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GLOBAL SOLVABILITY OF A MIXED PROBLEM FOR A SINGULAR SEMILINEAR HYPERBOLIC 1D SYSTEM

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Using the method of characteristics and the Banach fixed point theorem (for the Bielecki metric), in the paper it is established the existence and uniqueness of a global (continuous) solution of the mixed problem in the rectangle $\Pi = \{(x,t)\colon 0 < x < l < \infty,\ 0 < t < T < \infty\}$ for the first order hyperbolic system of semi-linear equations of the form

$$\frac{\partial u}{\partial t} + \Lambda(x,t)\frac{\partial u}{\partial x} = f(x,t,u,v,w), \frac{\partial v}{\partial x} = g(x,t,u,v,w), \frac{\partial w}{\partial t} = h(x,t,u,v,w),$$

for a singular with orthogonal (degenerate) and non-orthogonal to the coordinate axes characteristics and with nonlinear boundary conditions, where $\Lambda(x,t) = \operatorname{diag}(\lambda_1(x,t),\ldots,\lambda_k(x,t))$, $u = (u_1,\ldots,u_k)$, $v = (v_1,\ldots,v_m)$, $w = (w_1,\ldots,w_n)$, $f = (f_1,\ldots,f_k)$, $g = (g_1,\ldots,g_m)$, $h = (h_1,\ldots,h_n)$ and besides $\operatorname{sign}\lambda_i(0,t) = \mathrm{const} \neq 0$, $\operatorname{sign}\lambda_i(l,t) = \mathrm{const} \neq 0$ for all $t \in [0,T]$ and for all $i \in \{1,\ldots,k\}$. The presence of non-orthogonal and degenerate characteristics of the hyperbolic system for physical reasons indicates that part of the oscillatory disturbances in the medium propagates with a finite speed, and part with an unlimited one. Such a singularity (degeneracy of characteristics) of the hyperbolic system allows mathematical interpretation of many physical processes, or act as auxiliary equations in the analysis of multidimensional problems.

1. Introduction. The hyperbolic equations and systems are typically used to model processes having a finite speed of perturbation propagation. From a mathematical point of view, this means that the characteristics of the corresponding equations of the system are not orthogonal to the coordinate axes [1].

However, in many problems of solid-state physics, in intermediate equations, in the analysis of multidimensional problems, mathematical models in the form of hyperbolic equations are found, part of the family of characteristics of which is perpendicular, for example, to the time axis [1]. The presence of such characteristics indicates that the speed of oscillation propagation in one-dimensional continuous media is infinite [1, see the review of references].

It should be noted that the presence of characteristics orthogonal and non-orthogonal to the coordinate axes in mathematical models of applied processes is also closely related to the boundary layer effect [2, p. 353].

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In addition, many mathematical models, for example, biopopulation theory [3], epidemic theories [4], medicine [5, 6], solid state physics [7], in the hyperbolic case have equations with the characteristics orthogonal to the coordinate axes.

The study of global solutions of nonlinear one-dimensional hyperbolic equations poses certain difficulties, because even in the case of smooth initial and boundary data the solution of the problem may lose the required smoothness (continuity) over time [8].

In this paper, we consider a semi-linear hyperbolic system of the first order with orthogonal (degenerate) and non-orthogonal characteristics. The conditions of existence and uniqueness of the global generalized (continuous) solution of the mixed problem for this system are established. Our approach is borrowed from [1, 9, 10] and uses the Banach fixed point theorem for the Bielecki metric [11].

2. Statement of a problem. In the rectangle $\Pi = \{(x,t)\colon 0 < x < l < \infty,\ 0 < t < T < \infty\}$, we consider the hyperbolic system of semi-linear equations of the first order

$$\begin{cases} \frac{\partial u}{\partial t} + \Lambda(x,t)\dfrac{\partial u}{\partial x} = f(x,t,u,v,w), \\ \frac{\partial v}{\partial x} = g(x,t,u,v,w), \\ \frac{\partial w}{\partial t} = h(x,t,u,v,w), \end{cases} \tag{1}$$

where $u = (u_1,...,u_k)$, $v = (v_1,...,v_m)$, $w = (w_1,...,w_n)$, $f = (f_1,...,f_k)$, $g = (g_1,...,g_m)$, $h = (h_1,...,h_n)$, $\Lambda(x,t) = \operatorname{diag}(\lambda_1(x,t),...,\lambda_k(x,t))$ and for all $i \in \{1,\dots,k\}$ $\operatorname{sign} \lambda_i(0,t) = \text{const} \neq 0$, $\operatorname{sign} \lambda_i(l,t) = \text{const} \neq 0$ for all $t \in [0,T]$.

Let I_0, I_l be the sets of indices defined as follows:

$$I_0 = \big\{i \in \{1,\dots,k\} : \lambda_i(0,t) > 0,\ \forall t \in [0,T]\big\},$$
$$I_l = \big\{i \in \{1,\dots,k\} : \lambda_i(l,t) < 0,\ \forall t \in [0,T]\big\},$$

with $r_0 = \text{card } I_0,\ r_l = \text{card } I_l$.

For system (1), we set the initial conditions at $0 \le x \le l$

$$\begin{aligned} u_i(x,0) &= q_i(x), \quad i \in \{1,\dots,k\}, \\ w_s(x,0) &= r_s(x), \quad s \in \{1,\dots,n\} \end{aligned} \tag{2}$$

and the boundary conditions at $0 \le t \le T$:

$$\begin{aligned} u_i(0,t) &= \gamma_i^0\Big(t, \big(u_j(0,t)\big)_{j\in I_l}, w(0,t)\Big), \quad i \in I_0, \\ u_i(l,t) &= \gamma_i^l\Big(t, \big(u_j(l,t)\big)_{j\in\ I_0}, w(l,t)\Big), \quad i \in I_l, \\ v_i(0,t) &= \psi_i\Big(t, \big(u_j(0,t)\big)_{j\in I_l}, w(0,t)\Big), \quad i \in \{1,\dots,m\}. \end{aligned} \tag{3}$$

All given functions $f : \overline{\Pi} \times \mathbb{R}^{k+m+n} \to \mathbb{R}^k$, $g : \overline{\Pi} \times \mathbb{R}^{k+m+n} \to \mathbb{R}^m$, $h : \overline{\Pi} \times \mathbb{R}^{k+m+n} \to \mathbb{R}^n$, $\lambda_i : \overline{\Pi} \to \mathbb{R}$, $i \in \{1,\dots,k\}$, $q : [0,l] \to \mathbb{R}^k$, $r : [0,l] \to \mathbb{R}^n$, $\gamma^0 : [0,T] \times \mathbb{R}^{r_l} \times \mathbb{R}^n \to \mathbb{R}^{r_0}$, $\gamma^l : [0,T] \times \mathbb{R}^{r_0} \times \mathbb{R}^n \to \mathbb{R}^{r_l}$, $\psi : [0,T] \times \mathbb{R}^{r_l} \times \mathbb{R}^n \to \mathbb{R}^m$ are continuous and the functions λ_i satisfy the Lipschitz condition in the variable x,

2. Global solvability theorem. Assume $x = \varphi_i(t; x_0, t_0)$, $t \in \overline{J_i}$, $i \in \{1,\dots,k\}$ are the non-continuous solutions of the Cauchy problem

$$\frac{dx}{dt} = \lambda_i(x,t), \quad x(t_0) = x_0,$$

where $(x_0, t_0) \in (0, l) \times (0, T]$, J_i is interval of $[0, T]$. Note that these solutions are characteristics of the first group of equations in system (1). Moreover, this system also has horizontal characteristics of the form $t = t_0$ and vertical characteristics $x = x_0$. Denote by $\chi_i(x_0, t_0)$ the smallest t, for which is defined the solution $x = \varphi_i(t; x_0, t_0)$ in $\overline{\Pi}$, i.e., $\chi_i(x_0, t_0) = \min \overline{J_i}$.

Let us define the following sets

$$\Pi_q^i = \{(x,t) \in \overline{\Pi} : \chi_i(x,t) = 0\},\ i \in \{1, \dots, k\},$$
$$\Pi_0^i = \{(x,t) \in \overline{\Pi} : \chi_i(x,t) > 0,\ \varphi_i(\chi_i(x,t); x, t) = 0\},\ i \in I_0,$$
$$\Pi_l^i = \{(x,t) \in \overline{\Pi} : \chi_i(x,t) > 0,\ \varphi_i(\chi_i(x,t); x, t) = l\},\ i \in I_l.$$

Integrating the equations of system (1) along the corresponding characteristics, we obtain the following system of integro-operator equations:

$$u_i(x,t) = F_i[u,v,w](x,t) + \\ + \int\limits_{\chi_i(x,t)}^{t} f_i\Big(\varphi_i(\tau;x,t), \tau, u(\varphi_i(\tau;x,t),\tau), v(\varphi_i(\tau;x,t),\tau), w(\varphi_i(\tau;x,t),\tau)\Big) d\tau, \tag{4}$$

$$v_j(x,t) = \psi_j\Big(t, \big(u_j(0,t)\big)_{j\in I_l}, w(0,t)\Big) + \int\limits_0^x g_j\Big(y, t, u(y,t), v(y,t), w(y,t)\Big) dy, \tag{5}$$

$$w_s(x,t) = r_s(x) + \int\limits_0^t h_s\Big(x, \tau, u(x,\tau), v(x,\tau), w(x,\tau)\Big) d\tau, \tag{6}$$

where $i \in \{1, \dots, k\}$, $j \in \{1, \dots, m\}$, $s \in \{1, \dots, n\}$,

$$F_i[u,v,w](x,t) = \begin{cases} q_i(\varphi_i(0;x,t)), & (x,t) \in \Pi_q^i, \\ \gamma_i^0\Big(\chi_i(x,t), \big(u_j(0,\chi_i(x,t))\big)_{j\in I_l}, w(0,\chi_i(x,t))\Big), & (x,t) \in \Pi_0^i, \\ \gamma_i^l\Big(\chi_i(x,t), \big(u_j(l,\chi_i(x,t))\big)_{j\in I_0}, w(l,\chi_i(x,t))\Big), & (x,t) \in \Pi_l^i. \end{cases}$$

The *generalized solution of the problem* (1)-(3) is called a vector-valued function (u, v, w), whose components belong to the space $\mathrm{C}(\overline{\Pi})$ and satisfy the equations of the system (4)–(6).

Theorem. *Assume that the following conditions are satisfy:*
1) *the function Λ is continuous and Lipschitz on the set $\overline{\Pi}$ in the variable x;*
2) *f, g, h, q, r, γ^0, γ^l, ψ are continuous functions on the respective definition sets;*
3) *the functions f, g, h, γ^0, γ^l, ψ satisfy the Lipschitz condition in the variables u, v, w on the above defined sets;*
4) *the zero-order compatibility condition holds:*

$$q_i(0) = \gamma_i^0(0, (q_j(0))_{j\in I_l}, r(0)), \quad i \in I_0,$$
$$q_i(l) = \gamma_i^l(0, (q_j(l))_{j\in I_0}, r(l)), \quad i \in I_l,$$
$$v_i(0,0) = \psi_i(0, (q_j(0))_{j\in I_l}, r(0)), \quad i \in \{1, \dots, m\}.$$

Then there exists a unique generalized solution of problem (1)–(3).

Proof. Consider a metric space $\mathcal{Q}$ consisting of continuous functions $z = (u, v, w)$, whose components u, v, w belong to to the space $\mathrm{C}(\overline{\Pi})$, with $u_i(0,0) = q_i(0)$, $i \in I_0$ and $u_i(l,0) = q_i(l)$, $i \in I_l$.

Let $\{z^1, z^2\} \subset \mathcal{Q}$. Then we define the metric at the elements of $\mathcal{Q}$ by the formula from [9-11]

$$\rho(z^1,z^2) = \max\Big\{ \max_{i,x,t} |u_i^1(x,t) - u_i^2(x,t)|\alpha_i(x)e^{-at}, \\ \max_{i,x,t} |v_i^1(x,t) - v_i^2(x,t)|\beta_i(x)e^{-at}, \max_{i,x,t} |w_i^1(x,t) - w_i^2(x,t)|\eta_i(x)e^{-at}\Big\}, \tag{7}$$

where the constant $a > 0$ and the continuous positive functions $\alpha_i,\ \beta_i,\ \eta_i : [0,l] \to \mathbb{R}$ will be chosen later.

In the space $\mathcal{Q}$, we introduce the operator $\mathcal{A} = (\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3)$, where the operators $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ are defined, respectively, by the right sides of system (4)–(6), i.e.:

$$\mathcal{A}_i^1[z](x,t) = F_i[z](x,t) + \int\limits_{\chi_i(x,t)}^{t} f_i\Big(\varphi_i(\tau;x,t), \tau, u(\varphi_i(\tau;x,t),\tau), v(\varphi_i(\tau;x,t),\tau), w(\varphi_i(\tau;x,t),\tau)\Big)\,d\tau,$$

$$\mathcal{A}_j^2[z](x,t) = \psi_j\Big(t, \big(u_j(0,t)\big)_{j\in I_l}, w(0,t)\Big) + \int\limits_0^x g_j\Big(y,t,u(y,t),v(y,t),w(y,t)\Big)dy,$$

$$\mathcal{A}_s^3[z](x,t) = r_s(x) + \int\limits_0^t h_s\Big(x,\tau,u(x,\tau),v(x,\tau),w(x,\tau)\Big)d\tau,$$

where $i \in \{1,\dots,k\}$, $j \in \{1,\dots,m\}$, $s \in \{1,\dots,n\}$.

By L we denote the common constant in the Lipschitz conditions for the functions $f, g, h, \psi, \gamma^0, \gamma^l$, which is written, for example, for f, in the form

$$|f_i(x,t,u^1(x,t),v^1(x,t),w^1(x,t)) - f_i(x,t,u^2(x,t),v^2(x,t),w^2(x,t))| \le \\ \le L\max\Big\{ \max_{j,x,t}|u_j^1(x,t) - u_j^2(x,t)|,\ \max_{j,x,t}|v_j^1(x,t) - v_j^2(x,t)|,\ \max_{j,x,t}|w_j^1(x,t) - w_j^2(x,t)|\Big\}.$$

From the definition of the metric for admissible i, x, t, and $z \in \mathcal{Q}$, the inequalities follow

$$|u_i^1(x,t) - u_i^2(x,t)| \le \frac{\rho(z^1,z^2)}{\alpha_i(x)}e^{at}, \quad |v_i^1(x,t) - v_i^2(x,t)| \le \frac{\rho(z^1,z^2)}{\beta_i(x)}e^{at}, \\ |w_i^1(x,t) - w_i^2(x,t)| \le \frac{\rho(z^1,z^2)}{\eta_i(x)}e^{at}.$$

Let us show that the operator $\mathcal{A}$ is contractive. To do this, we will perform a series of estimates.

Let $z^1 \in \mathcal{Q}$, $z^2 \in \mathcal{Q}$, then for $j \in \{1,\dots,m\}$

$$|F_i[z^1](x,t) - F_i[z^2](x,t)| \le \begin{cases} L\max\{\max\limits_{j\notin I_0}\frac{e^{a\chi_i(x,t)}}{\alpha_j(0)}, \max\limits_j \frac{e^{a\chi_i(x,t)}}{\eta_j(x)}\}\rho(z^1,z^2),\ (x,t)\in\Pi_0^i, \\ L\max\{\max\limits_{j\notin I_l}\frac{e^{a\chi_i(x,t)}}{\alpha_j(l)}, \max\limits_j \frac{e^{a\chi_i(x,t)}}{\eta_j(x)}\}\rho(z^1,z^2),\ (x,t)\in\Pi_l^i. \end{cases} \tag{8}$$

Let us define $\mu = (\max_{i,x,t}|\lambda_i(x,t)|)^{-1}$. If the first inequality of (8) is satisfied, then for all $i \in I_0$, $\chi_i(x,t) \le t - \mu x$. Similarly, from the second part of (8), for all $i \in I_l$ we have $\chi_i(x,t) \le t - \mu(l-x)$.

Based on the obtained inequalities, we obtain estimates for the operators $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$:

$$\big|(\mathcal{A}_i^1[z^1])(x,t) - (\mathcal{A}_i^1[z^2])(x,t)\big|\,\alpha_i(x)e^{-at} \le$$

$$\le L\max\left\{\max_{\substack{i\in I_0,\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)},\max_{\substack{i\in I_l,\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)},\max_{\substack{i\in I_0,\\ j}}\frac{\alpha_i(x)e^{-a\mu x}}{\eta_j(x)},\right.$$
$$\left.\max_{\substack{i\in I_l,\\ j}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\eta_j(x)}\right\}\rho(z^1,z^2)+$$
$$+\int_0^t e^{a(\sigma-t)}d\sigma L\max\left\{\max_{i,j,y}\frac{\alpha_i(x)}{\alpha_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\beta_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\eta_j(y)}\right\}\rho(z^1,z^2)\le$$
$$\le L\max\left\{\max_{\substack{i\in I_0,\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)},\max_{\substack{i\in I_l,\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)},\max_{\substack{i\in I_0,\\ j}}\frac{\alpha_i(x)e^{-a\mu x}}{\eta_j(x)},\right.$$
$$\left.\max_{\substack{i\in I_l,\\ j}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\eta_j(x)}\right\}\rho(z^1,z^2)+$$
$$+\frac{L}{a}\max\left\{\max_{i,j,y}\frac{\alpha_i(x)}{\alpha_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\beta_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\eta_j(y)}\right\}\rho(z^1,z^2),$$

$$\left|\mathcal{A}_i^2[z^1](x,t)-\mathcal{A}_i^2[z^2](x,t)\right|\beta_i(x)e^{-at}\le\left(L\max\left\{\max_{i,\,j\notin I_0}\frac{\beta_i(x)}{\alpha_j(0)},\max_{i,j}\frac{\beta_i(x)}{\eta_i(x)}\right\}+\right.$$
$$\left.+\int_0^x L\max\left\{\max_{i,j}\frac{\beta_i(x)}{\alpha_j(y)},\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)},\max_{i,j}\frac{\beta_i(x)}{\eta_j(y)}\right\}dy\right)\rho(z^1,z^2),$$
$$\left|\mathcal{A}_i^3[z^1](x,t)-\mathcal{A}_i^3[z^2](x,t)\right|\eta_i(x)e^{-at}\le$$
$$\le\frac{L}{a}\max\left\{\max_{i,j,y}\frac{\eta_i(x)}{\alpha_j(y)},\max_{i,j,y}\frac{\eta_i(x)}{\beta_j(y)},\max_{i,j,y}\frac{\eta_i(x)}{\eta_j(y)}\right\}\rho(z^1,z^2).$$

From the latest estimates, we get

$$\rho(\mathcal{A}[z^1],\mathcal{A}[z^2])\le\max_x\left\{L\max\left\{\max_{\substack{i\in I_0,\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)},\max_{\substack{i\in I_l,\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)},\max_{\substack{i\in I_0,\\ j}}\frac{\alpha_i(x)e^{-a\mu x}}{\eta_j(x)},\right.\right.$$
$$\left.\max_{\substack{i\in I_l,\\ j}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\eta_j(x)}\right\}+\frac{L}{a}\max\left\{\max_{i,j,y}\frac{\alpha_i(x)}{\alpha_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\beta_j(y)},\max_{i,j,y}\frac{\alpha_i(x)}{\eta_j(y)}\right\}+$$
$$+L\max\left\{\max_{i,\,j\notin I_0}\frac{\beta_i(x)}{\alpha_j(0)},\max_{i,j}\frac{\beta_i(x)}{\eta_i(x)}\right\}+\int_0^x L\max\left\{\max_{i,j}\frac{\beta_i(x)}{\alpha_j(y)},\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)},\max_{i,j}\frac{\beta_i(x)}{\eta_j(y)}\right\}dy+$$
$$\left.+\frac{L}{a}\max\left\{\max_{i,j,y}\frac{\eta_i(x)}{\alpha_j(y)},\max_{i,j,y}\frac{\eta_i(x)}{\beta_j(y)},\max_{i,j,y}\frac{\eta_i(x)}{\eta_j(y)}\right\}\right\}\rho(z^1,z^2).$$

Let us choose the weighted functions of the space metric so that $\mathcal{A}$ is a contractive operator on $\mathcal{Q}$, $\eta_j(x)=e^{pl}$ $(j\in\{1,\dots,n\})$,

$$\alpha_i(x)=\begin{cases}e^{px(l-x)}, & i\in I_0,\ i\in I_l;\\ e^{px}, & i\in I_0,\ i\notin I_l;\\ e^{p(l-x)}, & i\notin I_0,\ i\in I_l;\\ e^{pl}, & i\notin I_0,\ i\notin I_l,\end{cases}\qquad \beta_i(x)=\varepsilon e^{-px},\ i\in\{1,\dots,m\},$$

where $p > 1$, $0 < \varepsilon < 1$ are some parameters. Suppose that the following conditions hold

$$p \leq a\mu, \quad pl \leq a\mu, \quad e^{-pl} \leq \varepsilon, \quad p \geq 1/\varepsilon. \tag{9}$$

Then we have

$$\max_x \left\{ \max_{\substack{i\in I_0,\\ j\notin I_0}} \frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)} \right\} = \max_x \left\{ \max_{i\in I_0} \frac{\alpha_i(x)e^{-a\mu x}}{e^{pl}} \right\} =$$
$$= \max_x \left\{ e^{px(l-x)}e^{-a\mu x - pl},\ e^{px}e^{-a\mu x-pl} \right\} = e^{-pl};$$
$$\max_x \left\{ \max_{\substack{i\in I_l,\\ j\notin I_l}} \frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)} \right\} = \max_x \left\{ \max_{i\in I_l} \frac{\alpha_i(x)e^{-a\mu(l-x)}}{e^{pl}} \right\} =$$
$$= \max_x \left\{ e^{px(l-x)}e^{-a\mu(l-x)-pl},\ e^{p(l-x)}e^{-a\mu(l-x)-pl} \right\} = e^{-pl};$$
$$\max_x \left\{ \max_{\substack{i\in I_0,\\ j}} \frac{\alpha_i(x)e^{-a\mu x}}{\eta_j(x)} \right\} = \max_x \left\{ \max_{i\in I_0} \frac{\alpha_i(x)e^{-a\mu x}}{e^{pl}} \right\} =$$
$$= \max_x \left\{ e^{px(l-x)}e^{-a\mu x-pl},\ e^{px}e^{-a\mu x-pl} \right\} = e^{-pl};$$
$$\max_x \left\{ \max_{i\in I_l} \frac{\alpha_i(x)e^{-a\mu(l-x)}}{\eta_j(x)} \right\} = \max_x \left\{ \max_{i\in I_l} \frac{\alpha_i(x)e^{-a\mu(l-x)}}{e^{pl}} \right\} =$$
$$= \max_x \left\{ e^{px(l-x)}e^{-a\mu(l-x)-pl},\ e^{p(l-x)}e^{-a\mu(l-x)-pl} \right\} = e^{-pl}.$$

In addition, we obtain the following inequalities

$$\max_x \{ \max_{i,\, j\notin I_l} \frac{\beta_i(x)}{\alpha_j(0)} \} = \max_x \frac{\varepsilon e^{-px}}{e^{pl}} = \max_x \varepsilon e^{-px-pl} = \varepsilon e^{-pl};$$
$$\max_x \{ \max_{i,j} \frac{\beta_i(x)}{\eta_j(x)} \} = \max_x \frac{\varepsilon e^{-px}}{e^{pl}} = \varepsilon \max_x e^{-px-pl} = \varepsilon e^{-pl};$$
$$\max_x \int_0^x \max_{i,j} \frac{\beta_i(x)}{\alpha_j(y)} dy \leq \max_x \int_0^x \varepsilon e^{-px} dy \leq \max_x \left\{ \varepsilon e^{-px} x \right\} \leq \varepsilon l e^{-pl};$$
$$\max_x \int_0^x \max_{i,j} \frac{\beta_i(x)}{\beta_j(y)} dy = \max_x \int_0^x \frac{\varepsilon e^{-px}}{\varepsilon e^{-py}} dy = \max_x \int_0^x e^{p(y-x)} dy = \max_x \frac{1-e^{-px}}{p} \leq \frac{1}{p} \leq \varepsilon;$$
$$\max_x \int_0^x \max_{i,j} \frac{\beta_i(x)}{\eta_j(y)} dy \leq \max_x \int_0^x \frac{\varepsilon e^{-px}}{e^{pl}} dy \leq \max_x \left\{ \varepsilon e^{-px-pl} x \right\} \leq \varepsilon l e^{-2pl}.$$

As a result, we get the inequality

$$\rho(A[z^1], A[z^2]) \leq \Big(Le^{-pl} + \frac{L}{a} \max_x \Big\{ \max_{i,j,y} \frac{\alpha_i(x)}{\alpha_j(y)}, \max_{i,j,y} \frac{\alpha_i(x)}{\beta_j(y)}, \max_{i,j,y} \frac{\alpha_i(x)}{\eta_j(y)} \Big\} +$$
$$+ L\varepsilon + L\varepsilon + \frac{L}{a} \max_x \Big\{ \max_{i,j,y} \frac{\eta_i(x)}{\alpha_j(y)}, \max_{i,j,y} \frac{\eta_i(x)}{\beta_j(y)}, \max_{i,j,y} \frac{\eta_i(x)}{\eta_j(y)} \Big\} \Big) \rho(z^1, z^2).$$

We fix the sufficiently small value of the parameter ε and sufficiently big value of the parameter $p > 1$: ($e^{-pl} \leq \varepsilon$, $p \geq 1/\varepsilon$) to satisfy the condition $Le^{-pl} + 2L\varepsilon < 1/2$. Then the functions α_i, β_i, η_i are defined, and

$$\max_x \left\{ \max_{i,j,y} \frac{\alpha_i(x)}{\alpha_j(y)}, \max_{i,j,y} \frac{\alpha_i(x)}{\beta_j(y)}, \max_{i,j,y} \frac{\alpha_i(x)}{\eta_j(y)} \right\} \equiv M,$$
$$\max_x \left\{ \max_{i,j,y} \frac{\eta_i(x)}{\alpha_j(y)}, \max_{i,j,y} \frac{\eta_i(x)}{\beta_j(y)}, \max_{i,j,y} \frac{\eta_i(x)}{\eta_j(y)} \right\} \equiv K,$$

where M and K are constant. Finally, we fix the value of the parameter a large enough to satisfy conditions (9) and the inequality $L(M+K)/a < 1/2$. Then the operator $\mathcal{A}$ is a contractive operator on the elements of the space $\mathcal{Q}$ with the selected functions α_i, β_i, η_i and the parameter a. Thus, based on Banach's contractive mapping theorem, there exists a unique fixed point of the operator $\mathcal{A}$ in the space $\mathcal{Q}$, which is a generalized continuous solution of problem (1)–(3). □

Remark 1. By increasing the smoothness of initial data for problem (1)–(3) with the fulfillment of the first-order compatibility conditions, it is easy to prove the corresponding theorem on the global classical solvability of this problem.

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