^{-1}$.*

Proof. Let $\delta := \pi/2 - \|K\|$, $W := e^{i\delta}(e^{2iK} + \Gamma)$. Since $\delta I \le 2K + \delta I \le (2\|K\| + \delta)I = (\pi - \delta)I$, we have that $\operatorname{Im} W = \sin(2K + \delta I) + (\sin\delta)\Gamma \ge (\sin\delta)I$, $\operatorname{Im}(-W^*) = \sin(2K + \delta I) + (\sin\delta)\Gamma \ge (\sin\delta)I$. Hence the operators W and W^* are bounded below, moreover,

$$\|Wf\| \ge (\sin\delta)\|f\|, \; f \in H.$$

From the above, we conclude that the operator W is invertible and $\|W^{-1}\| \le 1/(\sin\delta)$. As a result, $\|(e^{2iK} + \Gamma)^{-1}\| = \|W^{-1}\| \le (\sin\delta)^{-1} = [\cos(\|K\|)]^{-1}$. □

Lemma 9. *Let $K, \Gamma \in \mathcal{B}_+(H)$ and $K\Gamma + \Gamma K = R^*R$, where $R \in \mathcal{B}(H)$. If $\|K\| \le \pi/2$, then for the operator $F = R(e^{2iK} + \Gamma)^{-1}$ the inequalities*

$$\|FKF^*\| \le \frac{\|K\|^2}{2\cos^2(\|K\|)}, \quad \|FF^*\| \le \frac{\pi^2\|K\|}{4\cos^2(\|K\|)} \tag{34}$$

hold.

Proof. Let $\lambda > 0$ and $K_\lambda := (K - i\lambda I)^{-1}$, $B := FK_\lambda R^*$. It follows from the conditions of Lemma that

$$(K - i\lambda I)(e^{2iK} + \Gamma) + (e^{-2iK} + \Gamma)(K + i\lambda I) = R^*R + 2Kh(K), \tag{35}$$

where $h(t) := \cos 2t + \lambda \sin 2t/t$, $t \ge 0$. Multiplying the equality (35) on the left by the operator FK_λ and on the right by $(K_\lambda)^*F^*$, we have $B^* + B = BB^* + 2FKg(K)F^*$, where $g(t) := h(t)(t^2 + \lambda^2)^{-1}$, $t \ge 0$. Hence

$$2FKg(K)F^* = I - (I - B)(I - B)^* \le I. \tag{36}$$

Put $\beta := \|K\|$, $\gamma := \frac{\cos\beta}{\beta}$, $c(\lambda) := (\cos 2\beta + \lambda\frac{\sin 2\beta}{\beta})/(\beta^2 + \lambda^2)$.

Since the functions $\sin t/t$ and $\cos t$ decrease on the interval $[0, \pi]$, the function h and g decrease on the interval $[0, \pi/2]$. Therefore, $g(t) \ge c(\lambda)$ for all $t \in [0, \beta]$. Thus, taking into account (36), we get that $2c(\lambda)FKF^* \le 2FKg(K)F^* \le I$.

Since at $\lambda_0 = \beta \operatorname{tg}\beta$, $c(\lambda_0) = \frac{\cos 2\beta + \operatorname{tg}\beta \sin 2\beta}{\beta^2(1 + \operatorname{tg}^2\beta)} = \gamma^2$, we obtain $2\gamma^2 FKF^* \le I$. This inequality implies the first estimate in (34).

Let us prove the second inequality in (34). We consider the function

$$\varphi(t) := g(t)t + \frac{2t^3}{t^2+\lambda^2}, \quad t \in [0, \pi/2].$$

It follows from the definitions of the functions h and g that for all $t \in [0, \pi/2]$

$$\varphi(t) = \frac{t}{t^2+\lambda^2}\Big(\cos 2t + \lambda\frac{\sin 2t}{t} + 2t^2\Big) \geq \frac{t}{t^2+\lambda^2}\left(\cos 2t + 2t^2\right).$$

Note that the function $\cos 2t + 2t^2$ is monotonically increasing and $1 \leq \cos 2t + 2t^2 \leq \pi^2/2 - 1$, $\frac{t^2}{t^2+\lambda^2} \leq 1$, $t \in [0, \pi/2]$. Thus $\frac{t}{t^2+\lambda^2} \leq \varphi(t) \leq g(t)t + 2t$, $t \in [0, \pi/2]$. Therefore, $FK(K^2+\lambda^2 I)^{-1}F^* \leq FKg(K)F^* + 2FKF^*$. Using this inequality and the estimates (36) and $2\gamma^2 FKF^* \leq I$, we get $FK(K^2+\lambda^2 I)^{-1}F^* \leq \left(\frac{1}{2}+\gamma^{-2}\right)I$.

Since $\|K\|^{-1}K \leq I$, then

$$\|K\|^{-1}FK^2(K^2+\lambda^2 I)^{-1}F^* \leq FK(K^2+\lambda^2 I)^{-1}F^* \leq \Big(\frac{1}{2}+\gamma^{-2}\Big)I,$$

and hence $FK^2(K^2+\lambda^2 I)^{-1}F^* \leq \beta(\frac{1}{2}+\frac{\beta^2}{\cos^2\beta})$. By passing to the limit as $\lambda \to +0$, we obtain

$$FF^* \leq \beta\frac{\cos^2\beta + 2\beta^2}{2\cos^2\beta} = \beta\frac{\cos 2\beta + 4\beta^2 + 1}{4\cos^2\beta} \leq \frac{\pi^2\beta}{4\cos^2\beta}.$$

Therefore, the second inequality in (34) is proved. □

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REFERENCES

1. V.A. Marchenko, The Cauchy problem for the KdV equation with nondecreasing initial data, in What is integrability?, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1991, 273–318.
2. I. Hur, M. McBride, C. Remling, *The Marchenko representation of reflectionless Jacobi and Schrödinger operators*, Trans. AMS, **368** (2016), №2, 1251–1270.
3. F. Gesztesy, W. Karwowski, Z. Zhao, *Limits of soliton solutions*, Duke Math. J., **68** (1992), №1, 101–150.
4. S. Kotani, *KdV flow on generalized reflectionless potentials*, Zh. Mat. Fiz. Anal. Geom., **4** (2008), №4, 490–528.
5. R. Hryniv, B. Melnyk, Ya. Mykytyuk, *Inverse scattering for reflectionless Schrödinger operators with integrable potentials and generalized soliton solutions for the KdV equation*, Ann. Henri Poincaré, **22** (2021), 487–527.
6. Ya.V. Mykytyuk, N.S. Sushchyk, *Reflectionless Schrödinger operators and Marchenko parametrization*, Mat. Stud., **61** (2024), №1, 69–73. https://doi.org/10.30970/ms.61.1.79-83
7. F. Gesztesy, R. Weikard, M. Zinchenko, *On spectral theory for Schrödinger operators with operator-valued potentials*, J. Diff. Equat., **255** (2013), №7, 1784–1827.
8. Ya.V. Mykytyuk, N.S. Sushchyk, *The strip of analyticity of reflectionless potentials*, Mat. Stud., **57** (2022), №2, 186–191. https://doi.org/10.30970/ms.57.2.186-191

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