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KRONECKER PRODUCT OF MATRICES AND SOLUTIONS OF SYLVESTER-TYPE MATRIX POLYNOMIAL EQUATIONS

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We investigate the solutions of the Sylvester-type matrix polynomial equation

$$A(\lambda)X(\lambda)+Y(\lambda)B(\lambda)=C(\lambda),$$

where $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ are the polynomial matrices with elements in a ring of polynomials $\mathcal{F}[\lambda]$, $\mathcal{F}$ is a field, $X(\lambda)$ and $Y(\lambda)$ are unknown polynomial matrices. Solving such a matrix equation is reduced to the solving a system of linear equations

$$G\left\|\begin{matrix}\mathbf{x}\\ \mathbf{y}\end{matrix}\right\|=\mathbf{c}$$

over a field $\mathcal{F}$. In this case, the Kronecker product of matrices is applied. In terms of the ranks of matrices over a field $\mathcal{F}$, which are constructed by the coefficients of the Sylvester-type matrix polynomial equation, the necessary and sufficient conditions for the existence of solutions $X_0(\lambda)$ and $Y_0(\lambda)$ of given degrees to the Sylvester-type matrix polynomial equation are established. The solutions of this matrix polynomial equation are constructed from the solutions of the linear equations system. As a consequence of the obtained results, we give the necessary and sufficient conditions for the existence of the scalar solutions X_0 and Y_0, whose entries are elements in a field $\mathcal{F}$, to the Sylvester-type matrix polynomial equation.

1. Introduction and preliminary results. Linear matrix equations, in particular the Sylvester-type matrix equations, over different domains appear in various branches of mathematics and applied problems of the stability theory, the control theory, the dynamical systems theory, and so on [1–4]. In the many problems, in addition to the solvability of such matrix equations and the finding of general solutions, solutions with certain properties are needed. The solutions of bounded degrees to the linear matrix polynomial equations are indicated in [5–9]. In [10], integer solutions of such matrix equations over quadratic rings are found. Applying standard forms of matrices over quadratic rings with respect to (z,k)-equivalence [11,12], solutions with bounded Euclidean norms are described in [11,13]. The particular and general solutions of such matrix equation with the diagonalizable pair of matrices (A,B) over commutative Bezout domains are described in [14]. The solutions with other properties are obtained in [15].

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The paper deals with the Sylvester-type matrix polynomial equation

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda) \tag{1}$$

and its solutions, where

$$A(\lambda) \in M(n, m, \mathcal{F}[\lambda]),\ \ B(\lambda) \in M(p, q, \mathcal{F}[\lambda]) \ \text{ and } \ C(\lambda) \in M(n, q, \mathcal{F}[\lambda])$$

are known matrices over a ring of polynomials $\mathcal{F}[\lambda]$, $\mathcal{F}$ is a field, $M(n, m, \mathcal{F}[\lambda])$ is the set of $n \times m$ matrices over $\mathcal{F}[\lambda]$. Matrices $X(\lambda) \in M(m, q, \mathcal{F}[\lambda])$ and $Y(\lambda) \in M(n, p, \mathcal{F}[\lambda])$ are unknown polynomial matrices. Roth [16] proved that the equation (1) over a field $\mathcal{F}$ and over a ring of polynomials $\mathcal{F}[\lambda]$ has a solution if and only if the block matrices

$$M = \begin{Vmatrix} A & C \\ 0 & B \end{Vmatrix} \quad \text{and} \quad N = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix}$$

are equivalent.

In the present work, we establish the necessary and sufficient conditions for the existence of solutions $X(\lambda), Y(\lambda)$ of given degrees $\deg X(\lambda) = k$, $\deg Y(\lambda) = l$ to the matrix polynomial equation (1). We construct such solutions of the matrix polynomial equation (1) from the solutions of the linear equations system over a field $\mathcal{F}$, which we get by applying the Kronecker product of matrices.

Recall that $A \otimes B$ is the Kronecker product [2, 17] of a $n \times m$ matrix $A = \|a_{ij}\|_{i,j}^{n,m}$ and a $p \times q$ matrix $B = \|b_{ij}\|_{i,j}^{p,q}$, namely, if it is a $np \times mq$ block matrix in the form

$$A \otimes B = \begin{Vmatrix} a_{11}B & \dots & a_{1m}B \\ \cdots & \cdots & \cdots \\ a_{n1}B & \dots & a_{nm}B \end{Vmatrix}.$$

Further, $\text{row}_i(A)$ denotes the ith row of the matrix A.

Lemma 1. *Let $A \in M(n, m, \mathcal{F})$, $B \in M(p, q, \mathcal{F})$, and $C \in M(n, q, \mathcal{F})$ be known matrices, $X \in M(m, q, \mathcal{F})$ and $Y \in M(n, p, \mathcal{F})$ be unknown matrices over a field $\mathcal{F}$.*

Matrix equation

$$AX + YB = C \tag{2}$$

is solvable if and only if the matrix equation

$$G \begin{Vmatrix} \mathbf{x} \\ \mathbf{y} \end{Vmatrix} = \mathbf{c} \tag{3}$$

is solvable, namely, if and only if $\operatorname{rank} G = \operatorname{rank} \begin{Vmatrix} G & \mathbf{c} \end{Vmatrix}$, *where*

$$G = \begin{Vmatrix} A \otimes I_q & I_n \otimes B^{\mathsf{T}} \end{Vmatrix}, \tag{4}$$

the columns $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{c}$ have the forms $\mathbf{x} = \begin{Vmatrix} \text{row}_1(X) & \text{row}_2(X) & \dots & \text{row}_m(X) \end{Vmatrix}^{\mathsf{T}}$, $\mathbf{y} = \begin{Vmatrix} \text{row}_1(Y) & \text{row}_2(Y) & \dots & \text{row}_n(Y) \end{Vmatrix}^{\mathsf{T}}$, $\mathbf{c} = \begin{Vmatrix} \text{row}_1(C) & \text{row}_2(C) & \dots & \text{row}_n(C) \end{Vmatrix}^{\mathsf{T}}$, I_k is the $k \times k$ identity matrix, the symbol T denotes the transposition of corresponding matrix.

Proof. It is known [2, 17] that the solving the matrix equation $AX = C$ is reduced to the solving the matrix equation $(A \otimes I_q)\mathbf{x} = \mathbf{c}$, and the solving the matrix equation $YB = C$ is reduced to the solving the matrix equation $(I_n \otimes B^{\mathsf{T}})\mathbf{y} = \mathbf{c}$. Thus, from matrix equation (2) we obtain the equation

$$\begin{Vmatrix} A \otimes I_q & I_n \otimes B^{\mathsf{T}} \end{Vmatrix} \begin{Vmatrix} \mathbf{x} \\ \mathbf{y} \end{Vmatrix} = \mathbf{c}$$

that has the form (3) where the matrix G is in the form (4).

Equation (3) over a field $\mathcal{F}$ has a solution if and only if $\operatorname{rank} G = \operatorname{rank} \begin{Vmatrix} G & \mathbf{c} \end{Vmatrix}$. □

2. The main results. Let $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ be known polynomial matrices from the Sylvester-type matrix polynomial equation (1). Write them in the form of matrix polynomials:

$$\begin{aligned} A(\lambda) &= A_r\lambda^r + \ldots + A_1\lambda + A_0,\ \deg A(\lambda) = r,\ A_i \in M(n,m,\mathcal{F}),\ i \in \{0,1,\ldots,r\}, \\ B(\lambda) &= B_s\lambda^s + \ldots + B_1\lambda + B_0,\ \deg B(\lambda) = s,\ B_j \in M(p,q,\mathcal{F}),\ j \in \{0,1,\ldots,s\}, \\ C(\lambda) &= C_t\lambda^t + \ldots + C_1\lambda + C_0,\ \deg C(\lambda) = t,\ C_f \in M(n,q,\mathcal{F}),\ f \in \{0,1,\ldots,t\}. \end{aligned} \tag{5}$$

Matrices $X(\lambda)$, $Y(\lambda)$ are unknown polynomial matrices and write them in the form of matrix polynomials

$$\begin{aligned} X(\lambda) &= X_k\lambda^k + \ldots + X_1\lambda + X_0,\ \deg X(\lambda) = k, \\ Y(\lambda) &= Y_l\lambda^l + \ldots + Y_1\lambda + Y_0,\ \deg Y(\lambda) = l, \end{aligned}$$

where $X_v,\ v \in \{0,1,\ldots,k\},\ Y_w,\ w \in \{0,1,\ldots,l\}$, have appropriate sizes. Then, it is obvious that the necessary condition for the existence of solution $X(\lambda)$, $Y(\lambda)$ of given degrees k, l to the matrix equation (1) is

$$\max\{\deg A(\lambda) + \deg X(\lambda), \deg B(\lambda) + \deg Y(\lambda)\} \geq \deg C(\lambda),$$

i.e., $\max\{r+k, s+l\} \geq t$.

Using the Kronecker product of matrices, from the coefficients A_i and B_j of matrix polynomials $A(\lambda)$ and $B(\lambda)$ (5), we construct the block matrix

$$G_A = \left\| \begin{array}{cccccc} A_r \otimes I_q & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ A_{r-1} \otimes I_q & A_r \otimes I_q & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_0 \otimes I_q & A_1 \otimes I_q & A_2 \otimes I_q & \ldots & A_{k-1} \otimes I_q & A_k \otimes I_q \\ \mathbf{0} & A_0 \otimes I_q & A_1 \otimes I_q & \ldots & A_{k-2} \otimes I_q & A_{k-1} \otimes I_q \\ \mathbf{0} & \mathbf{0} & A_0 \otimes I_q & \ldots & A_{k-3} \otimes I_q & A_{k-2} \otimes I_q \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & A_0 \otimes I_q \end{array} \right\| \tag{6}$$

that contains $(r+k+1)$ block rows and $(k+1)$ block columns, i.e., it is the $((r+k+1)nq) \times ((k+1)mq)$ matrix, moreover, all $A_i = \mathbf{0}$ for $i > r$ if $k > r$, and

$$G_B = \left\| \begin{array}{cccccc} I_n \otimes B_s^\intercal & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ I_n \otimes B_{s-1}^\intercal & I_n \otimes B_s^\intercal & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ I_n \otimes B_0^\intercal & I_n \otimes B_1^\intercal & I_n \otimes B_2^\intercal & \ldots & I_n \otimes B_{l-1}^\intercal & I_n \otimes B_l^\intercal \\ \mathbf{0} & I_n \otimes B_0^\intercal & I_n \otimes B_1^\intercal & \ldots & I_n \otimes B_{l-2}^\intercal & I_n \otimes B_{l-1}^\intercal \\ \mathbf{0} & \mathbf{0} & I_n \otimes B_0^\intercal & \ldots & I_n \otimes B_{l-3}^\intercal & I_n \otimes B_{l-2}^\intercal \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & I_n \otimes B_0^\intercal \end{array} \right\| \tag{7}$$

that contains $(s+l+1)$ block rows and $(l+1)$ block columns, i.e., it is the $((s+l+1)nq) \times ((l+1)np)$ matrix, moreover, all $B_j = \mathbf{0}$ for $j > s$ if $l > s$, where $\mathbf{0}$ is a zero matrix of the suitable size.

From matrices G_A and G_B in the forms (6) and (7), respectively, we construct the matrices $\mathbf{G}_i$, $i \in \{1,2,3\}$, of the size $(\max\{r+k, s+l\}+1)nq \times ((k+1)mq+(l+1)np)$. With respect to relationships between the degrees of coefficients $A(\lambda)$, $B(\lambda)$ and solutions $X(\lambda)$, $Y(\lambda)$, the matrices G_A and G_B can have different amount of rows. Therefore, we get the following cases:
(i) if

$$r+k = s+l, \tag{8}$$

then

$$\mathbf{G}_1 = \left\| G_A \quad G_B \right\|, \tag{9}$$

(ii) if

$$r+k > s+l, \tag{10}$$

then

$$\mathbf{G}_2 = \left\| G_A \quad \widetilde{G}_B \right\|, \tag{11}$$

where the matrix $\widetilde{G}_B = \left\| \begin{matrix} \mathbf{0} \\ G_B \end{matrix} \right\|$, $\mathbf{0}$ is a zero matrix with $((r+k)-(s+l))nq$ rows,
(iii) if

$$r+k < s+l, \tag{12}$$

then

$$\mathbf{G}_3 = \left\| \widetilde{G}_A \quad G_B \right\|, \tag{13}$$

where the matrix $\widetilde{G}_A = \left\| \begin{matrix} \mathbf{0} \\ G_A \end{matrix} \right\|$, $\mathbf{0}$ is a zero matrix with $((s+l)-(r+k))nq$ rows.

From coefficients C_f of the matrix polynomial $C(\lambda)$ in (5) we construct the column $\mathbf{c}$

$$\mathbf{c} = \left\| \mathbf{c}_t \quad \ldots \quad \mathbf{c}_1 \quad \mathbf{c}_0 \right\|^{\mathsf{T}}, \tag{14}$$

where $\mathbf{c}_f = \left\| \mathrm{row}_1(C_f) \quad \mathrm{row}_2(C_f) \quad \ldots \quad \mathrm{row}_n(C_f) \right\|$, $f \in \{0,1,\ldots,t\}$.

Theorem 1. *Suppose that in the Sylvester-type matrix polynomial equation* (1)

$$\max\{\deg A(\lambda) + \deg X(\lambda), \deg B(\lambda) + \deg Y(\lambda)\} = \deg C(\lambda),$$

i.e., $\max\{r+k, s+l\} = t$. *Then the matrix equation* (1) *has solutions* $X(\lambda)$, $Y(\lambda)$ *of degrees* k, l, *respectively, if and only if* $\operatorname{rank} \mathbf{G}_i = \operatorname{rank} \left\| \ \mathbf{G}_i \quad \mathbf{c} \ \right\|$, *where the matrix* $\mathbf{G}_i$, $i \in \{1,2,3\}$, *has one of the forms* (9), (11) *or* (13) *with respect to relationships* (8), (10) *or* (12) *between the degrees of coefficients* $A(\lambda)$, $B(\lambda)$ *and solutions* $X(\lambda)$, $Y(\lambda)$, *the column* $\mathbf{c}$ *is of form* (14).

Proof. According to the condition of Theorem 1, $\max\{r+k, s+l\} = t$, and considering the representations of the matrices $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ in the form of matrix polynomials (5), from the matrix equation (1) we obtain the following system of linear matrix equations over a field $\mathcal{F}$

$$\sum_{i=0}^{f} A_{f-i}X_i + Y_{f-i}B_i = C_f, \quad f \in \{0,1,\ldots,t\}, \tag{15}$$

where $A_i = \mathbf{0}$ for all $i > r$ and $B_j = \mathbf{0}$ for all $j > s$.

We will apply Lemma 1 to the equations of this system. Thus, solving the system (15) is reduced to solving the matrix equation

$$\mathbf{G}\begin{Vmatrix}\mathbf{x}\\ \mathbf{y}\end{Vmatrix} = \mathbf{c} \tag{16}$$

over a field $\mathcal{F}$, where columns $\mathbf{x}$, $\mathbf{y}$ have the following forms

$$\mathbf{x} = \begin{Vmatrix}\mathbf{x}_k & \dots & \mathbf{x}_1 & \mathbf{x}_0\end{Vmatrix}^\mathsf{T}, \quad \mathbf{y} = \begin{Vmatrix}\mathbf{y}_l & \dots & \mathbf{y}_1 & \mathbf{y}_0\end{Vmatrix}^\mathsf{T}, \tag{17}$$

and

$$\mathbf{x}_v = \begin{Vmatrix}\mathrm{row}_1(X_v) & \mathrm{row}_2(X_v) & \dots & \mathrm{row}_m(X_v)\end{Vmatrix}^\mathsf{T}, \; v \in \{0, 1, \dots, k\},$$
$$\mathbf{y}_w = \begin{Vmatrix}\mathrm{row}_1(Y_w) & \mathrm{row}_2(Y_w) & \dots & \mathrm{row}_n(Y_w)\end{Vmatrix}^\mathsf{T}, \; w \in \{0, 1, \dots, l\},$$

the column $\mathbf{c}$ is in the form (14) and the matrix $\mathbf{G} = \mathbf{G}_1$ has the form (9) if the condition (8) between the degrees of coefficients $A(\lambda)$, $B(\lambda)$ and solutions $X(\lambda)$, $Y(\lambda)$ is fulfilled.

If the condition (10) is fulfilled, then the system (15) contains the equations

$$\sum_{j=0}^{f} A_{f-j}X_j = C_f, \quad f \in \{s+l+1, s+l+2, \dots, t\},$$

and so, the matrix $\mathbf{G} = \mathbf{G}_2$ has the form (11).

If the condition (12) is fulfilled, then the system (15) contains the equations

$$\sum_{j=0}^{f} Y_{f-j}B_j = C_f, \quad f \in \{r+k+1, r+k+2, \dots, t\},$$

and so, the matrix $\mathbf{G} = \mathbf{G}_3$ has the form (13).

It is clear that the matrix equation (16) has a solution if and only if $\operatorname{rank} \mathbf{G} = \operatorname{rank} \begin{Vmatrix}\mathbf{G} & \mathbf{c}\end{Vmatrix}$. This completes the proof. □

Further the ith block row of matrix G_A in form (6) will be denoted by $\mathbf{Row}_i(G_A)$, $i \in \{1, 2, \dots, (r+k+1)\}$, i.e.,

$$\underset{1}{\mathbf{Row}}(G_A) = \begin{Vmatrix}A_r \otimes I_q & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\end{Vmatrix},$$
$$\underset{2}{\mathbf{Row}}(G_A) = \begin{Vmatrix}A_{r-1} \otimes I_q & A_r \otimes I_q & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\end{Vmatrix},$$

and so on. Analogously $\mathbf{Row}_j(G_B)$ denotes the jth block row of matrix G_B in form (7), $j \in \{1, 2, \dots, (s+l+1)\}$.

Therefore, we rewrite matrices $\mathbf{G}_i$, $i \in \{1, 2, 3\}$, from (9), (11), (13) as follows

(i) if condition (8) is satisfied, then

$$\mathbf{G}_1 = \begin{Vmatrix}\mathbf{Row}_1(G_A) & \mathbf{Row}_1(G_B)\\ \mathbf{Row}_2(G_A) & \mathbf{Row}_2(G_B)\\ \dots & \dots\\ \mathbf{Row}_{r+k+1}(G_A) & \mathbf{Row}_{r+k+1}(G_B)\end{Vmatrix}, \tag{18}$$

(ii) if condition (10) is satisfied, then

$$\mathbf{G}_2 = \begin{Vmatrix}\mathbf{Row}_1(G_A) & \mathbf{0}\\ \mathbf{Row}_2(G_A) & \mathbf{0}\\ \dots & \dots\\ \mathbf{Row}_{(r+k)-(s+l)}(G_A) & \mathbf{0}\\ \mathbf{Row}_{(r+k)-(s+l)+1}(G_A) & \mathbf{Row}_1(G_B)\\ \dots & \dots\\ \mathbf{Row}_{r+k+1}(G_A) & \mathbf{Row}_{s+l+1}(G_B)\end{Vmatrix}, \tag{19}$$

(iii) if condition (12) is satisfied, then

$$\mathbf{G}_3 = \left\| \begin{array}{cc} \mathbf{0} & \mathbf{Row}_1(G_B) \\ \mathbf{0} & \mathbf{Row}_2(G_B) \\ \dots & \dots \\ \mathbf{0} & \mathbf{Row}_{(s+l)-(r+k)}(G_B) \\ \mathbf{Row}_1(G_A) & \mathbf{Row}_{(s+l)-(r+k)+1}(G_B) \\ \dots & \dots \\ \mathbf{Row}_{r+k+1}(G_A) & \mathbf{Row}_{s+l+1}(G_B) \end{array} \right\|, \tag{20}$$

where $\mathbf{0}$ is a zero matrix of suitable size.

Thus, $\mathbf{Row}_j(\mathbf{G}_i)$ is the jth block row of matrix $\mathbf{G}_i$, $i \in \{1,2,3\}$, that has one of the forms (18), (19) or (20) with respect to relationships (8), (10) or (12) between the degrees of coefficients $A(\lambda)$, $B(\lambda)$ and solutions $X(\lambda)$, $Y(\lambda)$, $j \in \{1,2,\dots,(\max\{r+k,s+l\}+1)\}$.

Theorem 2. *Suppose that in the Sylvester-type matrix polynomial equation* (1)

$$\max\{\deg A(\lambda)+\deg X(\lambda), \deg B(\lambda)+\deg Y(\lambda)\} > \deg C(\lambda),$$

i.e., $\max\{r+k,s+l\} > t$. *Then the matrix equation* (1) *has solutions* $X(\lambda)$, $Y(\lambda)$ *of degrees* k, l, *respectively, if and only if*

$$\operatorname{rank} \left\| \begin{array}{c} \mathbf{Row}_1(\mathbf{G}_i) \\ \mathbf{Row}_2(\mathbf{G}_i) \\ \dots \\ \mathbf{Row}_{(\max\{r+k,s+l\}-t)}(\mathbf{G}_i) \end{array} \right\| < (k+1)mq + (l+1)np \tag{21}$$

and

$$\operatorname{rank} \left\| \begin{array}{c} \mathbf{Row}_{(\max\{r+k,s+l\}-t+1)}(\mathbf{G}_i) \\ \mathbf{Row}_{(\max\{r+k,s+l\}-t+2)}(\mathbf{G}_i) \\ \dots \\ \mathbf{Row}_{(\max\{r+k,s+l\}+1)}(\mathbf{G}_i) \end{array} \right\| = \operatorname{rank} \left\| \begin{array}{c|c} \mathbf{Row}_{(\max\{r+k,s+l\}-t+1)}(\mathbf{G}_i) & \\ \mathbf{Row}_{(\max\{r+k,s+l\}-t+2)}(\mathbf{G}_i) & \\ \dots & \mathbf{c} \\ \mathbf{Row}_{(\max\{r+k,s+l\}+1)}(\mathbf{G}_i) & \end{array} \right\|, \tag{22}$$

where matrix $\mathbf{G}_i$, $i \in \{1,2,3\}$, *has one of forms* (18), (19) *or* (20) *with respect to relationships* (8), (10) *or* (12) *between the degrees of coefficients* $A(\lambda)$, $B(\lambda)$ *and solutions* $X(\lambda)$, $Y(\lambda)$, *the column* $\mathbf{c}$ *is of form* (14).

Proof. As in the proof of Theorem 1, solving the matrix equation (1) is reduced to solving the system of linear matrix equations (15) over a field $\mathcal{F}$ and this system has a solution if and only if the matrix equation (16) over a field $\mathcal{F}$ has a solution.

According to the condition of Theorem 2, the equation (16) has the form

$$\left\| \begin{array}{c} \mathbf{Row}_{(\max\{r+k,s+l\}-t+1)}(\mathbf{G}_i) \\ \mathbf{Row}_{(\max\{r+k,s+l\}-t+2)}(\mathbf{G}_i) \\ \dots \\ \mathbf{Row}_{(\max\{r+k,s+l\}+1)}(\mathbf{G}_i) \end{array} \right\| \left\| \begin{array}{c} \mathbf{x} \\ \mathbf{y} \end{array} \right\| = \mathbf{c},$$

where the matrix $\mathbf{G}_i$, $i \in \{1,2,3\}$, has one of forms (18), (19) or (20) with respect to relationships (8), (10) or (12) between the degrees of coefficients $A(\lambda)$, $B(\lambda)$ and solutions $X(\lambda)$, $Y(\lambda)$. This equation has a solution if and only if condition (22) is fulfilled.

Since $\max\{r+k,s+l\} > t$, it immediately follows that system (15) will be supplemented with matrix equations in the form

$$\sum_{i=0}^{f} A_{f-i}X_i + Y_{f-i}B_i = 0, \quad f \in \{t+1, t+2, \dots, \max\{r+k, s+l\}+1\}.$$

Applying Lemma 1 to such equations, we obtain that their solving is reduced to solving the matrix equation

$$\left\| \begin{matrix} \mathbf{Row}_1(\mathbf{G}_i) \\ \mathbf{Row}_2(\mathbf{G}_i) \\ \cdots \\ \mathbf{Row}_{(\max\{r+k,s+l\}-t)}(\mathbf{G}_i) \end{matrix} \right\| \left\| \begin{matrix} \mathbf{x} \\ \mathbf{y} \end{matrix} \right\| = \mathbf{0},$$

where columns $\mathbf{x}$ and $\mathbf{y}$ are in form (17), and the matrix $\mathbf{G}_i$, $i \in \{1,2,3\}$, has one of forms (18), (19) or (20) with respect to relationships (8), (10) or (12) between the degrees of coefficients $A(\lambda)$, $B(\lambda)$ and solutions $X(\lambda)$, $Y(\lambda)$, $\mathbf{0}$ is a zero column of suitable size. This equation has a nontrivial solution if and only if the condition (21) is fulfilled. □

In the next assertion, we formulate the necessary and sufficient conditions for the existence of scalar solutions to the matrix polynomial equation (1), namely, the entries of such solutions are elements in a field $\mathcal{F}$.

Corollary 1. *Sylvester-type matrix polynomial equation* (1) *whose coefficients* $A(\lambda)$, $B(\lambda)$, *and* $C(\lambda)$ *have degrees* $\deg A(\lambda) = r$, $\deg B(\lambda) = s$, *and* $\deg C(\lambda) = t$, *has scalar solutions* $X(\lambda) = X_0$, $Y(\lambda) = Y_0$, *i.e.,* X_0, Y_0 *are the matrices with elements in a field* $\mathcal{F}$,
1) if $\max\{r,s\} = t$, *if and only if* $\operatorname{rank} \mathbf{G}_i = \operatorname{rank} \left\| \mathbf{G}_i \quad \mathbf{c} \right\|$,
2) if $\max\{r,s\} > t$, *if and only if*

$$\operatorname{rank} \left\| \begin{matrix} \mathbf{Row}_1(\mathbf{G}_i) \\ \mathbf{Row}_2(\mathbf{G}_i) \\ \cdots \\ \mathbf{Row}_{(\max\{r,s\}-t)}(\mathbf{G}_i) \end{matrix} \right\| < nq(\max\{r,s\} - t)$$

and

$$\operatorname{rank} \left\| \begin{matrix} \mathbf{Row}_{(\max\{r,s\}-t+1)}(\mathbf{G}_i) \\ \mathbf{Row}_{(\max\{r,s\}-t+2)}(\mathbf{G}_i) \\ \cdots \\ \mathbf{Row}_{(\max\{r,s\}+1)}(\mathbf{G}_i) \end{matrix} \right\| = \operatorname{rank} \left\| \begin{matrix} \mathbf{Row}_{(\max\{r,s\}-t+1)}(\mathbf{G}_i) \\ \mathbf{Row}_{(\max\{r,s\}-t+2)}(\mathbf{G}_i) \\ \cdots \\ \mathbf{Row}_{(\max\{r,s\}+1)}(\mathbf{G}_i) \end{matrix} \quad \mathbf{c} \right\|,$$

where the matrix $\mathbf{G}_i$, $i \in \{1,2,3\}$, *has one of the forms*

(i) if $r = s = t$, *then* $\mathbf{G}_1 = \left\| \begin{matrix} A_r \otimes I_q & I_n \otimes B_r^{\intercal} \\ A_{r-1} \otimes I_q & I_n \otimes B_{r-1}^{\intercal} \\ A_{r-2} \otimes I_q & I_n \otimes B_{r-2}^{\intercal} \\ \cdots & \cdots \\ A_0 \otimes I_q & I_n \otimes B_0^{\intercal} \end{matrix} \right\|$, *(ii) if* $r > s$, *then*

$\mathbf{G}_2 = \left\| \begin{matrix} A_r \otimes I_q & \mathbf{0} \\ \cdots & \cdots \\ A_{r-s} \otimes I_q & \mathbf{0} \\ A_{r-s+1} \otimes I_q & I_n \otimes B_s^{\intercal} \\ \cdots & \cdots \\ A_0 \otimes I_q & I_n \otimes B_0^{\intercal} \end{matrix} \right\|$, *(iii) if* $r < s$, *then* $\mathbf{G}_3 = \left\| \begin{matrix} \mathbf{0} & I_n \otimes B_s^{\intercal} \\ \cdots & \cdots \\ \mathbf{0} & I_n \otimes B_{s-r}^{\intercal} \\ A_r \otimes I_q & I_n \otimes B_{s-r+1}^{\intercal} \\ \cdots & \cdots \\ A_0 \otimes I_q & I_n \otimes B_0^{\intercal} \end{matrix} \right\|$,

the column $\mathbf{c}$ *is of form* (14).

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