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A UNIQUENESS THEOREM FOR MEROMORPHIC FUNCTIONS

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In this paper, we prove the uniqueness theorem for a special class of meromorphic functions on the complex plane $\mathbb{C}$. In particular, we study the class of meromorphic functions f in the domain $\mathbb{C}\setminus K'$, where K' is the finite set of limit points of simple poles of the function f. In this class, we describe non-trivial subclasses in which every function f can be uniquely determined by the residues of the function f at its poles. The result covered in this paper is a part of a problem in a spectral operator theory.

1. Introduction. When studying the reflectionless Jacobi operators [1–3], the authors encountered the following problem. Let B_1 and B_2 be Blaschke products in the unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$, i.e.,

$$B_s(z) := \prod_{j\in\mathbb{N}} b_{\lambda_{j,s}}(z), \quad b_\lambda(z) := \frac{|\lambda|}{\lambda}\cdot\frac{\lambda - z}{1-\bar{\lambda}z} \quad (s\in\{1,2\})$$

with zeros $\lambda_{j,s}$ satisfying the Blaschke condition (see [4–7])

$$\sum_{j\in\mathbb{N}}(1-|\lambda_{j,s}|) < \infty$$

and $\{\lambda_{j,1}\}_{j\in\mathbb{N}} \cap \{\lambda_{j,2}\}_{j\in\mathbb{N}} = \varnothing$. We assume that these zeros are simple and lie in the set $(-1,1)\setminus\{0\}$. Clearly, the function

$$f(z) := \frac{B_1(z)}{B_2(z)} \tag{1}$$

is meromorphic in $\mathbb{C}\setminus\{-1;1\}$ and has only simple poles.

Problem 1. *Let f be the above meromorphic function, and*

$$r_f(\lambda) := \operatorname*{res}_{z=\lambda} f(z), \quad \lambda\in\mathbb{R}\setminus\{-1;1\}.$$

Are the Blaschke products B_1 and B_2 uniquely determined by the function r_f and the number $f(0)$?

In this paper, we prove a result that leads to a positive answer to the question formulated above. Note that Problem 1 is a special case of a more general problem. To formulate the main result, let us introduce some definitions.

Let $\boldsymbol{k} = (k_j)_{j\in\mathbb{N}}$ and $\boldsymbol{\lambda} = (\lambda_j)_{j\in\mathbb{N}}$ be sequences in $\mathbb{C}$. The pair $(\boldsymbol{k},\boldsymbol{\lambda})$ is called *regular* if the following conditions hold:

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1) the sequences $\boldsymbol{k}$ and $\boldsymbol{\lambda}$ are bounded and $\sum_{j\in\mathbb{N}}|\lambda_j-k_j|<\infty$;
2) $k_j\neq k_s$ if $j\neq s$;
3) the set K' of the limit points of the set $K=\{k_j\}_{j\in\mathbb{N}}$ is finite;
4) $K\cap K'=\varnothing$ and $\{\lambda_j\}_{j\in\mathbb{N}}\cap\overline{K}=\varnothing$.

Each regular pair $(\boldsymbol{k},\boldsymbol{\lambda})$ generates the function

$$f(z):=\prod_{j=1}^{\infty}\frac{z-\lambda_j}{z-k_j}, \tag{2}$$

which is meromorphic in the domain $\mathbb{C}\setminus K'$. In view of the relation $\frac{z-\lambda_j}{z-k_j}-1=\frac{k_j-\lambda_j}{z-k_j}$ and convergence of the series $\sum_{j\in\mathbb{N}}|\lambda_j-k_j|$, the product (2) converges uniformly on compact subsets of $\mathbb{C}$ not intersecting with $K'\cup K$ and tends to 1 as $|z|\to\infty$.

We denote by $\mathcal{F}$ the set of all functions f generated by the regular pairs via (2). Denote by $P(f):=\{k_j\}_{j\in\mathbb{N}}$ and $Z(f):=\{\lambda_j\}_{j\in\mathbb{N}}$ the sets of poles and zeros of the function $f\in\mathcal{F}$. For every finite non-empty set $A\subset\mathbb{C}$, we put

$$\mathcal{F}(A):=\{f\in\mathcal{F}\colon (P(f))'=A\}. \tag{3}$$

Here and hereafter, X' is the set of limit points of $X\subset\mathbb{C}$.

For every function $f\in\mathcal{F}(A)$, the equality

$$r_f(\lambda)=\lim_{z\to\lambda}(z-\lambda)f(z),\quad \lambda\in\mathbb{C}\setminus A \tag{4}$$

holds.

The main problem of this paper is to find non-trivial subclasses in $\mathcal{F}(A)$ in which every function f is uniquely determined by the function r_f. In other words, we study the problem of finding subclasses $\mathcal{B}\subset\mathcal{F}(A)$ for which the mapping $\mathcal{B}\ni f\mapsto r_f$ is injective.

Note that the mapping $\mathcal{F}(A)\ni f\mapsto r_f$ is not injective. Indeed, let us consider for simplicity the case $A=\{0\}$. Take two meromorphic Herglotz functions

$$f_1(z)=1+\sum_{j=1}^{\infty}\frac{a_j}{k_j-z},\quad f_2(z)=f_1(z)-\frac{a_0}{z},$$

where $(k_j)_{j=1}^{\infty}$ is a strictly decreasing sequence of positive numbers converging to 0 and $(a_j)_{j=0}^{\infty}$ is a sequence of positive numbers belonging to $\ell_1(\mathbb{Z}_+)$. With an appropriate enumeration, the zeros $\lambda_{j,s}$ of the functions f_s form strictly decreasing sequences of positive numbers converging to zero, moreover, $\lambda_{j+1,1}<\lambda_{j+1,2}<k_j<\lambda_{j,1}<\lambda_{j,2}$, $j\in\mathbb{N}$. It is known (see [8], [9]) that the functions f_s can be represented as

$$f_s(z)=\prod_{j=1}^{\infty}\frac{z-\lambda_{j,s}}{z-k_j},\quad z\in\mathbb{C}\setminus\{0\}.$$

Obviously, $f_1,f_2\in\mathcal{F}(A)$, in particular, $r_{f_1}=r_{f_2}$ and $f_1\neq f_2$.

We denote by $\mathcal{F}_1(A)$ the set of all functions $f\in\mathcal{F}(A)$ for which the sequence $\boldsymbol{k}=(k_j)_{j\in\mathbb{N}}$ from the regular pair generating f possesses the property

$$\sum_{j\in\mathbb{N}}d(k_j,A)<\infty,\quad \text{where}\quad d(x,A):=\min_{a\in A}|x-a|. \tag{5}$$

Let A be a finite non-empty set and $\varphi\colon A\to[0,\pi]$. Denote by $\mathcal{F}_1(A,\varphi)$ the set of all $f\in\mathcal{F}_1(A)$ which possess the following property (A,φ):

$$(\forall a\in A)\colon\qquad \varlimsup_{\mathbb{R}\ni t\to 0}|f(a+te^{i\varphi(a)})|<\infty.$$

The main result of this paper is:

Theorem 1. *Every function $f\in\mathcal{F}_1(A,\varphi)$ is uniquely determined in the class $\mathcal{F}_1(A,\varphi)$ by the function r_f.*

Using Theorem 1, we will prove the following theorem.

Theorem 2. *Let the function f be defined by formula* (1)*. Then the Blaschke products B_1 and B_2 from formula* (1) *are uniquely determined by the function r_f and the number $f(0)$.*

This paper is organized as follows. In Section 2, we introduce some modifications of well-known results of the theory of analytic functions. In Section 3, we present the proof of Theorem 1, and in Section 4, we prove Theorem 2.

2. Some auxiliary results. We denote by Ω the domain
$$\Omega := \Omega(\alpha, \beta) := \{z \in \mathbb{C}\colon |z| > 1,\ \alpha < \arg z < \beta\} \quad (-\pi \le \alpha < \beta < \pi).$$
We also denote by $\mathcal{A}(\Omega)$ the algebra of all functions that are analytic in Ω and continuous in the closure $\overline{\Omega}$. Put by definition
$$M_f(r, \Omega) := \max_{\alpha \le \theta \le \beta} |f(re^{i\theta})|, \quad r \ge 1, \quad f \in \mathcal{A}(\Omega),$$
and call the *order* ρ_f and the *type* σ_f of a function $f \in \mathcal{A}(\Omega)$ in a domain Ω the values
$$\rho_f := \varlimsup_{r\to\infty} \frac{\ln\ln M_f(r,\Omega)}{\ln r}, \quad \sigma_f := \varlimsup_{r\to\infty} \frac{\ln M_f(r,\Omega)}{r^\rho}, \quad \rho = \rho_f.$$
Define $\mathcal{A}_{\rho,\sigma}(\Omega) := \{f \in \mathcal{A}(\Omega)\colon \rho_f \le \rho,\ \sigma_f \le \sigma\}$, $\rho, \sigma \in [0, \infty])$. We also denote by $\mathcal{A}$ the algebra of all entire functions and put
$$\mathcal{A}_{\rho,\sigma} := \mathcal{A} \cap \mathcal{A}_{\rho,\sigma}(\Omega(-\pi, \pi)).$$
Note that $\mathcal{A}_{1,0}$ and $\mathcal{A}_{1,0}(\Omega)$ are also algebras of functions.

Lemma 1. *Let $f \in \mathcal{A}(\Omega)$, $g \in \mathcal{A}_{1,0}$ and $g(0) \ne 0$. If $fg \in \mathcal{A}_{1,0}(\Omega)$, then $f \in \mathcal{A}_{1,0}(\Omega)$.*

Lemma 2. *Let $f \in \mathcal{A}_{1,0}(\Omega)$, where $\Omega = \Omega(\alpha, \beta)$ and $\beta - \alpha \le \pi$. If f is bounded on $\partial\Omega$, then it is bounded in $\overline{\Omega}$.*

The proofs of Lemmas 1 and 2 with slight and obvious modifications repeat the main ideas of proofs of Theorems 12 and 22 in [8, Ch. I]. We only give the proof of Lemma 1.

The proof of Lemma 1. Assume that $f \notin \mathcal{A}_{1,0}(\Omega)$. Then there exist $\eta > 0$ and a sequence $(r_n)_{n\in\mathbb{N}}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and $M_f(r_n, \Omega) > e^{4\eta r_n}$, $n \in \mathbb{N}$.

Since $g \in \mathcal{A}_{1,0}$ and $g(0) \ne 0$, the theorem on lower bounds for the modulus of an entire function (see [8, Theorem 11, Ch. I, Pt. 8]) implies that there exists n_0 such that for an arbitrary integer number $n > n_0$ in the interval $[r_n, 2r_n]$ there exists a number τ_n such that
$$\min_{|z|=\tau_n} |g(z)| > e^{-\eta\tau_n}.$$
Then $M_{fg}(\tau_n, \Omega) > M_f(\tau_n, \Omega)e^{-\eta\tau_n} \ge e^{4\eta r_n}e^{-\eta\tau_n} \ge e^{\eta\tau_n}$. It means that $fg \notin \mathcal{A}_{1,0}$. We get a contradiction and thus $f \in \mathcal{A}_{1,0}(\Omega)$. □

3. The proof of Theorem 1. Assume that there exist functions $f_1, f_2 \in \mathcal{F}_1(A, \varphi)$ such that $r_{f_1} = r_{f_2}$. Then $P(f_1) = P(f_2)$. Thus, the functions f_s can be represented as
$$f_s(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_{j,s}}{z - k_j}, \quad z \in \mathbb{C} \setminus A, \quad s \in \{1, 2\}.$$
We consider the function $f(z) := f_1(z) - f_2(z)$. Obviously, this function can be uniquely extended by continuity to a function that is analytic in $\mathbb{C} \setminus A$ and has zero at infinity. It suffices to prove that the function f is bounded in the neighborhood of each point $a \in A$.

Indeed, in this case the function f can be extended to an entire function that is bounded and has zero at infinity. Thus, in view of Liouville's Theorem, the function f vanishes identically, and, hence, $f_1 = f_2$.

Fix an arbitrary $a \in A$. It follows from the definitions that

$$\overline{\lim_{\mathbb{R}\ni t\to 0}} |f(a + te^{i\varphi(a)})| < \infty.$$

Thus, $\exists\delta > 0$ such that $\sup_{z\in I(\delta)} |f(z)| < \infty$, where $I(\delta) := \{a + te^{i\varphi(a)} \colon t \in (-\delta, \delta) \setminus \{0\}\}$. Let $U = \{z \in \mathbb{C} \colon 0 < |z - a| < \delta\}$ and $K = \{k_j\}_{j\in\mathbb{N}}$. Choosing δ small enough, we can get that the following conditions hold:
1) $A \cap U = \varnothing$; 2) the distance from U to the set $K \setminus U$ is positive.

Consider the following functions in the domain U:

$$g_s(z) := \prod_{k_j\in K\setminus U} \frac{z - \lambda_{j,s}}{z - k_j}, \quad h_s(z) := \prod_{k_j\in U} \Big(1 - \frac{\lambda_{j,s} - a}{z - a}\Big), \quad p(z) := \prod_{k_j\in U} \Big(1 - \frac{k_j - a}{z - a}\Big).$$

It follows from the definitions that

$$\sum_{k_j\in U} |\lambda_{j,s} - a| < \infty, \ \sum_{k_j\in U} |k_j - a| < \infty.$$

Thus, the functions h_s and p are well defined. It is easy to see that the functions g_s, h_s and p are analytic in U and

$$f(z) = \frac{h_1(z)g_1(z) - h_2(z)g_2(z)}{p(z)}, \quad z \in U.$$

Moreover, the functions g_s are bounded on $\overline{U}$. We consider the linear fractional function

$$\theta(\zeta) = a + \frac{e^{i\varphi(a)}\delta}{\zeta}$$

and put by the definition $G_s = g_s \circ \theta$, $H_s = h_s \circ \theta$, $P = p \circ \theta$, $F = f \circ \theta$. Since θ maps the domain $V := \mathbb{C} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| > 1\}$ into U, the functions G_s, H_s, P and F are analytic in V. Obviously, they are also continuous on $\overline{V}$. Therefore, all these functions belong to the algebra $\mathcal{A}(V)$. Moreover, in view of condition $\sup_{z\in I(\delta)} |f(z)| < \infty$, such a supremum $\sup\{|F(t)| \colon t \in \mathbb{R} \setminus (-1, 1)\} < \infty$ is also finite. It implies that

$$\sup_{z\in\partial V^\pm} |F(z)| < \infty, \quad V^+ := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0, |z| > 1\}, \ V^- := \{z \in \mathbb{C} \colon \operatorname{Im} z < 0, |z| > 1\}.$$

Moreover, the functions G_s are bounded on $\overline{V}$ and the equality

$$F(\zeta) = \frac{H_1(\zeta)G_1(\zeta) - H_2(\zeta)G_2(\zeta)}{P(\zeta)}, \ \zeta \in \overline{V},$$

holds. To complete the proof, we only need to show that F is bounded on $\overline{V}$.

Note that the functions H_s and P can be represented in the form

$$H_s(\zeta) := \prod_{j\in\mathbb{N}} (1 - \nu_{j,s}\zeta), \quad P(\zeta) := \prod_{j\in\mathbb{N}} (1 - \mu_j\zeta),$$

where $\sum_{j\in\mathbb{N}} |\nu_{j,s}| < \infty$, $\sum_{j\in\mathbb{N}} |\mu_j| < \infty$. Thus (see [10, Theorem 7, Ch. II, Pt. I, Sec.4]) the functions H_s and P are entire functions of exponential type zero, i.e., they belong to the algebra $\mathcal{A}_{1,0}$, and, hence, belong to $\mathcal{A}_{1,0}(V)$ as well. Since the functions G_s are bounded on $\overline{V}$, they belong to $\mathcal{A}_{1,0}(V)$ too. It follows that the function $H = H_1G_1 - H_2G_2$ belongs to the algebra $\mathcal{A}_{1,0}(V)$ and $P \in \mathcal{A}_{1,0}$. Since $FP = H \in \mathcal{A}_{1,0}(V)$ and $F \in \mathcal{A}(V)$, then in view of Lemma 1, $F \in \mathcal{A}_{1,0}(V)$. In particular, $F \in \mathcal{A}_{1,0}(V^{\pm})$. Thus, taking into account the condition $\sup\limits_{z\in\partial V^{\pm}} |F(z)| < \infty$ and Lemma 2, we get that F is bounded in $\overline{V^{+}}$ and $\overline{V^{-}}$, and hence, F is bounded in $\overline{V}$. Therefore, Theorem 1 is proved.

4. The proof on Theorem 2. Let $(\boldsymbol{k}, \boldsymbol{\lambda})$ be a regular pair, and the sequences $\boldsymbol{\lambda} = (\lambda_j)_{j\in\mathbb{N}}$ and $\boldsymbol{k} = (k_j)_{j\in\mathbb{N}}$ are real-valued. We say that the sequence $\boldsymbol{\lambda}$ is *subordinated* to the sequence $\boldsymbol{k}$ (denoted as $\boldsymbol{\lambda} \prec \boldsymbol{k}$), if there exists a sequence $(c_j)_{j\in\mathbb{N}} \in \ell_1(\mathbb{N})$ of positive numbers such that $d(\lambda_j, A) \le (1 + c_j)d(k_j, A)$, $j \in \mathbb{N}$.

Lemma 3. *Let $f \in \mathcal{F}_1(A)$ be generated by a regular pair $(\boldsymbol{k}, \boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \prec \boldsymbol{k}$. Then the function f satisfies the condition (A, φ) with $\varphi \equiv \pi/2$, i.e., $f \in \mathcal{F}_1(A, \varphi)$.*

Proof. Let the conditions of the lemma be satisfied. We show that it suffices to prove the lemma for the case, when A is a singleton. Indeed, assume that in this case the lemma has been proved and $A = \{a_s\}_{s=1}^{p}$. Clearly, $\mathbb{N}$ can be represented as the union of the disjoint sets $N_s = \{n_{s,j}\}_{j\in\mathbb{N}},\ s \in \{1, \dots, p\}$, such that for every s:

1) the sequence $(n_{s,j})_{j\in\mathbb{N}}$ is strictly increasing;
2) the sequences $\boldsymbol{k}_s := (k_{n_{s,j}})_{j\in\mathbb{N}}$ and $\boldsymbol{\lambda}_s := (\lambda_{n_{s,j}})_{j\in\mathbb{N}}$ converge to a_s;
3) the pair $(\boldsymbol{k}_s, \boldsymbol{\lambda}_s)$ is regular and $\boldsymbol{\lambda}_s \prec \boldsymbol{k}_s$.

Denote by f_s the function from $\mathcal{F}_1(\{a_s\})$ generated by the pair $(\boldsymbol{k}_s, \boldsymbol{\lambda}_s)$. By conjecture, $\overline{\lim\limits_{\mathbb{R}\ni t\to 0}} |f_s(a_s + it)| < \infty$, $s \in \{1, \dots, p\}$. Clearly, $f = \prod\limits_{s=1}^{p} f_s$ and every function f_s is bounded in the neighborhood of each point a_m with $m \ne s$. Thus, we get that $\overline{\lim\limits_{\mathbb{R}\ni t\to 0}} |f(a_s + it)| < \infty$, $s \in \{1, \dots, p\}$, thus the condition (A, φ) holds for f with $\varphi \equiv \pi/2$.

Let $A = \{a\}$ be a singleton. By definition, the sequences $\boldsymbol{\lambda}$ and $\boldsymbol{k}$ are real-valued. Thus, $a \in \mathbb{R}$. In our case, the condition $\boldsymbol{\lambda} \prec \boldsymbol{k}$ means that there exists a sequence $(c_j)_{j\in\mathbb{N}} \in \ell_1(\mathbb{N})$ of positive numbers such that $\frac{|a-\lambda_j|}{|a-k_j|} \le 1 + c_j,\ j \in \mathbb{N}$. Note that for every $y \in \mathbb{R} \setminus \{0\}$ and every $j \in \mathbb{N}$, if $|a - \lambda_j| \le |a - k_j|$, then

$$\left|\frac{a - \lambda_j + iy}{a - k_j + iy}\right|^2 = \frac{|a - \lambda_j|^2 + y^2}{|a - k_j|^2 + y^2} \le 1 \le (1 + c_j)^2,$$

and if $|a - \lambda_j| > |a - k_j|$, then

$$\left|\frac{a - \lambda_j + iy}{a - k_j + iy}\right|^2 = \frac{|a - \lambda_j|^2 + y^2}{|a - k_j|^2 + y^2} \le \frac{|a - \lambda_j|^2}{|a - k_j|^2} \le (1 + c_j)^2.$$

Then for every $y \in \mathbb{R} \setminus \{0\}$ and every $j \in \mathbb{N}$ the inequality $\left|\dfrac{a - \lambda_j + iy}{a - k_j + iy}\right| \le 1 + c_j$ holds.

Therefore,

$$|f(a + iy)| = \prod_{j\in\mathbb{N}} \left|\frac{a - \lambda_j + iy}{a - k_j + iy}\right| \le \prod_{j\in\mathbb{N}} (1 + c_n) =: c < \infty, \quad y \in \mathbb{R} \setminus \{0\}.$$

Hence, the condition (A, φ) with $\varphi \equiv \pi/2$ holds. □

The proof of Theorem 2. Let $A := \{-1; 1\}$, $B_s(z) := \prod\limits_{j\in\mathbb{N}} b_{\lambda_{j,s}}(z),\ s \in \{1, 2\}$, and the zeros $\lambda_{j,s}$ of the Blaschke products B_s are simple, lie in the set $(-1, 1) \setminus \{0\}$ and $\{\lambda_{j,1}\}_{j\in\mathbb{N}} \cap \{\lambda_{j,2}\}_{j\in\mathbb{N}} = \varnothing$. It is clear that the condition $\sum\limits_{j\in\mathbb{N}} (1 - |\lambda_{j,s}|) < \infty$, $s \in \{1, 2\}$, holds as well.

Let us define the sequences $\boldsymbol{k} = \{k_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_j\}_{j\in\mathbb{N}}$ by the formulas $k_{2j-1} := \lambda_{j,1}^{-1}$, $k_{2j} := \lambda_{j,2}$, $\lambda_j := k_j^{-1}$. Obviously, the pair $(\boldsymbol{k}, \boldsymbol{\lambda})$ is regular. Moreover, since $\left|\dfrac{1-|\lambda_j|}{1-|k_j|}\right| = \left|\dfrac{1-|k_j|^{-1}}{1-|k_j|}\right| = |k_j|^{-1} \le 1 + |1-|k_j|^{-1}|$ and $\sum\limits_{j=1}^{\infty} |1-|k_j|^{-1}| < \infty$, we obtain that $\boldsymbol{\lambda} \prec \boldsymbol{k}$.

Let h is a function from $\mathcal{F}(A)$ generated by the pair $(\boldsymbol{k}, \boldsymbol{\lambda})$. Taking into account Lemma 3, we get $h \in \mathcal{F}_1(A, \varphi)$. Thus, in view of Theorem 1, the function h is uniquely determined by the function r_h. It is easy to see that the function $f := \frac{B_1}{B_2}$ (see (1)) is related to h by the formulas $h = f(0)f,\ r_h = f(0)r_f$. Therefore, the function f is uniquely determined by the function r_f and the number $f(0)$. Since all poles of the function f are simple, the Blaschke products B_1 and B_2 can be uniquely reconstructed by the set $P(f)$. Theorem 2 is proved. □

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REFERENCES

1. I. Hur, M. McBride, C. Remling, *The Marchenko representation of reflectionless Jacobi and Schrödinger operators*, Trans. AMS, **368** (2016), №. 2, 1251–1270.
2. A. Poltoratski, C. Remling, *Reflectionless Herglotz functions and Jacobi matrices*, Comm. Math. Phys., **288** (2009), №. 3, 1007–1021.
3. A. Poltoratski, C. Remling, *Approximation results for reflectionless Jacobi matrices*, Int. Math. Res. Not., **16** (2011), 3575–3617.
4. P. Duren, A. Schuster, Bergman spaces. American Mathematical Society, Providence, RI, 2004. https://doi.org/10.1090/surv/100
5. W. Blaschke, *Eine Erweiterung des Satzes von Vitali uber Folgen analytischer Funktionen*, Berichte, Leipzig, **67** (1915), 194–200.
6. Yu. S. Trukhan, M. M. Sheremeta, *On l-index boundedness of the Blaschke product*, Mat. Stud. **19** (2003), №1, 106–112.
7. Yu. S. Trukhan, M. M. Sheremeta, *On the boundedness of l-index of canonical product of zero genus and of Blaschke product*, Mat. Stud. **29** (2008), №1, 45–51. (in Ukrainian)
8. B. Ja. Levin, Distribution of zeros of entire functions, Revised ed., Transl. Math. Monographs, **5**, American Mathematical Society, Providence, R.I., 1980.
9. R. Hryniv, B. Melnyk, Ya. Mykytyuk, *Inverse scattering for reflectionless Schrödinger operators with integrable potentials and generalized soliton solutions for the KdV equation*, Ann. Henri Poincare, **22** (2021), 487–527. https: //doi.org/10.1007/s00023-020-01000-5
10. R. Young, *An introduction to non-harmonic Fourier series*, 2nd edition, Academic Press, 2001.

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