D. C. Pramanik, A. Sarkar

UNIQUENESS OF SHIFT AND DERIVATIVES OF MEROMORPHIC FUNCTIONS

D. C. Pramanik, A. Sarkar. *Uniqueness of shift and derivatives of meromorphic functions*, Mat. Stud. **61** (2024), 160–167.

This paper addresses the uniqueness problem concerning the j-th derivative of a meromorphic function $f(z)$ and the k-th derivative of its shift, $f(z+c)$, where j, k are integers with $0 \leq j < k$. In this regard, our work surpasses the achievements of [2], as we have improved upon the existing results and provided a more refined understanding of this specific aspect. We give some illustrative examples to enhance the realism of the obtained outcomes.

Denote by $E(a, f)$ the set of all zeros of $f-a$, where each zero with multiplicity m is counted m times. In the paper proved, in particular, the following statement:
Let $f(z)$ be a non-constant meromorphic function of finite order, c be a non-zero finite complex number and j, k be integers such that $0 \leq j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ have the same $a-$points for a finite value $a(\neq 0)$ and satisfy conditions

$$E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c)) \quad \text{and} \quad E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z)),$$

then $f^{(j)}(z) \equiv f^{(k)}(z+c)$ (Theorem 6).

1. Introduction. Consider two non-constant meromorphic functions, denoted as f and g, defined in the entire complex plane $\mathbb{C}$. We use the term “meromorphic” to describe well-behaved functions everywhere except at isolated points, where they might have poles.

If there exists a complex number a (including the point at infinity) such that both functions f and g have the same a-points, counting their multiplicities, we say that f and g share the value a under the *counting multiplicities* (CM) condition. On the other hand, if they only share the locations of a-points without considering their multiplicities, we say that f and g share the value a under the *ignoring multiplicities* (IM) condition.

While the conventional notations of the value distribution theory are covered comprehensively in [13], we will introduce specific notes essential for our subsequent discussion.

L. A. Rubel, C.-C. Yang [11] have explored the uniqueness of non-constant entire functions sharing two values with their first derivative. Subsequent work by E. Mues, N. Steinmetz [8], along with G. G. Gundersen [6], refined and expanded this idea to include meromorphic functions. This progression culminated in the establishment of the following theorem.

Theorem 1 ([11]). *Let f be a non-constant meromorphic function, and let a and b be two distinct finite values. If f and f' share a and b CM, then $f \equiv f'$.*

2020 *Mathematics Subject Classification:* 30D30, 30D35.
Keywords: order; shifts; partial sharing; Nevanlinna theory.
doi:10.30970/ms.61.2.160-167

In Theorem 1, Gundersen [6] presented a counter-example indicating that two shared values under the CM condition cannot be reduced to 1 CM and 1 IM. However, the condition of 2 CM can be replaced by 3 IM, as demonstrated in [5], [8]. Furthermore, Frank and Weissenborn [4] confirmed the continued validity of the conclusion when substituting f' with a higher-order derivative $f^{(k)}$.

Theorem 2 ([4])**.** *Let f be a non-constant entire function and $k \geq 2$ be a positive integer. If f shares two distinct finite values a and b CM with $f^{(k)}$, then $f \equiv f^{(k)}$.*

The paper [7] investigates the case where $f(z+c)$ shares two different CM values and one other IM value with $f(z)$.

In the article [10], in particular, the following **question** was considered: *Under what conditions does the equality $f'(z) = f(z+c)$ hold?*

Theorem 3 ([10])**.** *Let $f(z)$ be a transcendental entire function of finite order and a be a non-zero complex constant. If $f'(z)$ and $f(z+c)$ share $0, a$ CM, then $f'(z) \equiv f(z+c)$.*

Denote by $E(a, f)$ the set of all zeros of $f - a$, where each zero with multiplicity m is counted m times. Similarly, we denote by $\overline{E}(a, f)$ the set of zeros of $f - a$, where each zero is counted only once. If $\overline{E}(a, f) \subseteq \overline{E}(a, g)$, then we say that $f(z)$ *partially shares* a with $g(z)$. If $E(a, f) \subset E(a, g)$, then we say that $f(z)$ and $g(z)$ *partially shares* a CM.

In [1, 2, 9], the question formulated above is considered using the concept of partially shared values.

Their findings revealed that when $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share one value, IM, along with two values that are partially shared, the outcome regarding uniqueness remains valid. This conclusion is subject to additional assumptions or hypotheses outlined in their study.

Theorem 4 ([2])**.** *Let $f(z)$ be a transcendental entire function of finite order, and let c be a non-zero finite complex number, and j, k be integers with $0 \leq j < k$. Suppose that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a \neq 0$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$. If $N(r, \frac{1}{f^{(j)}(z)}) + \overline{N}(r, \frac{1}{f(z)}) = S(r, f)$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

Chen and Huang [2] further demonstrated that if the sharing of values IM is replaced with CM, the result regarding uniqueness still holds. Notably, this holds without requiring the additional hypotheses stated in the previous theorem.

Theorem 5 ([2])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order, and c be a non-zero finite complex number and j, k be integers with $0 \leq j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a(\neq 0)$ CM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

2. Main results. In this context, our study confirmed that the result presented in Theorem 4 remains valid without the additional conditions stated in this theorem.

Theorem 6. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let c be a non-zero finite complex number and j, k be integers with $0 \leq j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a(\neq 0)$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

Remark 1. The following examples are relevant to be more realistic about Theorem 6 and the validity of the condition in the theorem.

Example 1. Consider the function $f(z) = e^{-z}$ and set $c = \pi i$. When we take $j = 1$ and $k = 4$, we notice that both $f^{(j)}(z)$ and $f^{(k)}(z+c)$ meet the conditions outlined in Theorem 6. This leads us to the conclusion that $f^{(j)}(z)$ is indeed equivalent to $f^{(k)}(z+c)$.

Remark 2. The sharpness of the condition $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$ in Theorem 6 becomes evident when we consider the following example.

Example 2. Consider the function $f(z) = \frac{2}{1-e^{-2z}}$ and set $c = \pi i$. Choose $j = 0$ and $k = 1$. Despite the fact that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share 1 IM and fulfill the inclusion $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$, they are not equivalent. This discrepancy underscores the significance of the condition $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$ in Theorem 6. In this specific example, the absence of this condition results in the non-equivalence of $f^{(j)}(z)$ and $f^{(k)}(z+c)$.

Remark 3. The sharpness of the condition "$f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a IM, $a \in \mathbb{C}\backslash\{0\}$" in Theorem 6 becomes apparent through the following example.

Example 3. Consider the function $f(z) = e^{z}$ and set $c = \pi i$. Let us take $j = 1$ and $k = 2$. Here, $f^{(j)}(z) = e^{z}$ and $f^{(k)}(z+c) = -e^{z}$. It is noticeable that they fulfill the conditions $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$. However, despite meeting these conditions, $f^{(j)}(z)$ and $f^{(k)}(z+c)$ are not equivalent. This is because they do not share a IM.

Given a positive integer l, let us denote by $E_{l)}(a, f)$ the set of all a-points of the function $f(z)$ with multiplicity at most l. Here, each a-point of f with multiplicity $m \leq l$ is counted m times. The set $\overline{E}_{l)}(a, f)$ represents the a-points of $f(z)$ with multiplicity at most l, but each a-point is counted only once. Furthermore, we define the reduced counting function $\overline{N}_{l)}(r, \frac{1}{f-a})$ associated with the set $\overline{E}_{l)}(a, f)$.

Similarly, we define another reduced counting function $\overline{N}_{(l}(r, \frac{1}{f-a})$, which corresponds to the a-points of f having multiplicities greater than l.

Theorem 7. *Let $f(z)$ be a non-constant meromorphic function of finite order, and c be a non-zero finite complex number, and j, k be integers with $0 \leq j < k$. If for $l > 1$, $E_{l)}(r, f^{(j)}(z)) = E_{l)}(r, f^{(k)}(z+c))$ and satisfy*

$$E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c)) \text{ and } E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z)),$$

then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.

3. Lemmas. The subsequent lemmas from this paper are pertinent to the forthcoming discussions and are invoked in the subsequent analysis.

Lemma 1 ([12])**.** *Let $f(z)$ be a non-constant meromorphic function and $a_i(z)$ are meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, ..., n$, where $a_n(z) \not\equiv 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([3])**.** *Let $f(z)$ be a meromorphic function of finite order and $c \in \mathbb{C}$. Then we have*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+c)}\Big) = S(r, f),$$

where $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E with finite linear measure.

Lemma 3 ([3])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$
$$N(r, \infty; f(z+c)) = N(r, \infty; f) + S(r, f), \quad N(r, 0; f(z+c)) = N(r, 0; f(z)) + S(r, f),$$
$$\overline{N}(r, \infty; f(z+c)) = \overline{N}(r, \infty; f) + S(r, f), \quad \overline{N}(r, 0; f(z+c)) = \overline{N}(r, 0; f(z)) + S(r, f).$$

Lemma 4. *Let $f(z)$ be a non-constant meromorphic function of finite order, and j, k be integers with $0 \leq j < k$. If $f^{(k)}(z+c)$ and $f^{(j)}(z)$ satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then f is transcendental.*

Proof. On the contrary, suppose that $f(z)$ is a non-constant rational function. Set

$$f(z) = \frac{P(z)}{Q(z)}, \tag{1}$$

where $P(z)$ and $Q(z)$ are two mutually prime polynomials. Differentiating (1) we get

$$f'(z) = \frac{P'Q - PQ'}{Q^2} = \frac{P_1(z)}{Q_1(z)}.$$

Similarly, differentiating (1) s-th times we get

$$f^{(s)}(z) = \frac{P'_{s-1}Q_{s-1} - P_{s-1}Q'_{s-1}}{Q^2_{s-1}} = \frac{P_s(z)}{Q_s(z)}.$$

So, $f^{(k)}(z+c) = P_k(z+c)/Q_k(z+c)$. Since, $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, we have

$$E(0, Q_k(z+c)) \subset E(0, Q_j(z)). \tag{2}$$

If z be a pole of f, then z must be a pole of $f^{(k)}$ and $f^{(j)}$. Let z_0 be a zero of $Q_j(z)$, then we have $Q_j(z_0) = 0$, so, $Q_k(z_0) = 0$.

Now we can write $Q_k(z_0) = 0$ as $Q_k((z_0 - c) + c) = 0$, so $z_0 - c$ is a zero of $Q_k(z+c)$. So using condition (2) we can conclude that $Q_j(z_0 - c) = 0$.

Now, from $Q_j(z_0 - c) = 0$, we have $Q_k(z_0 - c) = 0$, which implies $Q_j(z_0 - 2c) = 0$. Continuing inductively, we get $Q_j(z_0 - nc) = 0$, which is impossible. Therefore, we can deduce that $Q(z)$ is a constant. Thus, $f(z)$ is a polynomial.

Again, it is given f is non-constant, so we can see that $\deg(f^{(k)}(z+c)) < \deg(f^{(j)}(z+c))$, which leads to a contradiction because we are given that $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$. Hence, f is transcendental. □

In this paper, we will use several notations that are pertinent to our study.

Let a be a complex number or infinity. We denote by

(i) $N_0\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) = a\right)$ the *reduced counting function* of common a-points of $f^{(k)}(z+c)$ and $f^{(j)}(z)$ of different multiplicities.

(ii) $N_E\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) = a\right)$ denotes the *counting function* of common a-points of $f^{(k)}(z+c)$ and $f^{(j)}(z)$ of equal multiplicities.

(iii) $N_0\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) \neq a\right)$ the counting function of a-points of $f^{(k)}(z+c)$ which are not the a-points of $f^{(j)}(z)$.

Now we define the following function

$$\psi(z) = \frac{f^{(k)}(z+c)}{f^{(j)}(z)}. \tag{3}$$

Lemma 5. *Suppose $f(z)$ is a non-constant meromorphic function of finite order, and let ψ be defined in (3). If $f^{(k)}(z+c)$ and $f^{(j)}(z)$ satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then the following conclusions can be drawn:*
(i) $\psi(z)$ is entire and $T(r,\psi) = S(r,f)$.
(ii) $S(r,f) = S(r, f^{(k)}(z+c)) = S(r, f^{(k)}(z)) = S(r, f^{(j)}(z))$.
(iii) $N_0\left(r, 0; f^{(k)}(z+c)|f^{(j)}(z) = 0\right) + N\left(r, 0; f^{(k)}(z+c)|f^{(j)}(z) \neq 0\right) = S(r,f)$ and $N_0(r, f^{(k)}(z+c)|f^{(j)}(z) = \infty) + N(r, f^{(k)}(z+c)|f^{(j)}(z) \neq \infty) = S(r,f)$.
(iv) $T(r, f^{(k)}(z+c)) = T(r, f^{(j)}(z)) + S(r,f)$.
(v) $\overline{N}(r,\infty; f^{(j)}(z)) = \overline{N}(r,\infty; f^{(k)}(z+c)) = S(r,f)$.

Proof. Based on the assumption that $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, allows us to deduce that $\psi(z)$ is entire.
(i) By applying the lemma concerning the logarithmic derivative and making use of lemma 2, we can establish the following.

$$m(r,\psi(z)) = m\Big(r, \frac{f^{(k)}(z+c)}{f^{(j)}(z)}\Big) \leq m\Big(r, \frac{f^{(k)}(z+c)}{f^{(k)}(z)}\Big) + m\Big(r, \frac{f^{(k)}(z)}{f^{(j)}(z)}\Big) = S(r,f).$$

Therefore, it follows that $T(r,\psi(z)) = S(r,f)$.
(ii) Now, equation (3) can be rewritten as $f^{(k)}(z+c) = \psi(z) f^{(j)}(z)$. So we have

$$T(r, f^{(k)}(z)) = T(r, f^{(k)}(z+c)) + S(r,f) \leq$$
$$T(r,\psi) + T(r, f^{(j)}(z)) + S(r,f) = T(r, f^{(j)}(z)) + S(r,f).$$

Similarly, we have $T(r, f^{(j)}(z)) \leq T(r, f^{(k)}(z+c)) + S(r,f) = T(r, f^{(k)}(z)) + S(r,f)$. Hence, $S(r,f) = S(r, f^{(k)}(z+c)) = S(r, f^{(k)}(z)) = S(r, f^{(j)}(z))$ follows.
(iii) Continuing, we have

$$N_0\left(r, 0; f^{(k)}(z+c)|f^{(j)}(z) = 0\right) + N\left(r, 0 f^{(k)}(z+c)|f^{(j)}(z) \neq 0\right) \leq$$
$$\leq N(r, 0; \psi) \leq T(r,\psi) = S(r,f).$$

Likewise, we obtain

$$N_0(r,\infty; f^{(k)}(z+c)|f^{(j)}(z) = \infty) + N(r,\infty; f^{(k)}(z+c)|f^{(j)}(z) \neq \infty) = S(r,f).$$

(iv) Deriving from equality (3), we obtain

$$T(r, f^{(k)}(z+c)) \leq T(r,\psi) + T(r, f^{(j)}(z)) + S(r,f) = T(r, f^{(j)}(z)) + S(r,f)$$
$$T(r, f^{(j)}(z)) \leq T(r, \frac{1}{\psi}) + T(r, f^{(k)}(z+c)) + S(r,f) = T(r, f^{(k)}(z+c)) + S(r,f).$$

Therefore, we can deduce that $T(r, f^{(k)}(z+c)) = T(r, f^{(j)}(z)) + S(r,f)$.
(v) Based on the given assumption, we have $N(r,\infty; f^{(k)}(z+c)) \leq N(r,\infty; f^{(j)}(z)) + S(r,f)$. By making use of lemma 3, it is important to note that

$$N(r,\infty; f^{(k)}(z+c)) = N(r,\infty; f^{(k)}(z)) + S(r,f) =$$
$$= N(r,\infty; f^{(j)}(z)) + (k-j)\overline{N}(r,\infty; f^{(j)}(z)) + S(r,f).$$

Hence, $N(r,\infty; f^{(k)}(z+c)) \leq N(r,\infty; f^{(j)}(z)) + S(r,f)$. As a result, we can conclude that

$$N(r,\infty; f^{(k)}(z+c)) = N(r,\infty; f^{(j)}(z)) + S(r,f),$$
$$\overline{N}(r,\infty; f^{(j)}(z)) = \overline{N}(r,\infty; f^{(k)}(z+c)) = S(r,f).$$

□

4. Proof of the main results.

Proof of Theorem 6. Given the definition of $\psi(z)$, we have

$$\psi(z) - 1 = (f^{(k)}(z+c) - f^{(j)}(z))/f^{(j)}(z).$$

By the hypothesis of the Theorem, $f^{(k)}(z+c)$ and $f^{(j)}(z)$ share a non-zero complex number a IM, we can therefore infer that

$$\overline{N}\left(r,a;f^{(k)}(z+c)\right)=\overline{N}\left(r,a;f^{(j)}(z)\right)\leq N\left(r,1;\psi\right)\leq T(r,\psi)=S(r,f). \tag{4}$$

Our assertion is that $\overline{N}_E(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)\neq S(r,f)$.

On contrary that $\overline{N}_E(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)=S(r,f)$. Now, from the derived outcome (ii) in Lemma 5, we can derive that

$$\overline{N}(r,0;f^{(k)}(z+c))=\overline{N}_0(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)+\overline{N}_E(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)+ \\ +\overline{N}(r,0;f^{(k)}(z+c)|f^{(j)}(z)\neq 0)=S(r,f).$$

Applying the Second Fundamental Theorem of Nevanlinna and using equation (4), we can deduce that

$$T(r,f^{(k)}(z+c))\leq \overline{N}(r,\infty;f^{(k)}(z+c))+\overline{N}(r,0;f^{(k)}(z+c))+ \\ +\overline{N}(r,a;f^{(k)}(z+c))+S(r,f)=S(r,f).$$

We get a contradiction. Consequently, we must conclude that $\overline{N}_E(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)\neq S(r,f)$. Let us proceed by considering

$$\phi(z)=\frac{f^{(k+1)}(z+c)}{f^{(k)}(z+c)-a}-\frac{f^{(j+1)}(z)}{f^{(j)}(z)-a}.$$

At this point, two distinct cases emerge.

Case-1. Assume that $\phi(z)\equiv 0$, then

$$f^{(k+1)}(z+c)/(f^{(k)}(z+c)-a)=f^{(j+1)}(z)/(f^{(j)}(z)-a).$$

Upon integrating both sides, we obtain $f^{(k)}(z+c)-a\equiv d(f^{(j)}(z)-a)$, where $d\in\mathbb{C}\backslash\{0\}$. As $\overline{N}_E(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0)\neq S(r,f)$, it follows that $d=1$. Consequently, we can deduce that $f^{(k)}(z+c)\equiv f^{(j)}(z)$.

Case-2. Assume that $\phi\not\equiv 0$. In this scenario, the poles of ϕ arise from the poles of $f^{(k)}(z+c)$, $f^{(j)}(z)$, and the a-points of $f^{(k)}(z+c)$ and $f^{(j)}(z)$. Therefore, drawing from the implications of Lemma 5 and relation (4), we can derive that

$$N(r,\infty;\phi)=\overline{N}(r,\infty;f^{(k)}(z+c))+\overline{N}(r,a;f^{(k)}(z+c))+\overline{N}(r,\infty;f^{(j)}(z))+ \\ +\overline{N}(r,a;f^{(j)}(z))=S(r,f).$$

Note that $m(r,\phi)=S(r,f)$. Consequently, we can establish that $T(r,\phi)=S(r,f)$. Since $f^{(k)}(z+c)=\psi(z)f^{(j)}(z)$ and $f^{(k+1)}(z+c)=\psi'(z)f^{(j)}(z)+\psi(z)f^{(j+1)}(z)$, by definition of $\phi(z)$, we obtain that

$$\phi(z)=\frac{\psi' f^{(j)}(z)+\psi f^{(j+1)}(z)}{\psi f^{(j)}(z)-a}-\frac{f^{(j+1)}(z)}{f^{(j)}(z)-a}= \\ =\frac{(\psi' f^{(j)}(z)+\psi f^{(j+1)}(z))(f^{(j)}(z)-a)-(\psi f^{(j)}(z)-a)f^{(j+1)}(z)}{(\psi f^{(j)}(z)-a)(f^{(j)}(z)-a)}= \\ =\frac{\psi'(f^{(j)}(z))^2-a\psi'(f^{(j)}(z))+(a-a\psi)f^{(j+1)}(z)}{(\psi f^{(j)}(z)-a)(f^{(j)}(z)-a)}.$$

Hence,

$$\psi\phi(f^{(j)}(z))^2-(a+a\psi)\phi f^{(j)}(z)+a^2\phi=\psi'(f^{(j)}(z))^2-a\psi'(f^{(j)}(z))+(a-a\psi)f^{(j+1)}(z).$$

Therefore,

$$(a-a\psi)f^{(j+1)}(z)=(\psi\phi-\psi')(f^{(j)}(z))^2+(a\psi'-a\phi-a\phi\psi)f^{(j)}(z)+a^2\phi \tag{5}$$

Let us define $g(z):=f^{(j)}(z)$. Subsequently, equation (5) transforms into

$$Ag'(z) = Bg^2(z) + Cg(z) + D,$$

where $A = a(1-\psi), B = \psi\phi - \psi', C = a(\psi' - \phi - \phi\psi), D = a^2\phi$. In this context, it's evident that A, B, C, and D are small functions of $f(z)$. Assume that $B \not\equiv 0$. Using Lemmas 1 and 5, we consistently obtain that

$$2T(r, g(z)) \le T(r, g'(z)) + S(r, f) = T(r, \frac{g'(z)}{g(z)}) + T(r, g(z)) + S(r, f) =$$

$$= N(r, \infty; \frac{g'(z)}{g(z)}) + T(r, g(z)) + S(r, f) = \overline{N}(r, \infty; g(z)) + T(r, g(z)) + S(r, f).$$

Hence $T(r, g(z)) \le S(r, f)$, thus $T(r, f^{(j)}(z)) \le S(r, f)$, which is not possible. Therefore, we conclude that $B \equiv 0$, thus $\psi\phi - \psi' \equiv 0 \implies \phi \equiv \frac{\psi'}{\psi} \implies$

$$\frac{f^{(k+1)}(z+c)}{f^{(k)}(z+c) - a} - \frac{f^{(j+1)}(z)}{f^{(j)}(z) - a} \equiv \frac{\psi'(z)}{\psi(z)}.$$

On integration, we have $\psi(z) \equiv \lambda \frac{f^{(k)}(z+c)-a}{f^{(j)}(z)-a}$, where $\lambda \in \mathbb{C}\backslash\{0\}$. So, by the definition of $\psi(z)$ we have

$$\psi(z) = \frac{f^{(k)}(z+c)}{f^{(j)}(z)} \equiv \lambda \frac{f^{(k)}(z+c) - a}{f^{(j)}(z) - a}. \tag{6}$$

Let us consider an a-point denoted by z_0 of $f^{(j)}(z)$ with multiplicity p, where $p \in \mathbb{N}$. Given that $f^{(k)}(z+c)$ and $f^{(j)}(z)$ share a IM, it can be inferred that z_0 is also an a-point of $f^{(k)}(z+c)$ with multiplicity q. Because ψ is an entire function, the multiplicity of an a-point of $f^{(j)}(z)$ is either equal to or less than the multiplicity of the corresponding a-point of $f^{(k)}(z+c)$. Thus, two possibilities exist (i) $p = q$, (ii) $p < q$.

In the case $p < q$, it follows from equation (6) that z_0 must be a zero of $f^{(k)}(z+c)$. However, since $a \neq 0$, the conditions $f^{(k)}(z_0+c) = 0$ and $f^{(k)}(z_0+c) = a$ cannot both hold simultaneously, leading to a contradiction. Consequently, the only plausible scenario is $p = q$. Thus, we can deduce that $f^{(k)}(z+c)$ and $f^{(j)}(z)$ share a CM. □

Proof of Theorem 7. Let us introduce the following notation. For $a \in \mathbb{C} \cup \{\infty\}$, denote by

(i) $N_0^{l)}\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) = a\right)$ the reduced counting function of common a-points of $f^{(k)}(z+c)$ and $f^{(j)}(z)$ of different multiplicities.

(ii) $N_E^{l)}\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) = a\right)$ denotes the counting function of common a-points of $f^{(k)}(z+c)$ and $f^{(j)}(z)$ of equal multiplicities.

(iii) $N_0^{l)}\left(r, a; f^{(k)}(z+c) | f^{(j)}(z) \neq a\right)$ the counting function of a-points of $f^{(k)}(z+c)$ which are not the a-points of $f^{(j)}(z)$.

In the above cases, the a-points have multiplicity at most l.

Since, $E_{l)}(r, f^{(j)}(z)) = E_{l)}(r, f^{(k)}(z+c))$, we have

$$\overline{N}_{l)}(r, a; f^{(k)}(z+c)) = \overline{N}_{l)}(r, a; f^{(j)}(z)) \le N(r, 1, \psi) \le T(r, \psi) = S(r, f). \tag{7}$$

We claim that $\overline{N}_E^{l)}(r, 0; f^{(k)}(z+c) | f^{(j)}(z) = 0) \neq S(r, f)$. If not suppose that $\overline{N}_E^{l)}(r, 0; f^{(k)}(z+c) | f^{(j)}(z) = 0) = S(r, f)$. Now using lemma 5 and inequality (7) we have

$$\overline{N}(r, 0; f^{(k)}(z+c)) = \overline{N}_0^{l)}(r, 0; f^{(k)}(z+c) | f^{(j)}(z) = 0) + \overline{N}_E^{l)}(r, 0; f^{(k)}(z+c) | f^{(j)}(z) = 0) +$$

$$+ \overline{N}^{l)}(r, 0; f^{(k)}(z+c) | f^{(j)}(z) \neq 0) = S(r, f). \tag{8}$$

Now applying the second fundamental theorem and using lemma 3 and equation (7) and (8), we get

$$T(r,f^{(k)}(z+c)) \le \overline{N}(r,\infty;f^{(k)}(z+c)) + \overline{N}(r,0;f^{(k)}(z+c)) + \overline{N}(r,a;f^{(k)}(z+c)) + \\ +S(r,f) \le \overline{N}_{(l+1}(r,0;f^{(k)}(z+c)) + \overline{N}_{(l+1}(r,a;f^{(k)}(z+c)) + S(r,f) \le \\ \le \frac{2}{l+1}T(r,f^{(k)}(z+c)) + S(r,f),$$

which is a contradiction since $l > 1$. So, $\overline{N}_E^{l)}(r,0;f^{(k)}(z+c)|f^{(j)}(z)=0) \ne S(r,f)$. From here, we can proceed as the Theorem 6 and will get the desired result. □

Acknowledgments. The second author is thankful to the Council of Scientific and Industrial Research(CSIR) for their financial support under file no. 09/0285(11142)/2021-EMR-I.

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Department of Mathematics, University of North Bengal
Raja Rammohunpur, Siliguri, Darjeeling-734013
West Bengal, India
dcpramanik.nbu2012@gmail.com
apurba1995sarkar2@gmail.com

Received 28.03.2024
Revised 14.06.2024