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ON CLOSE-TO-PSEUDOCONVEX DIRICHLET SERIES

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For a Dirichlet series of form $F(s)=\exp\{s\lambda_1\}+\sum_{k=2}^{+\infty} f_k\exp\{s\lambda_k\}$ **(1)** absolutely convergent in the half-plane $\Pi_0=\{s\colon \operatorname{Re} s<0\}$ new sufficient conditions for the close-to-pseudoconvexity are found and the obtained result is applied to studying of solutions linear differential equations of second order with exponential coefficients. In particular, are proved the following statements: 1) Let $\lambda_k=\lambda_{k-1}+\lambda_1$ and $f_k>0$ for all $k\ge 2$. If $1\le \lambda_2 f_2/\lambda_1\le 2$ and $\lambda_k f_k-\lambda_{k+1}f_{k+1}\searrow q\ge 0$ as $k\to+\infty$ then function of form **(1)** is close-to-pseudoconvex in Π_0 (Theorem 3). This theorem complements Alexander's criterion obtained for power series. 2) If either $-h^2\le\gamma\le 0$ or $\gamma=h^2$ then differential equation $(1-e^{hs})^2w''-h(1-e^{2hs})w'+\gamma e^{2hs}=0$ $(h>0,\gamma\in\mathbb{R})$ has a solution $w=F$ of form **(1)** with the exponents $\lambda_k=kh$ and the the abscissa of absolute convergence $\sigma_a=0$ that is close-to-pseudoconvex in Π_0 (Theorem 4).

1. Introduction. An analytic function $f(z)=z+\sum_{k=1}^{+\infty} f_k z^k$ univalent in $\mathbb{D}=\{z\colon |z|<1\}$ is said to be *convex* [1, p.203] if $1+\operatorname{Re}\{zf''(z)/f'(z)\}>0$ for all $z\in\mathbb{D}$. According to W. Kaplan [2] an analytic function f is said to be *close-to-convex* in $\mathbb{D}$ if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}\{f'(z)/\Phi'(z)\}>0$ for all $z\in\mathbb{D}$. Every close-to-convex function is univalent in $\mathbb{D}$. From the results obtained by J. Alexander [3] it follows that if $1\ge 2g_2\ge 3g_3\ge\dots\ge(k-1)g_{k-1}\ge kg_k\ge\dots>0$, then f is close-to-convex in $\mathbb{D}$. Using the Alexander criterion, S. Shah [4] indicated the conditions for real parameters γ_0, γ_1, γ_2, under which a differential equation $z^2w''+zw'+(\gamma_0z^2+\gamma_1z+\gamma_2)w=0$ has an entire transcendental solution f such that the function f and all its derivatives are close-to-convex in $\mathbb{D}$. Many authors (see, for example, [5–8]) continued Shah's research.

A direct generalization of power development of an analytic function is a Dirichlet series with exponents increasing to $+\infty$. By $SD(\Lambda,0)$ we denote a class of Dirichlet series

$$F(s)=\exp\{s\lambda_1\}+\sum_{k=2}^{+\infty} f_k\exp\{s\lambda_k\},\quad s=\sigma+it, \tag{1}$$

with a given sequence $\Lambda=(\lambda_k)$ of positive exponents and the abscissa of absolute convergence $\sigma_a[F]=0$, that is, in particular, every Dirichlet series $F\in SD(\Lambda,0)$ is absolutely convergent in $\Pi_0=\{s\colon \operatorname{Re} s<0\}$. The geometric properties of functions from the class $SD(\Lambda,0)$ were studied in [9] (see also [10, p.135–154]). Every function $F\in SD(\Lambda,0)$ is non-univalent in Π_0. However, if $\sum_{k=2}^{+\infty}\lambda_k|f_k|\le\lambda_1$ then the function $F\in SD(\Lambda,0)$ is conformal at every point $z\in\Pi_0$.

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A function $F \in SD(\Lambda, 0)$, conformal at every point $z \in \Pi_0$, is called [9] *pseudoconvex* if

$$\mathrm{Re}\{F''(s)/F'(s)\} > 0$$

for all $s \in \Pi_0$ and is called *close-to-pseudoconvex* if there exists a pseudoconvex in Π_0 function Ψ such that $\mathrm{Re}\{F'(s)/\Psi'(s)\} > 0$ for all $s \in \Pi_0$. In [9] it is proved that if

$$\lambda_1 \geq \lambda_2 f_2 \geq \dots \geq \lambda_k f_k \geq \lambda_{k+1} f_{k+1} \geq \dots, \tag{2}$$

then a function $F \in SD(\Lambda, 0)$ of form 1 is close-to-pseudoconvex in Π_0. Using this statement, in [9] it is proved also that if $h > 0$, $\gamma_0 < 0$, $\gamma_1 < 0$, $\gamma_2 < 0$ and

$$|\gamma_1| \leq \frac{2\sqrt{|\gamma_2|} + h}{\sqrt{|\gamma_2|} + h} h\sqrt{|\gamma_2|}, \quad |\gamma_0| \leq \Big(\frac{4h(\sqrt{|\gamma_2|} + h)^2}{\sqrt{|\gamma_2|} + 2h} - |\gamma_1|\Big)\frac{|\gamma_1|}{h(2\sqrt{|\gamma_2|} + h)},$$

then the differential equation $\frac{d^2w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0$ has an entire solution of form (1) with the exponents $\lambda_k = \sqrt{|\gamma_2|} + (k-1)h$ $(k \geq 1)$ which is close-to-pseudoconvex in Π_0.

In this note we will find sufficient conditions for the close-to-pseudoconvexity function $F \in SD(\Lambda, 0)$ of form (1) that differ from (2), and we will point out the application to the studying the properties of solutions differential equations.

2. Sufficient conditions for close-to-pseudoconvexity. Let's start with this theorem.

Theorem 1. *Let $\lambda_k = \lambda_{k-1} + \lambda_1$ and $f_k > 0$ for all $k \geq 2$. If $\lambda_k f_k/\lambda_1 \nearrow q \leq 2$ as $k \to +\infty$ then function* (1) *is close-to-pseudoconvex in Π_0.*

Proof. It is easy to prove [9] that the function $\Psi(s) = \ln \frac{1}{1-\exp\{s\lambda_1\}}$ belongs to class SD_0, pseudoconvex and $\Psi'(s) = \frac{\lambda_1 \exp\{s\lambda_1\}}{1-\exp\{s\lambda_1\}}$. Since $f_1 = 1$ and $\lambda_{k+1} = \lambda_k + \lambda_1$, for F and Ψ we have

$$\frac{F'(s)}{\Psi'(s)} = (1 - \exp\{s\lambda_1\})\Big(1 + \sum_{k=2}^{+\infty} \frac{\lambda_k f_k}{\lambda_1} \exp\{s(\lambda_k - \lambda_1)\}\Big) =$$

$$= 1 - \exp\{s\lambda_1\} + \sum_{k=2}^{+\infty} \frac{\lambda_k f_k}{\lambda_1} \exp\{s(\lambda_k - \lambda_1)\} - \sum_{k=2}^{+\infty} \frac{\lambda_k f_k}{\lambda_1} \exp\{s\lambda_k\} =$$

$$= 1 - \frac{\lambda_1 f_1}{\lambda_1} \exp\{s\lambda_1\} + \sum_{k=1}^{+\infty} \frac{\lambda_{k+1} f_{k+1}}{\lambda_1} \exp\{s\lambda_k\} - \sum_{k=2}^{+\infty} \frac{\lambda_k f_k}{\lambda_1} \exp\{s\lambda_k\} =$$

$$= 1 + \frac{1}{\lambda_1} \sum_{k=1}^{+\infty} (\lambda_{k+1} f_{k+1} - \lambda_k f_k) \exp\{s\lambda_k\}.$$

We put $F_m(s) = 1 + \frac{1}{\lambda_1} \sum\limits_{k=1}^{m} (\lambda_{k+1} f_{k+1} - \lambda_k f_k) \exp\{s\lambda_k\}$. Then for $\mathrm{Re}\, s < 0$ we obtain

$$\mathrm{Re}\, F_m(s) \geq 1 - \frac{1}{\lambda_1}\Big|\sum_{k=1}^{m} (\lambda_{k+1} f_{k+1} - \lambda_k f_k) \exp\{s\lambda_k\}\Big| >$$

$$> 1 - \frac{1}{\lambda_1} \sum_{k=1}^{m} (\lambda_{k+1} f_{k+1} - \lambda_k f_k) = 1 - \frac{\lambda_{m+1} f_{m+1} - \lambda_1}{\lambda_1} = 2 - \frac{\lambda_{m+1} f_{m+1}}{\lambda_1} \geq 0.$$

Since $F'(s)/\Psi'(s) = \lim\limits_{m\to+\infty} F_m(s)$. □

The following theorem is true also.

Theorem 2. *Let $\lambda_k = \lambda_{k-1} + \lambda_1$ and $f_k > 0$ for all $k \geq 2$. If $\lambda_2 f_2/\lambda_1 \geq 2$ and $(\lambda_{k+1} f_{k+1} - \lambda_k f_k)/\lambda_1 \nearrow q \leq 2$ as $k \to +\infty$ then function* (1) *is close-to-pseudoconvex in Π_0.*

Proof. At first we remark that the function $\Psi(s)=\frac{\exp\{s\lambda_1\}}{1-\exp\{s\lambda_1\}}$ is pseudoconvex. For F and Ψ now we have

$$\frac{F'(s)}{\Psi'(s)}=(1-\exp\{\lambda_1 s\})^2\Big(1+\sum_{k=2}^{+\infty}\frac{\lambda_k f_k}{\lambda_1}\exp\{s(\lambda_k-\lambda_1)\}\Big)=$$

$$=1+\sum_{k=2}^{+\infty}\frac{\lambda_k f_k}{\lambda_1}\exp\{s(\lambda_k-\lambda_1)\}-2\exp\{\lambda_1 s\}-\sum_{k=2}^{+\infty}2\frac{\lambda_k f_k}{\lambda_1}\exp\{s\lambda_k\}+$$

$$+\exp\{2\lambda_1 s\}+\sum_{k=2}^{+\infty}2\frac{\lambda_k f_k}{\lambda_1}\exp\{s(\lambda_k+\lambda_1)\}=$$

$$=1+\sum_{k=1}^{+\infty}\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}\exp\{s\lambda_k\}-2\exp\{\lambda_1 s\}-\sum_{k=2}^{+\infty}2\frac{\lambda_k f_k}{\lambda_1}\exp\{s\lambda_k\}+$$

$$+\exp\{2\lambda_1 s\}+\sum_{k=3}^{+\infty}2\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\exp\{s\lambda_k\}=$$

$$=1+\Big(\frac{\lambda_2 f_2}{\lambda_1}-2\Big)\exp\{\lambda_1 s\}+\Big(1-2\frac{\lambda_2 f_2}{\lambda_1}+\frac{\lambda_3 f_3}{\lambda_1}\Big)\exp\{\lambda_2 s\}+$$

$$+\sum_{k=3}^{+\infty}\Big(\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}-2\frac{\lambda_k f_k}{\lambda_1}+\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\Big)\exp\{s\lambda_k\}=$$

$$=1+\Big(\frac{\lambda_2 f_2}{\lambda_1}-2\Big)\exp\{\lambda_1 s\}+\sum_{k=2}^{+\infty}\Big(\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}-2\frac{\lambda_k f_k}{\lambda_1}+\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\Big)\exp\{s\lambda_k\}.$$

Putting

$$F_m(s)=1+\Big(\frac{\lambda_2 f_2}{\lambda_1}-2\Big)\exp\{\lambda_1 s\}+\sum_{k=2}^{m}\Big(\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}-2\frac{\lambda_k f_k}{\lambda_1}+\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\Big)\exp\{s\lambda_k\},$$

for $\operatorname{Re} s<0$ we obtain

$$\operatorname{Re} F_m(s)\ge 1-\Big|\Big(\frac{\lambda_2 f_2}{\lambda_1}-2\Big)\exp\{\lambda_1 s\}\Big|-\sum_{k=2}^{m}\Big|\Big(\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}-2\frac{\lambda_k f_k}{\lambda_1}+\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\Big)\exp\{s\lambda_k\}\Big|>$$

$$>1-\Big(\frac{\lambda_2 f_2}{\lambda_1}-2\Big)-\sum_{k=2}^{m}\Big(\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}-\frac{\lambda_k f_k}{\lambda_1}-\Big(\frac{\lambda_k f_k}{\lambda_1}-\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}\Big)\Big)=$$

$$=3-\frac{\lambda_2 f_2}{l_1}-\Big(\frac{\lambda_{m+1} f_{m+1}}{\lambda_1}-\frac{\lambda_m f_m}{\lambda_1}\Big)+\Big(\frac{\lambda_2 f_2}{\lambda_1}-\frac{\lambda_1 f_1}{\lambda_1}\Big)=2-\Big(\frac{\lambda_{m+1} f_{m+1}}{\lambda_1}-\frac{\lambda_m}{\lambda_1}\Big)\ge 2-q\ge 0.$$

□

Finally, the following theorem complements the Alexander criterion and Theorem 2.

Theorem 3. *Let $\lambda_k=\lambda_{k-1}+\lambda_1$ and $f_k>0$ for all $k\ge 2$. If $1\le \lambda_2 f_2/\lambda_1\le 2$ and $\lambda_k f_k-\lambda_{k+1} f_{k+1}\searrow q\ge 0$ as $k\to+\infty$ then function* (1) *is close-to-pseudoconvex in* Π_0.

Proof. Choose Ψ as in the proof of Theorem 2. Then for $F_m(s)$ now we have

$$\operatorname{Re} F_m(s)\ge 1-\Big|\Big(2-\frac{\lambda_2 f_2}{\lambda_1}\Big)\exp\{\lambda_1 s\}\Big|-$$

$$-\sum_{k=2}^{m}\Big|\Big(\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}-2\frac{\lambda_k f_k}{\lambda_1}+\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}\Big)\exp\{s\lambda_k\}\Big|>$$

$$>1-\Big(2-\frac{\lambda_2 f_2}{\lambda_1}\Big)-\sum_{k=2}^{m}\Big(\frac{\lambda_{k-1} f_{k-1}}{\lambda_1}-\frac{\lambda_k f_k}{\lambda_1}-\Big(\frac{\lambda_k f_k}{\lambda_1}-\frac{\lambda_{k+1} f_{k+1}}{\lambda_1}\Big)\Big)=$$

$$= 1 - \Big(2 - \frac{\lambda_2 f_2}{\lambda_1}\Big) \sum_{k=2}^{m} \Big(\frac{\lambda_k f_k}{\lambda_1} - \frac{\lambda_{k+1} f_{k+1}}{\lambda_1} - \Big(\frac{\lambda_{k-1} f_{k-1}}{\lambda_1} - \frac{\lambda_k f_k}{\lambda_1}\Big)\Big) =$$

$$= 1 - \Big(2 - \frac{\lambda_2 f_2}{\lambda_1}\Big) + \frac{\lambda_m f_m}{\lambda_1} - \frac{\lambda_{m+1} f_{m+1}}{\lambda_1} - \Big(\frac{\lambda_1 f_1}{\lambda_1} - \frac{\lambda_2 f_2}{\lambda_1}\Big) \geq 2\frac{\lambda_2 f_2}{\lambda_1} - 2 \geq 0,$$

and, thus, Theorem 3 is proved. □

3. Close-to-pseudoconvexity of a solution of some differential equation. The solution to the equation $w'' + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0$ considered in [9] is an entire Dirichlet series. Here we will consider the equation

$$(1 - e^{hs})^2 w'' - h(1 - e^{2hs})w' + \gamma e^{2hs} = 0 \quad (h > 0, \gamma \in \mathbb{R}), \tag{3}$$

and show that it has solution [1] with the abscissa of absolute convergence $\sigma_a = 0$ and find the conditions, under which this solution is close-to-pseudoconvex in Π_0.

Theorem 4. *If either $-h^2 \leq \gamma \leq 0$ or $\gamma = h^2$ then differential equation (3) has a solution $F \in SD(\Lambda, 0)$ of form 1 with the exponents $\lambda_k = kh$ and the the abscissa of absolute convergence $\sigma_a = 0$ that is close-to-pseudoconvex in Π_0.*

Proof. Function(1) is a solution for differential equation (3) if and only if

$$(1 - 2e^{hs} + e^{2hs}) \sum_{k=1}^{+\infty} \lambda_k^2 f_k \exp\{s\lambda_k\} - h(1 - e^{2hs}) \sum_{k=1}^{+\infty} \lambda_k f_k \exp\{s\lambda_k\} + \gamma e^{2hs} \equiv 0,$$

and if $\lambda_k = kh$ then

$$\sum_{k=1}^{+\infty} h^2 k^2 f_k \exp\{skh\} - 2\sum_{k=1}^{+\infty} h^2 k^2 f_k \exp\{s(k+1)h\} + \sum_{k=1}^{+\infty} h^2 k^2 f_k \exp\{s(k+2)h\} -$$

$$- \sum_{k=1}^{+\infty} h^2 k f_k \exp\{skh\} + \sum_{k=1}^{+\infty} h^2 k f_k \exp\{s(k+2)h\} + \gamma e^{2hs} \equiv 0,$$

that is,

$$\sum_{k=1}^{+\infty} h^2 (k^2 - k) f_k \exp\{skh\} - 2\sum_{k=2}^{+\infty} h^2 (k-1)^2 f_{k-1} \exp\{skh\} +$$

$$+ \sum_{k=3}^{+\infty} h^2((k-2)^2 + k - 2) f_{k-2} \exp\{skh\} + \gamma e^{2hs} \equiv 0.$$

Therefore,

$$(2h^2 f_2 - 2h^2 + \gamma)e^{2hs} +$$

$$+ \sum_{k=3}^{+\infty} (h^2(k^2 - k) f_k + h^2(k-1)^2 f_{k-1} + h^2((k-2)^2 + k - 2) f_{k-2}) \exp\{skh\} \equiv 0,$$

whence it follows that $2h^2 f_2 - 2h^2 + \gamma = 0$ and $h^2(k^2 - k)f_k + h^2(k-1)^2 f_{k-1} + h^2((k-2)^2 + k - 2) f_{k-2} = 0$, that is $f_2 = 1 - \frac{\gamma}{2h^2}$ and

$$k f_k - 2(k-1) f_{k-1} + (k-2) f_{k-2} = 0, \quad k \geq 3. \tag{4}$$

Suppose at first that $-h^2 \leq \gamma \leq 0$. Then $\lambda_2 f_2 / \lambda_1 = 2 f_2 = 2 - \gamma h^{-2} \geq 2$ and in view of (4)

$$(k+1) f_{k+1} - k f_k = k f_k - (k-1) f_{k-1} = 2 f_2 - f_1 = 1 - \gamma h^{-2} \leq 2, \quad k \geq 2, \tag{5}$$

i.e. the conditions of Theorem 2 hold with $q = 1 - \gamma h^{-2}$ and the close-to-pseudoconvexity of (1) is proved.

If $\gamma = h^2$ then similarly $\lambda_2 f_2/\lambda_1 = 2f_2 = 1$ and

$$kf_k - (k+1)f_{k+1} = f_1 - 2f_2 = \gamma h^{-2} - 1 = 0, \quad k \geq 2, \tag{6}$$

i.e. the conditions of Theorem 3 hold with $q = 0$ and the close-to-pseudoconvexity of (1) is proved again.

Now we prove that $\sigma_a = 0$. From (5) it follows that $kf_k = (k-1)f_{k-1} + 2f_2 - f_1 = \cdots = (k-1)(2f_2 - f_1)$. Therefore, according to Valiron's formula [11]

$$\sigma_a = \varliminf_{k\to+\infty} \frac{1}{\lambda_k} \ln \frac{1}{|f_k|} = \varliminf_{k\to+\infty} \frac{1}{kh} \ln \frac{1}{2f_2 - f_1} = 0.$$

Also (6) implies $\frac{1}{f_{k+1}} = \frac{k+1}{k}\frac{1}{f_k} = \frac{1}{f_2}\prod_{j=2}^{k}(1+\frac{1}{j})$ and according to Valiron's formula $\sigma_a = 0$. □

REFERENCES

1. G.M. Golusin, Geometrical theory of functions of complex variables, M.: Nauka, 1966. (in Russian); Engl. transl.: AMS: Translations of Mathematical monograph, V.26, 1969.
2. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J., **1** (1952), №2, 169–185.
3. J.M. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Annals Math., (1915), 12–22.
4. S.M. Shah, *Univalence of a function f and its successive derivatives when f satisfies a differential equation*, II, J. Math. Anal. Appl., **142** (1989), 422–430.
5. Z.M. Sheremeta, *On entire solutions of a differential equation*, Mat. Stud., **14** (2000), №1, 54–58.
6. Ya.S. Mahola, M.M. Sheremeta, *Properties of entire solutions of a linear differential equation of $n-th$ order with polynomial coefficients of $n-th$ degree*, Mat. Stud., **30** (2008), №2, 153–162.
7. K.I. Dosyn, M.M. Sheremeta, *On the existence of meromorphically starlike and meromorphically convex solutions of Shah's differential equation*, Mat. Stud., **42** (2014), №1, 44–53.
8. O.M. Mulyava, Yu.S. Trukhan, *On meromorphically starlike functions of the order α and the type β, which satisfy Shah's differential equations*, Carpatian Math. Publ., **9** (2017), №2, 154–162. doi:10.15330/cmp.9.2.154-162.
9. O.M. Holovata, O.M. Mulyava, M.M. Sheremeta, *Pseudostarlike, pseudoconvex and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients*, Math. methods and phys-mech. fields, **61** (2018), №1, 57–70. (in Ukrainian)
10. M.M. Sheremeta, Geometric properties of analytic solutions of differential equations, Lviv: Publisher I.E. Chyzhykov, 2019.
11. S. Mandelbrojt, Dirichlet series: Principles and methods. Springer, Netherlands, 1972.

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