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THE UNIQUENESS AND VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS WITH DIFFERENT TYPES OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS SHARING A SMALL FUNCTION IM

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This paper delves into the uniqueness of finite-order meromorphic functions $f(z)$ and $g(z)$ over the extended complex plane, particularly when these functions share a small function $a(z)$ under specific conditions. The study reveals new insights with significant applications, such as classifying different complexes within $\mathbb{C}$ based on their uniqueness. The primary goal is to explore the uniqueness of meromorphic functions that share a small function $a(z)$ in the sense of IM (ignoring multiplicities) while constrained by finite order, alongside certain types of differential-difference polynomials. We focus on two non-constant meromorphic functions $f(z)$ and $g(z)$ of finite order, under the assumption that a small function $a(z)$, relative to $f(z)$, plays a crucial role in the analysis. The investigation centers on the uniqueness properties of a specific type of differential-difference polynomial of the form $[f^n P[f] H(z,f)]$, where $P[f]$ is a differential polynomial of $f(z)$ and $H(z,f)$ is a difference polynomial of $f(z)$, both defined in the equations (2) and (3), respectively. Importantly, these polynomials do not vanish identically and do not share common zeros or poles with either $f(z)$ or $g(z)$. The paper establishes conditions on several parameters, including k, n, $\overline{d}(P)$, Ψ, Q, t, and ξ, under which the shared value properties between $f(z)$ and $g(z)$ lead to two possible outcomes: either $f(z)$ is a constant multiple of $g(z)$, or $f(z)$ and $g(z)$ satisfy a specific algebraic difference equation. This result contributes to a deeper understanding of the relationship between shared values and the structural properties of meromorphic functions. As an application, the paper extends several previous results on meromorphic functions, including those by Dyavanal and M. M. Mathai, published in the Ukr. Math. J. (2019). Furthermore, by citing a particular example, we demonstrate that the established results hold true only under specific cases, highlighting the precision of the theorem. Finally, we offer a more compact version of the main theorem as an enhancement, providing a refined perspective on the uniqueness problem in the context of meromorphic functions.

1. Introduction. Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. We are using the notations of Nevanlinna theory of meromorphic functions (see [1–3]) such as, $m(r,f)$ $N(r,f)$, $T(r,f)$, $m(r,a)=m(r,\frac{1}{f-a})$, $N(r,a)=N(r,\frac{1}{f-a})$ etc. We denote by $S(r,f)$ the arbitrary quantity such that $S(r,f)=o(T(r,f))$ as $r\to\infty$ without restriction, if $f(z)$ is of finite order, and otherwise as $r\to\infty$ except possibly of some set of finite linear Lebesgue measure. A meromorphic function $a(z)$ is said to be a small function of f, if $T(r,a)=S(r,f)$.

Two non-constant meromorphic functions f and g share the value $a\in\mathbb{C}\cup\{\infty\}$, if $f^{-1}(a)=g^{-1}(a)$. We say that f and g share the value a CM (counting multiplicities) if in

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addition to the sharing of values if $f(z_0) = a$ with multiplicity p implies $g(z_0) = a$ with multiplicity p. If we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities). When $a = \infty$, the zeros of $f - a$ means the poles of f. Throughout this paper, we need the following definition

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\Big(r, \dfrac{1}{f-a}\Big)}{T(r, f)},$$

where a is a value in the extended complex plane.

Assume that $f(z)$ and $g(z)$ share IM the value 1 and that z_0 is a 1-point of $f(z)$ of order p and a 1-point of $g(z)$ of order q. We denote the counting function of the 1-points of both $f(z)$ and $g(z)$ with $p > q$ by $\overline{N}_L(r, \frac{1}{f-1})$. In the same way, we can define $\overline{N}_L(r, \frac{1}{g-1})$.

Let $f(z)$ be a non-constant meromorphic function. We denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicities atleast k, and by $N_{(k}(r, \frac{1}{f-a})$ the one for which multiplicity is not counted. Similarly, we denote by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicities atmost k, and by $\overline{N}_{(k}(r, \frac{1}{f-a})$ the one for which multiplicity is not counted. Then

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + ... + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

Further, we define the order $\rho(f)$ and the hyperorder $\rho_2(f)$ of a meromorphic function $f(z)$ by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}, \qquad \rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r},$$

respectively. Let m be a non-negative integer and let $a_0(\neq 0), a_1, a_2, ..., a_{m-1}, a_m(\neq 0)$ be complex constants. Define

$$P(w) = a_m w^m + a_{m-1} w^{m-1} + ... + a_1 w + a_0. \tag{1}$$

Any expression of the form

$$P[f] = \sum_{i=1}^{m} \alpha_i(z)(f)^{n_{i0}}(f')^{n_{i1}}(f'')^{n_{i2}} \dots (f^s)^{n_{is}} \tag{2}$$

is called *differential polynomial* in f of degree $\overline{d}(P)$, lower degree $\overline{d}(P)$, where $n_{i0}, n_{i1}, \dots, n_{is}$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$

$$\overline{d}(P) = \max\Bigg\{\sum_{j=0}^{s} n_{ij} : l \le i \le m\Bigg\}, \qquad \underline{d}(P) = \min\Bigg\{\sum_{j=0}^{s} n_{ij} : l \le i \le m\Bigg\}.$$

Further, if $\overline{d}(P) = \overline{d}(P) = l$, then the differential polynomial $P[f]$ is called a homogeneous differential polynomial in f of degree 1. Also we define $Q = \max\limits_{1\le i\le m}\{n_{i0} + n_{i1} + ... + n_{il}\}$.

We define the following difference polynomial:

$$H(z, f) = \sum_{i=1}^{t} \prod_{j=1}^{\xi} b_j(f(z + c_{ij}))^{v_{ij}}, \tag{3}$$

where $c_{ij}(i \in \{1,2,\dots,t\}; j \in \{1,2,\dots,\xi\})$ be distinct finite complex numbers, the degree of item i is

$$\theta_i = \sum_{j=1}^{\xi}(1,2,\dots,t), \quad \Psi = \sum_{i=1}^{t}\theta_i$$

and v_{ij} are a non-negative integer.

In 2006, Halburd and Korhonen ([25]), and Chiang and Feng ([26]) independently gave the difference logarithmic derivative lemma, then Halburd and Korhonen ([27]) established a version of Nevanlinna theory based on difference operators. With this development, many researchers studied the zero distribution ans unicity problem of different types of difference polynomials and obtained many results.

For certain types difference polynomial of meromorphic functions and its certain properties, we refer to the papers ([4, 22, 23]). For recent developments in difference polynomials and different aspects of it, we refer to the papers ([5–8, 24]).

In 2010, X. G. Qi, L. Z. Yang and K. Liu ([9]) studied the uniqueness problems of the difference polynomials with entire functions and obtained the following result.

Theorem 1. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and let c be a nonzero complex constant. If $n \geq 6$ and, in addition, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share the value 1 CM, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 such that $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2011, X. M. Li, W. L. Li, H. X. Yi, and Z. T. Wen ([10]) improved the result presented above and obtained the following result.

Theorem 2. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, let $\alpha(z)$ be a meromorphic function such that $\rho(\alpha) < \rho(f)$, let c be a nonzero complex constant, and let $n \geq 7$ be an integer. If $f(z)^n(f(z)-1)f(z+c) - \alpha(z)$ and $g(z)^n(g(z)-1)g(z+c) - \alpha(z)$, share the value 0 CM, then $f(z) \equiv g(z)$.*

Further, K. Liu, X. L. Liu, and T. B. Cao ([11, 12]) established the following results.

Theorem 3. *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions of finite order. Suppose that c is a nonzero constant and $n \in \mathbb{N}$. If $n \geq 26$ and, in addition, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share the value 1 IM, then $f = tg$ or $fg = t$, where $t^{n+1} = 1$.*

Theorem 4. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and let $n \geq 5k + 12$. If $[f(z)^n f(z+c)]^{(k)}$ and $[g(z)^n g(z+c)]^{(k)}$ share the value 1 IM, then either $f(z) = c_1 e^{Cz}$ and $g(z) = c_2 e^{-Cz}$, where c_1, c_2, and C are constants satisfying the equality $(-1)^k (c_1 c_2)^{n+1}[(n+1)C]^{2k} = 1$ or $f = tg$, where $t^{n+1} = 1$.*

Theorem 5. *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) > 1$ and let $n \geq 5k + 4m + 12$. If $[f^n(f^m - 1)f(z+c)]^{(k)}$ and $[g^n(g^m - 1)g(z+c)]^{(k)}$ share the value 1 IM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

In 2019, R. S. Dyavanal and M. M. Mathai ([13]) extended the above results and obtained the following results.

Theorem 6. *Let $f(z)$ and $g(z)$ be two nonconstant finite order meromorphic functions. Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$, which has no common* *zeros or poles with $f(z)$ and $g(z)$. Let $k(>0)$ and $m(>0)$ be two integers satisfying the inequality $n > 4m + 13k + 19$, let $P(w)$ be defined in (1), and let c be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c, the poles of $f(z)$ are not zeros of $f(z+c)$, and the poles of $g(z)$ are not zeros of $g(z+c)$. If $[f^nP(f)f(z+c)]^{(k)}$ and $[g^nP(g)g(z+c)]^{(k)}$ share $a(z)$ IM, $f(z)$ and $g(z)$ share the value 1 IM and then one of the following two cases is realized:*

(i) $f \equiv tg$ *for a constant t such that $t^d = 1$, where $d = GCD(n+m+1, ..., n+m+1-i, ..., n+1)$ and $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, ..., m\}$;*

(ii) $f(z)$ *and* $g(z)$ *satisfy the algebraic difference equation $R(f, g) \equiv 0$, where*

$$R(f(z), g(z)) = (f(z))^n(a_m(f(z))^m + a_{m-1}(f(z))^{m-1} + ... + a_0)f(z+c)- \\ -(g(z))^n(a_m(g(z))^m + a_{m-1}(g(z))^{m-1} + ... + a_0)g(z+c) = 0.$$

Regarding the result of R. S. Dyavanal and M. M. Mathai stated above it is natural to ask the following question which is the motivation of the present paper.

Question 1. What happen if we replace the differential-difference polynomial $[f^nP(f)f(z+c)]^{(k)}$ by $[f^nP[f]H(z,f)]^{(k)}$ in Theorem 6?

In the paper, our main concern is to find the possible answer to the above question. We prove the following theorem which extends and improves Theorem 6.

2. Main Results.

Theorem 7. *Let $f(z)$ and $g(z)$ be two non-constant finite-order meromorphic functions. Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let k, n, $\overline{d}(P)$, Ψ, Q, t, ξ be positive integers satisfying the inequality $n > 4\overline{d}(P) + 4k(Q+2) + \Psi(5k+5t+5\xi+6) + 8Q + 11$. Let $P[f]$ and $H(z,f)$ be defined as in (2) and (3), $c_{ij} (i \in \{1, 2, \ldots, t\}; j \in \{1, 2, \ldots, \xi\})$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c_{ij}, the poles of $f(z)$ (resp., $g(z)$) are not zeros of $H(z,f)$ (resp., $H[z,g]$). If $[f^nP[f]H(z,f)]^{(k)}$ and $[g^nP[g]H(z,g)]^{(k)}$ share $a(z)$ IM and $f(z)$ and $g(z)$ share the value ∞ IM, then one of the following two cases is realized:*

(i) $f \equiv tg$, *where t a constant sunch that $t^d = 1$, $d = GCD(\lambda_0, \lambda_1, \ldots, \lambda_m)$, where $\lambda_i^{`}s$ are defined by* $\lambda_i = \begin{cases} n_{i0} + n_{i1} + \cdots + n_{is} + n + 1, & \text{if } \alpha_i \neq 0; \\ n_{m0} + n_{m1} + \cdots + n_{ms} + n + 1, & \text{if } \alpha_i = 0. \end{cases}$

(ii) $f(z)$ *and* $g(z)$ *satisfy the algebraic difference equation $R(f, g) \equiv 0$, where*

$$R(w_1, w_2) = w_1^nP[w_1]H(z, w_1) - w_2^nP[w_2]H(z, w_2).$$

Example 1. Let $P[z] = z^m - 1$. Suppose t is a non-zero constant such that $t^d = 1$, where $d = GCD(n+m+1, n+1)$. Let $f(z) = e^z$, $g(z) = te^z$, $H(z,f) = f(z+c) - f(z)$, $H(z,g) = g(z+c) - g(z)$ and $\alpha(z)$ be a small function of both f and g, where c is a non-zero constant such that $e^z \neq 1$. Then it can be easliy verified that $[f^zP[f]H(z,f)]^{(k)}$ and $[g^zg[f]H(z,g)]^{(k)}$ share $\alpha(z)$ CM.

Example 2. Let $P[z] = (z^2-1)^q$, where q is a positive integer such that $n+1 = 2q$. Suppose that $f(z) = \cos z$, $g(z) = \sin z$, $H(z,f) = f(z+c) - f(z)$, $H(z,g) = g(z+c) - g(z)$ and $\alpha(z)$ be a small function of both f and g, where $c = 2p\pi$, p is an integer. Then it can be easily verified that $[f^zP[f]H(z,f)]^{(k)}$ and $[g^zg[f]H(z,g)]^{(k)}$ share $\alpha(z)$ CM.

3. Some Lemmas. In this section, we summarize some lemmas, which will be used to prove our main results. Henceforth, let F and G be two non-constant meromorphic functions defined by

$$F=\frac{[f^nP[f]H(z,f)]^{(k)}}{a(z)},\quad G=\frac{[g^nP[g]H(z,g)]^{(k)}}{a(z)}. \tag{4}$$

Henceforth, we shall denote by H and V in the following

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big), \tag{5}$$

$$V=\Big(\frac{F'}{F-1}-\frac{F'}{F}\Big)-\Big(\frac{G'}{G-1}-\frac{G'}{G}\Big). \tag{6}$$

Lemma 1 ([18])**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then, for any $\epsilon>0$*

$$m\Big(r,\frac{f(z+c)}{f(z)}\Big)+m\Big(r,\frac{f(z)}{f(z+c)}\Big)=O(r^{\rho-1+\epsilon}).$$

Lemma 2 ([19])**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then, for any $\epsilon>0$*

$$T(r,f(z+c))=T(r,f)+O(r^{\rho-1+\epsilon}).$$

It is evident that $S(r,f(z+c))=S(r,f)$.

Lemma 3 ([20])**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then*
(i) $N\Big(r,\dfrac{1}{f(z+c)}\Big)\le N\Big(r,\dfrac{1}{f}\Big)+S(r,f)$, (ii) $N(r,f(z+c))\le N(r,f)+S(r,f)$,
(iii) $\overline{N}\Big(r,\dfrac{1}{f(z+c)}\Big)\le \overline{N}\Big(r,\dfrac{1}{f}\Big)+S(r,f)$, (iv) $\overline{N}(r,f(z+c))\le \overline{N}(r,f)+S(r,f)$,
outside an exceptional set of finite logarithmic measure.

Lemma 4 ([21])**.** *Let $f(z)$ be a non-constant meromorphic function and let p and k be two positive integers. Then*

$$N_p\Big(r,\frac{1}{f^{(k)}}\Big)\le T(r,f^{(k)})-T(r,f)+N_{p+k}\Big(r,\frac{1}{f}\Big)+S(r,f),$$
$$N_p\Big(r,\frac{1}{f^{(k)}}\Big)\le k\overline{N}(r,f)+N_{p+k}\Big(r,\frac{1}{f}\Big)+S(r,f).$$

Lemma 5 (Valiron–Mohon'ko Theorem)**.** *Let f be a non-constant meromorphic function and*

$$R(f)=\frac{\sum\limits_{i=0}^{n}a_if^i}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with the constant coefficients a_i and b_j, where $a_n\neq 0$ and $a_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f).$$

Lemma 6 ([22])**.** *Let $f(z)$ and $g(z)$ be a non-constant meromorphic functions. If $f(z)$ and $g(z)$ share the value 1 CM, then one of the following three cases is realized:*
(i) $T(r,f)\le N_2\Big(r,\dfrac{1}{f}\Big)+N_2\Big(r,\dfrac{1}{g}\Big)+N_2(r,f)+N_2(r,g)+S(r,f)+S(r,g)$, the same inequality holds for $T(r,g)$, (ii) $fg=1$, (iii) $f\equiv g$.

Lemma 7 ([21]). *Let $f_1(z)$, $f_2(z)$ be two non-constant meromorphic functions such that $c_1f_1 + c_2f_2 = c_3$, where c_1, c_2, c_3 are three nonzero constants. Then*

$$T(r, f_1) \le \overline{N}(r, f_1) + \overline{N}\Big(r, \frac{1}{f_1}\Big) + \overline{N}\Big(r, \frac{1}{f_2}\Big) + S(r, f_1).$$

Lemma 8 ([23]). *Let F, G, and H be defined as in (4) and (5). If F and G share 1 IM and ∞ IM and, moreover, $H \not\equiv 0$, then $F \not\equiv G$,*

$$T(r,F) \le N_2\Big(r, \frac{1}{F}\Big) + N_2\Big(r, \frac{1}{G}\Big) + 2\overline{N}\Big(r, \frac{1}{F}\Big) + \overline{N}\Big(r, \frac{1}{G}\Big) + 7\overline{N}(r, F) + S(r, F) + S(r, G).$$

and the same inequality holds for $T(r, G)$.

Lemma 9 ([24]). *Let F, G, and V be defined as in (4) and (6). If F and G share ∞ IM and $V \equiv 0$, then $F \equiv G$.*

Lemma 10 ([24]). *If F and G share IM 1, then*

$$\overline{N}_L\Big(r, \frac{1}{F-1}\Big) \le \overline{N}\Big(r, \frac{1}{F}\Big) + \overline{N}(r, F) + S(r, F) + S(r, G).$$

Lemma 11. *Let $f(z)$ is a non-constant meromorphic function with finite order. Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$, $P[f]$ and $H(z, f)$ be defined as in (2) and (3). Then we have*

$$(n + \overline{d}(P) - \Psi)T(r, f) + S(r, f) \le T(r, F) \le (n + \overline{d}(P) - \Psi)T(r, f) + S(r, f).$$

Proof. Let $F = f^nP[f]H(z, f)$, we know that

$$\begin{aligned} T(r, F) = T(r, f^nP[f]H(z, f)) \le T(r, f^nP[f]) + T(r, H(z, f)) + S(r, f) \le \\ \le (n + \overline{d}(P) + \Psi)T(r, f) + S(r, f). \end{aligned} \tag{7}$$

A quick calculation reveals that

$$\begin{aligned} (n + \overline{d}(P) + 1)T(r, f) + S(r, f) = T(r, f^nP[f]f) + S(r, f) \le \\ \le m(r, f^nP[f]f) + N(r, f^nP[f]f) + S(r, f) \le \\ \le m\Big(r, F\frac{f}{H(z, f)}\Big) + N\Big(r, F\frac{f}{H(z, f)}\Big) + S(r, f) \le T(r, F) + (1 + \Psi)T(r, f) + S(r, f). \end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$(n + \overline{d}(P) - \Psi)T(r, f) + S(r, f) \le T(r, F) \le (n + \overline{d}(P) + \Psi)T(r, f) + S(r, f).$$

□

Lemma 12. *If $f(z)$ and $g(z)$ are two non-constant meromorphic functions with finite order. If $c_{ij}(i \in \{1, 2, \ldots, t\}; j \in \{1, 2, \ldots, \xi\})$ is a nonzero complex constant, f and g are not periodic functions of period c_{ij} and k, n, $\overline{d}(P)$, Ψ, Q, t, ξ be positive integers satisfying the inequality $n > k + 2Q + \Psi(k + t + \xi) + 2$. Let $P[f]$ and $H(z, f)$ be defined as in (2) and (3). If $a(z)(\not\equiv 0, \infty)$is a small function with respect to f. If $[f^nP[f]H(z, f)]^{(k)}$ and $[g^nP[f]H(z, f)]^{(k)}$ share $a(z)$ IM, then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.*

Proof. Let $F_1 = f^n P[f]H(z,f)$. By the Second Fundamental Theorem for small functions and for all $\varepsilon > 0$, we get

$$T(r,F^{(k)}) \leq \overline{N}(r,F_1) + \overline{N}\Big(r,\frac{1}{F_1^{(k)}}\Big) + \overline{N}\Big(r,\frac{1}{F_1^{(k)}-a(z)}\Big) + (\varepsilon + O(1))T(r,F_1) \leq$$
$$\leq \overline{N}(r,f) + \overline{N}(r,H(z,f)) + \overline{N}\Big(r,\frac{1}{F_1^{(k)}}\Big) + \overline{N}\Big(r,\frac{1}{F_1^{(k)}-a(z)}\Big) + (\varepsilon + O(1))T(r,F).$$

In view of Lemma 4, with $s = 1$ and Lemma 11, applying to the function F, we obtain

$$(n+\overline{d}(P)-\Psi)T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,H(z,f)) + (k+1)\overline{N}\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{P[f]}\Big)+$$
$$+N_{k+1}\Big(r,\frac{1}{H(z,f)}\Big) + \overline{N}\Big(r,\frac{1}{g^nP[f]H(z,f)-a}\Big) + (\varepsilon + O(1))T(r,f) \leq$$
$$\leq \overline{N}(r,f) + \overline{N}(r,H(z,f)) + (k+1)\overline{N}\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{P[f]}\Big)+$$
$$+Q\overline{N}(r,f) + N_{k+t+\xi+1}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{g^nP[g])H(z,g)-a}\Big) + (\varepsilon + O(1))T(r,f) \leq$$
$$\leq (k+\overline{d}(P)+2Q+\Psi(k+l+1)+2)T(r,f)+$$
$$+(n+m+\Psi)(k+1)T(r,g) + (\varepsilon + O(1))T(r,f).$$

A quick calculation reveals that

$$(n-k-2Q-\Psi(k+t+\xi)-2)T(r,f) \leq (n+m+\Psi)(k+1)T(r,g) + (\varepsilon + O(1))T(r,f).$$

Since $n > k+2Q+\Psi(k+t+\xi)+2$, taking $\varepsilon > 1$, we obtain $T(r,f) = O(T(r,g))$. Similarly, we can prove that $T(r,g) = OT((r,f))$. $\square$

Lemma 13. *Let $f(z)$, $g(z)$ be two non-constant finite-order meromorphic functions such that the poles of $f(z)$ are not zeros of $H(z,f)$ and the poles of $g(z)$ are not zeros of $H(z,g)$, F, G and V are defined as in* (4) *and* (6)*, let $P[f]$ and $H(z,f)$ be defined as in* (2) *and* (3)*, and $k(>0)$, $n(>3)$, $\overline{d}(P)(\geq 0)$ Q be positive integers. Also let $c_{ij}(i \in \{1,2,\dots,t\}; j \in \{1,2,\dots,\xi\})$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_{ij}. If $V \not\equiv 0$, F and, in addition, G share IM the values 1 and ∞. Then*

$$(n+\overline{d}(P)+k-2Q-3)\overline{N}(r,f) \leq 2\overline{N}\Big(r,\frac{1}{F}\Big) + 2\overline{N}\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g),$$
$$(n+\overline{d}(P)+k-2Q-3)\overline{N}(r,g) \leq 2\overline{N}\Big(r,\frac{1}{F}\Big) + 2\overline{N}\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g).$$

Proof. Let poles of $f(z)$ and $g(z)$ be not zeros of $H(z,f)$ and $H(z,f)$ respectively. If z_0 is a pole of $f(z)$ and $g(z)$ of order p and q respectively then z_0 must be a pole of F and G of order $(n+\overline{d}(P))p+k$ and $(n+\overline{d}(P))q+k$, respectively. Thus, z_0 is a zero of $\frac{F'}{F-1}-\frac{F'}{F}$ of order $(n+\overline{d}(P))p+k-1 \geq n+\overline{d}(P)+k-1$. Moreover, z_0 is a zero of $\frac{G'}{G-1}-\frac{G'}{G}$ of order $(n+\overline{d}(P))q+k-1 \geq n+\overline{d}(P)+k-1$. Hence, z_0 is a zero of V of order at least $n+\overline{d}(P)+k-1$. Therefore, we obtain

$$(n+\overline{d}(P)+k-1)\overline{N}(r,f) \leq N\Big(r,\frac{1}{V}\Big) \tag{9}$$

and

$$(n+\overline{d}(P)+k-1)\overline{N}(r,g) \leq N\Big(r,\frac{1}{V}\Big). \tag{10}$$

By the Lemma on logarithmic derivative, we get $m(r, V) = S(r, f) + S(r, g)$. We now consider

$$N\Big(r, \frac{1}{V}\Big) \le T(r, V) \le \overline{d}(P)(r, V) + N(r, V) \le N(r, V) + S(r, f) + S(r, g). \qquad (11)$$

Since $F(z)$ and $G(z)$ share the value 1 IM, the zeros of $F(z) - 1$ and the zeros of $G(z) - 1$ with different multiplicities contribute to the poles of V. Furthermore, since $F(z)$ and $G(z)$ share the value 1 IM, the poles of $F(z)$ and $G(z)$ with different multiplicities contribute to the zeros of V. Thus, it follows from (9) and (11) that

$$N\Big(r, \frac{1}{V}\Big) \le \overline{N}\Big(r, \frac{1}{F}\Big) + \overline{N}\Big(r, \frac{1}{G}\Big) + \overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G-1}\Big) + \\ + S(r, f) + S(r, g). \qquad (12)$$

Since F and G share 1 IM, by Lemma 10 and (12), we get

$$N\Big(r, \frac{1}{V}\Big) \le 2\overline{N}\Big(r, \frac{1}{F}\Big) + 2\overline{N}\Big(r, \frac{1}{G}\Big) + \overline{N}(r, F) + \overline{N}(r, G) + S(r, f) + S(r, g). \qquad (13)$$

By Lemma 3, we obtain

$$\overline{N}(r, F) = \overline{N}\Big(r, \frac{[f^n P[f] H(z, f)]^{(k)}}{a(z)}\Big) \le \overline{N}(r, f) + \overline{N}(r, H(z, f)) + S(r, f) \le \\ \le (Q+1)\overline{N}(r, f) + S(r, f). \qquad (14)$$

Similarly,

$$\overline{N}(r, G) \le (Q+1)\overline{N}(r, g) + S(r, g). \qquad (15)$$

In view of (13)–(15) and the fact that $f(z)$ and $g(z)$ share IM ∞, we find

$$N\Big(r, \frac{1}{V}\Big) \le 2\overline{N}\Big(r, \frac{1}{F}\Big) + 2\overline{N}\Big(r, \frac{1}{G}\Big) + (1+Q)\overline{N}(r, f) +)(Q+1)\overline{N}(r, g) + \\ + S(r, f) + S(r, g) \le 2\overline{N}\Big(r, \frac{1}{F}\Big) + 2\overline{N}\Big(r, \frac{1}{G}\Big) + 2(Q+1)\overline{N}(r, f) + S(r, f) + S(r, g). \qquad (16)$$

It follows from (9) and (16) that

$$(n + \overline{d}(P) + k - 1)\overline{N}(r, f) \le 2\overline{N}\Big(r, \frac{1}{F}\Big) + 2\overline{N}\Big(r, \frac{1}{G}\Big) + 2(Q+1)\overline{N}(r, f) + S(r, f) + S(r, g),$$

i.e., $(n + \overline{d}(P) + k - 2Q - 3)\overline{N}(r, f) \le 2\overline{N}(r, \frac{1}{F}) + 2\overline{N}(r, \frac{1}{G}) + S(r, f) + S(r, g)$.
Similarly, $(n + \overline{d}(P) + k - 2Q - 3)\overline{N}(r, g) \le 2\overline{N}(r, \frac{1}{F}) + 2\overline{N}(r, \frac{1}{G}) + S(r, f) + S(r, g)$. □

Lemma 14. *Let $f(z)$ be a transcendental finite-order meromorphic function, k, n, $\overline{d}(P)$, Ψ, Q, t, ξ be positive integers satisfying the inequality $n > k + 2Q + \Psi(k + l) + 2$ and $c_{ij}(i \in \{1, 2, \dots, t\}; j \in \{1, 2, \dots, \xi\})$ be a nonzero complex constant such that $f(z)$ is not a periodic function of period c_{ij}, let $P(w)$ and $H[f]$ be defined as in (2) and (3). Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$. Then $[f^n P[f] H(z, f)]^{(k)} - a(z)$ has infinitely many zeros.*

Proof. Suppose $[f^nP[f]H(z,f)]^{(k)} - a(z)$ has only finitely many zeros. Let $F_1 = f^nP(f)H[f]$ and $F = F_1^{(k)}$. By the Second Fundamental Theorem, we obtain

$$T(r,F^{(k)}) \le \overline{N}\Big(r,\frac{1}{F_1^{(k)}}\Big) + \overline{N}\Big(r,\frac{1}{F_1^{(k)}-a}\Big) + \overline{N}(r,F_1^{(k)}) + S(r,F_1) \le$$
$$\le T(r,F_1^{(k)}) - T(r,F_1) + \overline{N}_{k+1}\Big(r,\frac{1}{F_1}\Big) + \overline{N}(r,F_1) + S(r,F_1),$$

it reveals that

$$T(r,F) \le \overline{N}_{k+1}\Big(r,\frac{1}{F_{k+1}}\Big) + \overline{N}(r,F_1) + S(r,F_1). \tag{17}$$

Hence, we have $T(r,f) = T(r,f) + S(r,f)$. Therefore, we obtain

$$\overline{N}_{k+1}\Big(r,\frac{1}{F_1}\Big) = \overline{N}_{k+1}\Big(r,\frac{1}{f^nP[f]H(z,f)}\Big) \le \overline{N}_{k+1}\Big(r,\frac{1}{f^n}\Big) + \overline{N}_{k+1}\Big(r,\frac{1}{P[f]}\Big) +$$
$$+ N_{k+1}\Big(r,\frac{1}{H(z,f)}\Big) + S(r,f) \le (k+1)\overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}_{k+1}\Big(r,\frac{1}{P[f]}\Big) + Q\overline{N}(r,f) +$$
$$+ \Psi N_{k+t+\xi+1}\Big(r,\frac{1}{f}\Big) + S(r,f) \le (k+m+Q+\Psi(k+t+\xi+1)+1)T(r,f) + S(r,f). \tag{18}$$

and

$$\overline{N}(r,F_1) = \overline{N}(r,f^nP[f]H(z,f)) \le (Q+1)T(r,f) + S(r,f). \tag{19}$$

By Lemma 11, using (18) and (19), from (17) we obtain

$$(n + \overline{d}(P) - \Psi)T(r,f) \le (k+m+2Q+\Psi(k+t+\xi+1)+2)T(r,f) + S(r,f).$$

It is a contradiction for $n > k + 2Q + \Psi(k+t+\xi) + 2$. □

Lemma 15. *Let $f(z)$ and $g(z)$ be two non-constant finite-order meromorphic functions, $P[f]$ and $H(z,f)$ be defined as in (2) and (3) and k, n, $\overline{d}(P)$, Ψ, Q, t, ξ be positive integers satisfying the inequality $n > \overline{d}(P)+3\Psi+Q+2k+1$, and let $c_{ij}(i \in \{1,2,\dots,t\}; j \in \{1,2,\dots,\xi\})$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_{Ij}. If*

$$[f^nP[f]H(z,f)]^{(k)} \equiv [g^nP[g]H(z,g)]^{(k)},$$

then

$$f^nP[f]H(z,f) \equiv g^nP[g]H(z,g).$$

Proof. Let $[f^nP[f]H(z,f)]^{(k)} \equiv [g^nP[g]H(z,g)]^{(k)}$. Integrating k times above we get $f^nP[f]H(z,f) \equiv g^nP[g]H(z,g) + Q(z)$, where $Q(z)$ is a polynomial of degree at most $k-1$. If $R(z) \not\equiv 0$, then this equation can be expressed as $\frac{f^nP[f]H(z,f)}{R} = \frac{g^nP[g]H(z,g)}{R} + 1$.

Then from the above equation and Lemma 7, we have

$$T\Big(r,\frac{f^nP[f]H(z,f)}{R}\Big) \le \overline{N}\Big(r,\frac{f^nP[f]H(z,f)}{R}\Big) + \overline{N}\Big(r,\frac{R}{f^nP[f]H(z,f)}\Big) +$$
$$+ \overline{N}\Big(r,\frac{R}{g^nP[g]H(z,g)}\Big) + S(r,f).$$

Using the above equation, we obtain

$$T(r,f^nP[f]H(z,f)) \le \overline{N}\Big(r,f^nP[f]H(z,f)\Big) + \overline{N}\Big(r,\frac{1}{f^nP[f]H(z,f)}\Big) +$$
$$+ \overline{N}\Big(r,\frac{1}{g^nP[g]H(z,g)}\Big) + 2(k-1)\log r + S(r,f) \le \overline{N}(r,f) + \overline{N}(r,H(z,f)) + \overline{N}\Big(r,\frac{1}{f}\Big) +$$

$$+\overline{N}\Big(r,\frac{1}{P[f]}\Big)+\overline{N}\Big(r,\frac{1}{H(z,f)}\Big)+\overline{N}\Big(r,\frac{1}{g}\Big)+\overline{N}\Big(r,\frac{1}{P[g]}\Big)+$$
$$+\overline{N}\Big(r,\frac{1}{H(z,f)}\Big)+2(k-1)\log r+S(r,f).$$

Using the aforementioned equation and Lemma 11, we deduce

$$(n+\overline{d}(P)-\Psi)T(r,f)\le(\overline{d}(P)+Q+\Psi+2)T(r,f)+(\overline{d}(P)+\Psi+1)T(r,g)+$$
$$+2(k-1)\log r+S(r,f)+S(r,g). \tag{20}$$

Similarly, we obtain

$$(n+\overline{d}(p)-\Psi)T(r,g)\le(\overline{d}(P)+Q+\Psi+2)T(r,g)+(\overline{d}(P)+\Psi+1)T(r,f)+$$
$$+2(k-1)\log r+S(r,f)+S(r,g). \tag{21}$$

Since f and g are non-constant, we have

$$T(r,f)\ge\log r+S(r,f),T(r,g)\le\log r+S(r,g). \tag{22}$$

It follows from (20), (21) and (22) that $(n+\overline{d}(P)-\Psi)\{T(r,f)+T(r,g)\}\le(2k+2\overline{d}(P)+2\Psi+Q+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$, which contradicts $n>\overline{d}(P)+3\Psi+Q+2k+1$. Thus we have $Q(z)\equiv 0$ and therefore, we obtain $f^nP(f)H[f]\equiv g^nP(g)H[g]$. □

Lemma 16. *Let $f(z)$ and $g(z)$ be two non-constant finite-order meromorphic functions, let $c_{ij}(i\in\{1,2,\dots,t\};j\in\{1,2,\dots,\xi\})$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_{ij}, and let $k(>0)$ be an integer satisfying $n>k+1$. Also let $P[f]$ and $H(z,f)$ be defined as in (2) and (3). Suppose that $a(z)$ $(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$ with finitely many zeros and poles. If*

$$[f^nP[f]H(z,f)]^{(k)}\equiv[g^nP[g]H(z,g)]^{(k)},\quad [f^nP[f]H(z,f)]\equiv[g^nP[g]H(z,g)]$$

and, in addition, $f(z)$ and $g(z)$ share 1 IM, then $P[f]$ reduces to a nonzero monomial, namely, $P(w)=a_iw^i\not\equiv 0$ for some $i\in 0,1,...,m$.

Proof. Using the same reasoning as in Lemma 3.13 [13], we can easily obtain Lemma 16. □

4. Proof of the main results.

Proof of Theorem 7. Let F, G, H and V be defined as in (4), (5) and (6). We suppose that $F_1=f^nP[f]H(z,f)$ and $G_1=g^nP[g]H(z,g)$. By the assumption, $[f^nP[f]H(z,f)]^{(k)}$ and $[g^nP[g]H(z,g)]^{(k)}$ share a small function $a(z)$ and 1 IM, hence F and G share the values 1 and ∞ IM. Suppose that $H\not\equiv 0$. It is easy to see that $F\not\equiv G$. We must have $V\not\equiv 0$. It follows from Lemmas 8 and 9 that

$$T(r,F)\le N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+2\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{G}\Big)+7\overline{N}(r,F)+S(r,F)+S(r,G). \tag{23}$$

By Lemma 4 with $s=2$, Lemma 3 and (23), we obtain

$$T(r,F_1)\le N_2\Big(r,\frac{1}{G}\Big)+2\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{G}\Big)+N_{k+2}\Big(r,\frac{1}{F_1}\Big)+7\overline{N}(r,F)+S(r,F)+$$
$$+S(r,G)\le N_{k+2}\Big(r,\frac{1}{G_1}\Big)+k\overline{N}(r,G_1)+2N_{k+1}\Big(r,\frac{1}{F_1}\Big)+2k\overline{N}(r,F_1)+N_{k+1}\Big(r,\frac{1}{G_1}\Big)+$$

$$
\begin{gathered}
+k\overline{N}(r,G_1)+N_{k+2}\Big(r,\frac{1}{F_1}\Big)+7\overline{N}(r,F)+S(r,F)+S(r,G)\le(k+2)\overline{N}\Big(r,\frac{1}{g}\Big)+\\
+N\Big(r,\frac{1}{P[g]}\Big)+N_{k+2}\Big(r,\frac{1}{H(z,g)}\Big)+k(Q+1)\overline{N}(r,g)+2(k+1)\overline{N}\Big(r,\frac{1}{f}\Big)+2N\Big(r,\frac{1}{P[f]}\Big)+\\
+2N_{k+1}\Big(r,\frac{1}{H(z,f)}\Big)+2k(Q+1)\overline{N}(r,f)+(k+1)\overline{N}\Big(r,\frac{1}{g}\Big)+N\Big(r,\frac{1}{P[g]}\Big)+\\
+N_{k+1}\Big(r,\frac{1}{H(z,g)}\Big)+k(Q+1)\overline{N}(r,g)+(k+2)\overline{N}\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{P[f]}\Big)+\\
+N_{k+2}\Big(r,\frac{1}{H(z,f)}\Big)+7(Q+1)\overline{N}(r,f)+S(r,f)+S(r,g)\le\\
\le(k+2)\overline{N}\Big(r,\frac{1}{g}\Big)+N\Big(r,\frac{1}{P[g]}\Big)+Q\overline{N}(r,g)+\Psi N_{k+t+\xi+2}\Big(r,\frac{1}{g}\Big)+k(Q+1)\overline{N}(r,g)+\\
+2(k+1)\overline{N}\Big(r,\frac{1}{f}\Big)+2N\Big(r,\frac{1}{P[f]}\Big)+2Q\overline{N}(r,f)+2\Psi N_{k+t+\xi+1}\Big(r,\frac{1}{f}\Big)+\\
+2k(Q+1)\overline{N}(r,f)+(k+1)\overline{N}\Big(r,\frac{1}{g}\Big)+N\Big(r,\frac{1}{P[g]}\Big)+Q\overline{N}(r,g)+\Psi N_{k+t+\xi+1}\Big(r,\frac{1}{g}\Big)+\\
+k(Q+1)\overline{N}(r,g)+(k+2)\overline{N}\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{P[f]}\Big)+Q\overline{N}(r,f)+\Psi N_{k+t+\xi+2}\Big(r,\frac{1}{f}\Big)+\\
+7(Q+1)\overline{N}(r,f)+S(r,f)+S(r,g).
\end{gathered}
$$

Therefore, we have

$$
\begin{gathered}
T(r,F_1)\le(3k+4)\overline{N}\Big(r,\frac{1}{f}\Big)+(2k+3)\overline{N}\Big(r,\frac{1}{g}\Big)+3N\Big(r,\frac{1}{P[f]}\Big)+\\
+2N\Big(r,\frac{1}{P[g]}\Big)+\Psi(3k+3t+3\xi+4)N\Big(r,\frac{1}{f}\Big)+\\
+\Psi(2k+2t+2\xi+3)N\Big(r,\frac{1}{g}\Big)+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,f)+S(r,f)+S(r,g).
\end{gathered}
$$

By Lemma 11, the above inequality can be reduced as

$$
\begin{gathered}
(n+\overline{d}(P)-\Psi)T(r,f)\le(3k+3\overline{d}(P)+\Psi(3k+3t+3\xi+4)+4)T(r,f)+\\
+(2k+2\overline{d}(P)+\Psi(2k+2t+2\xi+3)+3)T(r,g)+\\
+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,f)+S(r,f)+S(r,g).
\end{gathered}\tag{24}
$$

Similarly, we obtain

$$
\begin{gathered}
(n+\overline{d}(P)-\Psi)T(r,g)\le(3k+3\overline{d}(P)+\Psi(3k+3t+3\xi+4)+4)T(r,g)+\\
+(2k+2\overline{d}(P)+\Psi(2k+2t+2\xi+3)+3)T(r,f)+\\
+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,g)+S(r,f)+S(r,g).
\end{gathered}\tag{25}
$$

Combining (24) and (25), we obtain

$$
\begin{gathered}
(n+\overline{d}(P)-\Psi)\{T(r,f)+T(r,g)\}\le\\
\le(5k+5\overline{d}(P)+\Psi(5k+5t+5\xi+7)+7)\{T(r,f)+T(r,g)\}+\\
+\{(4k+7)(Q+1)+5Q\}\{\overline{N}(r,f)+\overline{N}(r,g)\}+S(r,f)+S(r,g).
\end{gathered}
$$

Thus we have

$$
(n-5k-4\overline{d}(P)-\Psi(5k+5t+5\xi+6)-7)\{T(r,f)+T(r,f)\}\le
$$

$$\leq 2((4k+7)(Q+1)+5Q)\overline{N}(r,f)+S(r,f)+S(r,g). \tag{26}$$

Since $V \not\equiv 0$ and F and G share the values 1 and ∞ IM, by Lemma 12, we obtain

$$(n+\overline{d}(P)+k-2Q-3)\overline{N}(r,f) \leq 2\overline{N}\Big(r,\frac{1}{F}\Big)+2\overline{N}\Big(r,\frac{1}{G}\Big)+S(r,f)+S(r,g). \tag{27}$$

By Lemma 4 with $s=1$, (27) takes form

$$\begin{gathered}(n+\overline{d}(P)+k-2Q-3)\overline{N}(r,f) \leq 2(k+1)\overline{N}\Big(r,\frac{1}{f}\Big)+2N\Big(r,\frac{1}{P[f]}\Big)+2N_{k+1}\Big(r,\frac{1}{H(z,f)}\Big)+\\ +2k\overline{N}(r,f)+2k\overline{N}(r,H(z,f))+2(k+1)\overline{N}\Big(r,\frac{1}{g}\Big)+2N\Big(r,\frac{1}{P[g]}\Big)+2N_{k+1}\Big(r,\frac{1}{H(z,g)]}\Big)+\\ +2k\overline{N}(r,g)+2k\overline{N}(r,H(z,g))+S(r,f)+S(r,g) \leq 2(k+1)\overline{N}\Big(r,\frac{1}{f}\Big)+2N\Big(r,\frac{1}{P[f]}\Big)+\\ +2Q\overline{N}(r,f)+2\Psi N_{k+t+\xi+1}\Big(r,\frac{1}{f}\Big)+2k\overline{N}(r,f)+2k\overline{N}(r,H(z,f))+2(k+1)\overline{N}\Big(r,\frac{1}{g}\Big)+\\ +2N\Big(r,\frac{1}{P[g]}\Big)+2Q\overline{N}(r,g)+2\Psi N_{k+t+\xi+1}\Big(r,\frac{1}{g}\Big)+2k\overline{N}(r,g)+2k\overline{N}(r,H(z,g))+\\ +S(r,f)+S(r,g) \leq 2(k+\overline{d}(P)+\Psi(k+t+\xi+1)+1)T(r,f)+2(k+\overline{d}(P)+\\ +\Psi(k+t+\xi+1)+1)T(r,g)+\{4k(Q+1)+4Q\}\overline{N}(r,f)+S(r,f)+S(r,g).\end{gathered}$$

Thus we have

$$\begin{gathered}(n+\overline{d}(P)-k(4Q+3)-6Q-3)\overline{N}(r,f) \leq 2(k+\overline{d}(P)+\\ +\Psi(k+t+\xi+1)+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{gathered} \tag{28}$$

Since $\overline{N}(r,f)=\overline{N}(r,g)$, combining (26) and (28), we deduce

$$\begin{gathered}(n-5k-4\overline{d}(P)-\Psi(5k+5t+5\xi+6)-7)(n+\overline{d}(P)-k(4Q+3)-6Q-3)-\\ -4((4k+7)(Q+1)+5Q)(k+\overline{d}(P)+\Psi(k+t+\xi+1)+1)\{T(r,f)+T(r,g)\} \leq \\ \leq S(r,f)+S(r,g).\end{gathered}$$

Which contradicts $n>4\overline{d}(P)+4k(Q+2)+\Psi(5k+5t+5\xi+6)+8Q+11$. As in the proof of Lemma 6 applied to the functions F and G, we obtain the following cases:

(i) $T(r,F) \leq N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+N_2(r,F)+N_2(r,F)+S(r,F)+S(r,G)$,

(ii) $FG=1$,

(iii) $F\equiv G$.

By the condition imposed on n, case (i) is impossible. By Lemma 16, case (ii) is impossible. Hence, we get only the case (iii), i.e., $[f^nP[f]H(z,f)]^{(k)} \equiv [g^nP[g]H(z,g)]^{(k)}$. Thus, by Lemma 15, we obtain

$$[f^nP[f]H(z,f)] \equiv [g^nP[g]H(z,g)], \tag{29}$$

i.e.,

$$\sum_{i=1}^{m}\alpha_i(z)f^{n_{i0}+n}\prod_{j=1}^{s}(f^{(j)})^{n_{ij}} \equiv \sum_{i=1}^{m}\alpha_i(z)g^{n_{i0}+n}\prod_{j=1}^{s}(g^{(j)})^{n_{ij}}.$$

Let $h=\frac{f}{g}$. If h is a constant, from (29), we can get that f and g satisfy the algebraic equation $R(w_1,w_2)=0$, where $R(w_1,w_2)=w_1^nP[w_1]H(z,w_1)-w_2^nP[w_2]H(z,w_2)$. if h is a constant, then the above equation can be written as

$$\sum_{i=1}^{m}(h^{n_{i0}+n_{i1}+\cdots+n_{is}+n+1}-1)\alpha_i(z)g^{n_{i0}+n}\prod_{j=1}^{s}(g^{(j)})H(z,g)\equiv 0.$$

Since $H(z,g)\not\equiv 0$, we must have

$$\sum_{i=1}^{m}(h^{n_{i0}+n_{i1}+\cdots+n_{is}+n+1}-1)\alpha_i(z)g^{n_{i0}+n}\prod_{j=1}^{s}(g^{(j)})\equiv 0. \tag{30}$$

Let $g^{n_{i0}+n}(g')(g'')\dots(g^{(s)})^{n_{is}}=V_i(z)$. Without loss of generality, let

$$\sum_{j=0}^{s}n_{mj}+n+1\geq\sum_{j=0}^{s}n_{(m-1)_j}+n+1\geq\cdots\geq\sum_{j=0}^{s}n_{1j}+n+1.$$

Since g is a transcendental function, we can find that $V_i(z)\not\equiv 0$. If $\alpha_m(z)\not\equiv 0$ and $\alpha_{m-1}(z)=\alpha_{m-2}(z)=\cdots=\alpha_1(z)=0$, then from (30), we can get taht $h^{n_{m0}+n_{m1}+\cdots+n_{ms}+n+1}=1$. If $\alpha_m(z)\not\equiv 0$ and there exists $\alpha_i(z)\not\equiv 0$ $(i\in\{1,2,\dots,m-1\})$. If $h^{n_{m0}+n_{m1}+\cdots+n_{ms}+n+1}\neq 1$, by Lemma 5 and (30), we have $T(r,g)=S(r,g)$, which contradicts with a transcendental function g. Therefore, $h^{n_{m0}+n_{m1}+\cdots+n_{ms}+n+1}=1$. If $\alpha_i(z)\not\equiv 0$ for some $i\in\{1,2,\dots,m\}$, we can also get that $h^{n_{m0}+n_{m1}+\cdots+n_{ms}+n+1}=1$. Thus, from the definition of d, we can obtain that $f\equiv tg$, where t a constant such that $t^d=1$, $d=GCD(\lambda_0,\lambda_1,\dots,\lambda_m)$, where $\lambda_i^{`}s$ are defined by

$$\lambda_i=\begin{cases}n_{i0}+n_{i1}+\ldots+n_{is}+n+1, & \text{if } \alpha_i\neq 0;\\ n_{m0}+n_{m1}+\ldots+n_{ms}+n+1, & \text{if } \alpha_i=0.\end{cases}$$

□

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