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MEROMORPHIC FUNCTIONS SHARING THE ZEROS OF LOWER DEGREE SYMMETRIC POLYNOMIALS IN WEIGHTED WIDER SENSE

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In this paper, we establish some mathematical rules for determining the initial and terminal numbers of non-zero terms in any arbitrary polynomial. These rules lead to the definitions of index s and reverse index $\hat{s}$ of a polynomial. Further, building on these concepts, we introduce the order of a polynomial $P(z)$ as $(s, \hat{s})$. If $P_*(z)$ is another polynomial of order $(\hat{s}, s)$, then the pair $P(z)$ and $P_*(z)$ are referred to as symmetric polynomials. The concept of symmetric polynomials is central in this work, as we investigate the effects of weighted sharing in the wider sense (see Adv. Stud: Euro-Tbilisi Math. J., 16(4)(2023), 175-189) on the zeros of symmetric polynomials along with the sharing of poles. Our study focuses on symmetric polynomials of degree 3, analyzing their intrinsic properties. Sharing of zeros of polynomials of lower degree are critical in nature and at the same time it exhibits sophisticated structural characteristics, making them an ideal subject for such analysis. Our exploration of the sharing of zeros of symmetric polynomials establishes connections between two non-constant meromorphic functions. The article includes examples of both a general nature and specific, partial cases that serve to illustrate and validate our theoretical results.

1. Introduction and definitions. Let f and g be the meromorphic functions in $\mathbb{C}$, S be a set of distinct elements of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We begin by recalling the following definitions.

For a non-constant meromorphic function f and $a \in \mathbb{C}$, we denote

$$E_f(a) = \{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a, \text{with multiplicity } p\},$$
$$\overline{E}_f(a) = \{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}.$$

Then we say that f and g *share* CM (IM) a value a, if $E_f(a) = E_g(a)$ ($\overline{E}_f(a) = \overline{E}_g(a)$). For $a = \infty$, we define $E_f(\infty) = E_{\frac{1}{f}}(0)$ ($\overline{E}_f(\infty) = \overline{E}_{\frac{1}{f}}(0)$).

Let us denote $E_f(S) = \bigcup_{a\in S} E_f(a)$, $\overline{E}_f(a) = \bigcup_{a\in S} \overline{E}_f(a)$ for a non-constant meromorphic function f and a set $S\subset\overline{\mathbb{C}}$. Then we say that f and g share CM (IM) a set S, if $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$).

If the readers need further information or a detailed explanation about the standard notations of set sharing, we suggest consulting the original sources cited in the text [1]. There is provided pertinent information for further insight in the second paragraph, which includes the definition of value sharing as well. The conventional notations of value distribution theory are outlined in [2, 3].

Now, we are going to demonstrate some relevant definitions, which are closely related to the subject of this article. First, we slightly modify the definition of [4] as follows, as this will be necessary in the following steps.

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Definition 1 (Initial term non-gap and gap polynomials)**.** A polynomial $P(z)=\sum_{k=0}^{n}a_kz^k$ of degree n is called an *initial term non-gap polynomial* or *polynomial without gap concerning initial term* (ITNGP), if for some $1\leq i<n$ one has $a_{n-1}\neq 0$, $a_{n-2}\neq 0,\ldots,a_{n-i+1}\neq 0$, $a_{n-i}\neq 0$. Otherwise, $P(z)$ is called an *initial term gap polynomial* (ITGP).

A polynomial $P(z)$ is said to be a *non-degenerate polynomial*, if $0\notin\{z\colon P(z)=0\}$, otherwise it is called *degenerate polynomial*.

We are now introducing analogs of *Definition* 1, vis-a-vis terminal term non gap and gap polynomials.

Definition 2 (Terminal term non-gap and gap polynomials)**.** A non-degenerate polynomial $P(z)=\sum_{k=0}^{n}a_kz^k$ is called a *terminal term non-gap polynomial* or *polynomial without gap concerning terminal term* (TTNGP) if for some $t\in\{1,2,\ldots n-1\}$ one has $a_t\neq 0$, $a_{t-1}\neq 0$, $a_{t-2}\neq 0,\ldots,a_1\neq 0$. Otherwise, the polynomial is said to be a *terminal term gap polynomial* (TTGP).

Next, in view of Definitions 1 and 2, we want to propose the definitions of *index* and *reserve index* of a polynomial in the following manner.

Definition 3 (Index of a polynomial)**.** Let $P^{[s]}(z)=\sum_{k=0}^{n}a_kz^k$ be a polynomial.
A. If $P^{[s]}(z)$ is an ITNGP. The integer s is called the *index* of $P^{[s]}(z)$ (and $P^{[s]}(z)$ is said to be *initial term* non-gap polynomials of index s or ITNGP$_s$ in short) if one of the followings is satisfied:
i) for some greatest $1\leq s\leq n$, one has $a_{n-s+1}\neq 0$ but $a_0=0$;
ii) for all $i=0,1,2,\ldots,n$ one has $a_i\neq 0$, then we put the index $s=n+1$.
B. If $P^{[s]}(z)$ is an ITGP, then it is of index 1.

Note 1. Any polynomial of degree n is of index $s\geq 1$.

Definition 4 (Reverse index of a polynomial)**.** Suppose $P_{[\widehat{s}]}(z)=\sum_{k=0}^{n}b_kz^k$, $b_0\neq 0$.
A. Let $P_{[\widehat{s}]}(z)$ be a TTNGP. The integer $\widehat{s}$ is called the reverse index of the polynomial $P_{[\widehat{s}]}(z)$ (and $P_{[\widehat{s}]}(z)$ is said to be terminal term non-gap polynomials of reverse index $\widehat{s}$ or TTNGP$_{\widehat{s}}$ in short) if one of the followings is satisfied:
i) for some greatest $1<\widehat{s}<n$, one has $b_{\widehat{s}-1}\neq 0$
ii) for all $i=0,1,2,\ldots,n-1$ one has $b_i\neq 0$, then we put the reverse index $\widehat{s}=n+1$.
B. If $P_{[\widehat{s}]}(z)$ is a TTGP, then reverse index is 1.

Note 2. For a polynomial of degree n, n cannot be reverse index of the polynomial.

Definition 5 (Order of a polynomial)**.** For a polynomial $P(z)$ with index s and reverse index $\widehat{s}$, we will call it polynomial of order $(s,\widehat{s})$.

Definition 6 (Symmetric polynomials)**.** Consider a polynomial $P(z)$ with order $(s,\widehat{s})$. If $P_*(z)$ be another polynomial with order $(\widehat{s},s)$, then $P(z)$ and $P_*(z)$ are called symmetric polynomials. For example, $P(z)=z^3+z+1$ and $P_*(z)=z^3+z^2+1$. $P(z)$ is a polynomial of order $(1,2)$ and $P_*(z)$ is a polynomial of order $(2,1)$. Hence, the pair $P(z)$ and $P_*(z)$ are symmetric polynomials.

Note 3. In view of the definitions of index and reverse index of a polynomial we can see that the polynomial defined in Theorem A is of index 1 and reverse index 3, whereas the same in Theorems B, C are of index 2 and reverse index 1.

In 2001, the concept of weighted sharing of sets was introduced, which contributed to the uniqueness theory in complex analysis. The specific details and implications of this notion can be found in the paper by Lahiri [5]. We have recently refined the definition of weighted sharing of sets and termed it as 'weighted sharing of sets in a wider sense' for meromorphic functions.

Definition 7 ([6])**.** Let f and g be two non-constant meromorphic functions and $P(z)$ and $Q(z)$ be two polynomials of degree n without any multiple zero. Let

$$S_P = \{z : P(z) = 0\} \quad \text{and} \quad S_Q = \{z : Q(z) = 0\}.$$

We say that f and g *share the sets* S_P and S_Q with weight l in the wider sense if $E_f(S_P, l) = E_g(S_Q, l)$ and we denote it by f, g share $(S_P, S_Q; l)$. We note that, if $P = Q$, then we get the traditional definition of weighted sharing of sets.

2. Background, motivations and main results. There are a lot of papers in versatile directions in connection to the different category of set sharing problems as well as characterization of the functions (e.g. [7–10]). However, in this paper we are taken into account the investigations of three set sharing problem only.

In connection to the famous question of Gross ([11]), in 1994, regarding sharing of three sets and uniqueness of meromorphic function the following question was asked by Yi ([12]):

Question A([12]). *Can one find three finite sets* S_j $(j \in \{1, 2, 3\})$ *such that any two non constant meromorphic functions* f *and* g *satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j \in \{1, 2, 3\}$ *must be identical?*

Many research articles were published to find the possible answers of the above question.

In 2018, Banerjee-Mallick ([13]) employed three weighted set sharing to obtain a uniqueness result, where all the range sets were exclusively chosen from $\mathbb{C}$. The result was the first in the three set sharing genre exclusively on $\mathbb{C}$.

Theorem A ([13])**.** *Let* $S_1 = \{w : P(z) = 0\}$, $S_2 = \{d\}$, *and* $S_3 = \{0\}$, *where*

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)dz - (n-1)(n-2)d^2,$$

with $ad^{(n-2)} \notin \{0, 1, 2\}$ *and* $n \geq 5$. *Suppose that* f *and* g *are two non-constant meromorphic functions satisfying* $E_f(S_1, l) = E_g(S_1, l)$, $E_f(S_2, k) = E_g(S_2, k)$ *and* $E_f(S_3, m) = E_g(S_3, m)$ *then* $f \equiv g$ *for* $(l, k, m) = (3, 0, 2), (2, 1, 3)$.

When $n < 5$, then the situation is not so easy to tackle. In fact, in this paper we will solely confine our attention on the case $n = 3$, due to the scarcity of the results for the same case. Foremost, consider the following polynomial of order $(2, 1) : P^1(z) = a_3z^3 + a_2z^2 + a_0$, where $a_3 \cdot a_2 \neq 0$. The next example shows that two meromorphic functions sharing the zeros of $P^1(z)$ can have the same zero and pole set still, they may not be identical. So, additional assumptions are required for uniqueness.

Example 1. Let $f \equiv e^z g$, where $g(z) \equiv -\frac{a_2}{a_3}(\frac{e^z+1}{e^{2z}+e^z+1})$, and $S^1 = \{z \colon P^1(z) = 0\}$. Note that as $f^2(f + \frac{a_2}{a_3}) \equiv g^2(g + \frac{a_2}{a_3})$, we have $E_f(S^1, \infty) = E_g(S^1, \infty)$. Also $E_f(\{0\}, \infty) = E_g(\{0\}, \infty)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. Here both f and g have simple poles, but $f \not\equiv g$.

For three degree polynomial the investigations were started long before the appearance of Theorem A. In 2002, Qiu-Fang ([14]) answered the Question A and determined the additional conditions in the following manner.

Theorem B ([14])**.** *Let $n \geq 3$ be a positive integer, $S_1^n = \{z : z^n - z^{n-1} - 1 = 0\}$, f and g be two non-constant meromorphic functions with multiple poles. If $E_f(S_1^n, \infty) = E_g(S^*, \infty)$, $E_f(\{0\}, \infty) = E_g(\{0\}, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, then $f \equiv g$.*

In 2010, the second author ([15]) further improved Theorem B in the following direction for 3 degree polynomial and sharing a double tone set.

Theorem C ([15])**.** *Let $S_1^3 = \{z : z^3 - z^2 - 1 = 0\}$. If $E_f(S_1^3, 3) = E_g(S_1^3, 3)$, $E_f(\{0, \frac{2}{3}\}, 0) = E_g\left(\{0, \frac{2}{3}\}, 0\right)$ and $E_f(\{\infty\}, 1) = E_g(\{\infty\}, 1)$, then $f \equiv g$.*

Note that, in Theorems A–C, presence of 0 is essential, either individually or as an element of a set. So, the following question is inevitable:

Question 1. *Can the similar results of Theorems B and C be obtained for 3 sets without the element 0?*

In Theorems B–C, the polynomials are of order $(2, 1)$. So, it will be natural to investigate its counterpart in terms of symmetric polynomials, that is to say, to find the similar results to Theorems B–C, in terms of the following polynomial of order $(1, 2)$:

$$\mathrm{P}_*^1(z) = a_3 z^3 + a_1 z + a_0.$$

However, the following example shows that, the same thing happens here like Example 1, i.e., there exist two meromorphic functions sharing the zeros of the polynomial $\mathrm{P}_*{}^1(z)$ together with same 0 and pole sets, still the do not become identical.

Example 2. Take two functions $f(z) \equiv \frac{\sqrt{a_1}\,(-\sin z + i\sqrt{3})}{\sqrt{3a_3}\,\cos z}$ and $g(z) \equiv \frac{2\sqrt{a_1}\,\sin z}{\sqrt{3a_3}\,\cos z}$ and the set $S_*^1 = \{z\colon P_*^1(z) = 0\}$. We have $E_f(S_*^1, \infty) = E_g(S_*^1, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. Here both f and g have simple poles, but $f \not\equiv g$.

To resolve all the problems discussed earlier in a compact and convenient way is the main motivation of writing this paper. In fact, with respect to Definition 6 and 7, we will investigate the situation under the purview of symmetric polynomials in a more generalized frame.

First, let us take two complex numbers χ_2 and χ_1 such that $\chi_2\chi_1 = 0$ and $\chi_2 + \chi_1 \neq 0$. We now define the following polynomials

$$P(z) = \frac{z^3}{3} + \frac{\chi_2 z^2}{2} + \chi_1 z - c, \quad \widehat{P}(z) = k\left(\frac{z^3}{3} + \frac{\chi_2 z^2}{2} + \chi_1 z\right) - \widehat{c}, \tag{1}$$

where k, c and $\widehat{c}$ be non-zero constants such that both $P(z)$ and $\widehat{P}(z)$ do not have any multiple zero.

With respect to the above defined polynomials (1), we now present the main results of this paper.

Theorem 1. *Let $S = \{z\colon P(z) = 0\}$ and $\widehat{S} = \{z\colon \widehat{P}(z) = 0\}$, where $P(z)$ and $\widehat{P}(z)$ are given by (1). Suppose f and g are two non-constant meromorphic functions possessing multiple poles. Further,*
i) if $\chi_1 = 0$, $(\widehat{c} + ck) \neq \frac{k\chi_2^3}{6}$; f, g share $(S, \widehat{S}; 3)$, $(\{0, -\chi_2\}, 0)$ and $(\{\infty\}, 1)$, then $f \equiv g$;
ii) if $\chi_2 = 0$, $(\widehat{c}+ck) \neq 0$; f, g share $(S, \widehat{S}; 2)$, $(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\}, 0)$ and $(\{\infty\}, \infty)$, then $f \equiv g$.

Note 4. For $\chi_2 = 0$, $\chi_1 = 1$ and $c = \frac{1}{6}$, the set S defined in Theorem 1 becomes:

$$S = \{z\colon P(z) = 0\} = \Big\{\frac{1}{2}\Big(-1 + {t_2}^{-\frac{1}{3}} + {t_2}^{\frac{1}{3}}\Big), -\frac{1}{2} - \frac{(1 + i\sqrt{3})}{4{t_2}^{\frac{1}{3}}} - \frac{1}{4}(1 - i\sqrt{3}){t_2}^{\frac{1}{3}},$$
$$-\frac{1}{2} - \frac{(1 - i\sqrt{3})}{4{t_2}^{\frac{1}{3}}} - \frac{1}{4}(1 + i\sqrt{3}){t_2}^{\frac{1}{3}}\Big\},$$

where $t_2 = (1 + 2\sqrt{6})^{\frac{1}{3}}$. Similarly, choosing $\chi_2 = 0$, $\chi_1 = 1$ and $k = 1$, $\widehat{c} = \frac{1}{3}$ and replacing t_2 by $\widehat{t_2} = (3 + 2\sqrt{2})^{\frac{1}{3}}$, we get $\widehat{S} = \{z\colon \widehat{P}(z) = 0\}$. Now, from Theorem 1 (i) and in view of the sets S and $\widehat{S}$, we know that there are no two distinct meromorphic functions f, g such that $E_f(S, m) = E_g(\widehat{S}, m)$ along with $E_f(\{i, -i\}, k_1) = E_g(\{i, -i\}, k_1)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ hold.

Note 5. For $\chi_2 = 1$, $\chi_1 = 0$ and $c = \frac{1}{3}$, the set S defined in Theorem 1 becomes:

$$S = \Big\{\Big(\frac{2}{t_1}\Big)^{\frac{1}{3}} + \Big(\frac{t_1}{2}\Big)^{\frac{1}{3}}, \frac{(1 + i\sqrt{3})}{(4t_1)^{\frac{1}{3}}} - \frac{(1 - i\sqrt{3}){t_1}^{\frac{1}{3}}}{2.2^{\frac{1}{3}}}, \frac{(1 - i\sqrt{3})}{(4t_1)^{\frac{1}{3}}} - \frac{(1 + i\sqrt{3}){t_1}^{\frac{1}{3}}}{2.2^{\frac{1}{3}}}\Big\},$$

where $t_1 = \frac{1+\sqrt{37}}{3}$. Similarly, choosing $\chi_2 = 1$, $k = 1$, $\widehat{c} = \frac{2}{3}$ and replacing t_1 by $\widehat{t_1} = \frac{2(1+\sqrt{10})}{3}$, we get $\widehat{S} = \{z\colon \widehat{P}(z) = 0\}$. Now, from Theorem 1 (ii) and in view of the sets S and $\widehat{S}$, we know that there are no two distinct meromorphic functions f, g such that $E_f(S, m) = E_g(\widehat{S}, m)$ along with $E_f(\{0, -1\}, k_1) = E_g(\{0, -1\}, k_1)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ hold.

Corollary 1. *Under the same situation of Theorem 1 (ii), if f and g are two entire functions such that, they share $(S, \widehat{S}; 2)$, $(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\}, 0)$, then $f \equiv g$.*

For the standard definitions and notations of the value distribution theory we refer to [3] and for the definitions of $N(r, a; f\colon \geq s)$, $N(r, a; f\colon = s)$ for $s \geq 1$, $\overline{N}_L(r, 1; f)$, $\overline{N}_L(r, 1; g)$ and $\overline{N}_*(r, a; f, g)$ we refer to [16–19].

3. Lemmas. In this section we present some lemmas, which will be needed in the sequel. Henceforth, unless and otherwise stated, let us assume f and g be two non constant meromorphic functions. Let F and G be two non constant meromorphic functions defined in $\mathbb{C}$ as follows:

$$F \equiv \frac{1}{c}\Big(\frac{f^3}{3} + \frac{\chi_2 f^2}{2} + \chi_1 f\Big) \quad \text{and} \quad G \equiv \frac{k}{\widehat{c}}\Big(\frac{g^3}{3} + \frac{\chi_2 g^2}{2} + \chi_1 g\Big). \tag{2}$$

Henceforth we shall denote by H and Φ the following functions

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F - 1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G - 1}\Big), \tag{3}$$

$$\Phi = \frac{F'}{F - 1} - \frac{G'}{G - 1}. \tag{4}$$

Lemma 1 ([19]). *If F, G are two non constant meromorphic functions such that they share $(1,1)$ and $H \not\equiv 0$ then*

$$N(r,1;F:\ =1) = N(r,1;G:\ =1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2 ([15]). *If the meromorphic functions f and g share $(1,m)$, where $1 \leq m < \infty$, then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N(r,1;f:\ =1) + \left(m - \frac{1}{2}\right)\overline{N}_*(r,1;f,g) \leq \frac{1}{2}[N(r,1;f) + N(r,1;g)].$$

Lemma 3 ([2], Theorem 6.2). *Let f be meromorphic function, $P(f) = \sum_{k=0}^{n} a_k f^k$, where a_k's are constant and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$ $(r \to +\infty)$.*

Lemma 4. *Let S, $\widehat{S}$ be defined as in Theorem 1, with $\chi_2 = 0$; F, G be defined as in (2). If $E_f(S,0) = E_g(\widehat{S},0)$, $E_f(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\},p) = E_g(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\},p)$, $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, where $0 \leq p < \infty$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,\infty;H) \ \leq\ & \overline{N}(r,i\sqrt{\chi_1};f:\ \geq p+1) + \overline{N}(r,-i\sqrt{\chi_1};f:\ \geq p+1) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f', which are not zeros of $(f^2+\chi_1)(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined. Similar result is valid for $\chi_1 = 0$.

Proof. From the definition of H and (2), by a simple calculation we can write

$$\begin{aligned} H \equiv & \left[\frac{f'}{f - i\sqrt{\chi_1}} - \frac{g'}{g - i\sqrt{\chi_1}}\right] + \left[\frac{f'}{f + i\sqrt{\chi_1}} - \frac{g'}{g + i\sqrt{\chi_1}}\right] + \left[\frac{f''}{f'} - \frac{g''}{g'}\right] - \\ & -2\left[\frac{(f^2+\chi_1)f'}{\frac{f^3}{3} + \chi_1 f - c} - \frac{(g^2+\chi_1)g'}{\frac{g^3}{3} + \chi_1 g - \widehat{c}}\right]. \end{aligned}$$

Since, $E_f(S,0) = E_g(\widehat{S},0)$, it follows that F and G share $(1,0)$. We can easily verify that possible poles of H occur at (i) those $i\sqrt{\chi_1}$ of f, whose multiplicities are greater than p, (ii) those $-i\sqrt{\chi_1}$ of f whose multiplicities are greater than p, (iii) those 1-points of F and G with different multiplicities, (iv) those poles of f and g whose multiplicities are distinct, (v) zeros of f' which are not the zeros of $(f^2+\chi_1)(F-1)$, (v) zeros of g' which are not zeros of $(g^2+\chi_1)(G-1)$. Since H has only simple poles, the lemma follows from above. □

Lemma 5 ([1]). *Let S, $\widehat{S}$ be defined as in Theorem 1 and F, G be given by (2). Let $E_f(S,m) = E_g(\widehat{S},m)$, where $0 \leq m < \infty$ Then*

$$(i)\ \ \overline{N}_L(r,1;F) \leq \frac{1}{m+1}\left(\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\otimes}(r,0;f')\right) + S(r,f),$$

$$(ii)\ \ \overline{N}_L(r,1;G) \leq \frac{1}{m+1}\left(\overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\otimes}(r,0;g')\right) + S(r,g),$$

where $N_{\otimes}(r,0;f') = N(r,0;f':\ f \neq 0, w_1, w_2, w_3)$ and w_1, w_2, w_3 are the roots of the equation $P(z) = 0$, $N_{\otimes}(r,0;g')$ is defined similarly to $N_{\otimes}(r,0;f')$.

Lemma 6. *Let F and G be given by (2) such that $E_f(S,m)=E_g(\widehat{S},m)$, $E_f(\{\infty\},k)=E_g(\{\infty\},k)$, $0\le k<\infty$ and $\Phi\not\equiv 0$.*
(i) If $\chi_2=0$, $0\le p<\infty$ and $E_f(\{i\sqrt{\chi_1},-i\sqrt{\chi_1}\},p)=E_g(\{i\sqrt{\chi_1},-i\sqrt{\chi_1}\},p)$ then,

$$(2p+1)\left\{\overline{N}(r,i\sqrt{\chi_1};f:\ \ge p+1)+\overline{N}(r,-i\sqrt{\chi_1};f:\ \ge p+1)\right\}\le$$
$$\le \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).$$

(ii) Similarly, if $\chi_1=0$, $0\le p_1<\infty$ and $E_f(\{0,-\chi_2\},p_1)=E_g(\{0,-\chi_2\},p_1)$ then,

$$(2p+1)\left\{\overline{N}(r,0;f:\ \ge p_1+1)+\overline{N}(r,-\chi_2;f:\ \ge p_1+1)\right\}\le$$
$$\le \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).$$

Proof. Proof of the theorem is similar to the proof of lemma from [15]. □

Lemma 7. *Under the same supposition of Theorem 1 (ii), let*

$$P(f)\equiv\frac{c}{\widehat{c}}\widehat{P}(g),$$

i.e.,

$$\left(\frac{f^3}{3}+\chi_1 f\right)\equiv\frac{kc}{\widehat{c}}\left(\frac{g^3}{3}+\chi_1 g\right),$$

then we have $f\equiv g$.

Proof. Given

$$\left(\frac{f^3}{3}+\chi_1 f\right)\equiv\frac{kc}{\widehat{c}}\left(\frac{g^3}{3}+\chi_1 g\right),$$

which implies that

$$\frac{1}{3}\left(f^3-\frac{kc}{\widehat{c}}\right)+\chi_1\left(f-\frac{kc}{\widehat{c}}g\right)\equiv 0.$$

We have to prove that, $f\equiv g$. If possible, let $f\not\equiv g$. Taking $f\equiv gh$, by a simple calculation we have

$$\frac{g^2}{3}\left(h^3-\frac{kc}{\widehat{c}}\right)+\chi_1\left(h-\frac{kc}{\widehat{c}}\right)\equiv 0. \tag{5}$$

Now, we wish to show that $\frac{kc}{\widehat{c}}=1$ i.e., $P(z)$ and $\widehat{P}(z)$ become linearly dependent. Hence, $S=\widehat{S}$ and then we have only one polynomial $P(z)$, whose zeros are shared with respect to the traditional weighted sharing of sets. Suppose on the contrary, $\frac{kc}{\widehat{c}}\ne 1$.
Case 1. Assume that h is constant.
Subcase 1.1. If $h=\frac{kc}{\widehat{c}}$, then from (5) we have,

$$\frac{g^2kc}{3\widehat{c}}\left[\left(\frac{kc}{\widehat{c}}\right)^2-1\right]\equiv 0,$$

which implies that $\frac{kc}{\widehat{c}}=-1$, a contradiction, by the hypothesis of the theorem.
Subcase 1.2. Let $h\ne\frac{kc}{\widehat{c}}$, then (5) gives $g^2\equiv-\frac{3\chi_1\left(h-\frac{kc}{\widehat{c}}\right)}{\left(h^3-\frac{kc}{\widehat{c}}\right)}$, which is not possible as g is non

constant.

Case 2. Suppose, h is non constant.

First, take g as an entire function. Then all three distinct zeros of the polynomial $\left(z^3 - \frac{kc}{\hat{c}}\right)$ are Picard's exceptional values of h, which is not possible.

Next, if g is non constant meromorphic function, then as g has multiple poles, a simple calculation yields that each zero of the polynomial $\left(z^3 - \frac{kc}{\hat{c}}\right)$ say β_i, $i \in \{1,2,3\}$ are of multiplicities at least 4. By the Second Fundamental Theorem we have

$$2T(r,h) \leq \sum_{i=1}^{3} \overline{N}(r,\beta_i;h) + \overline{N}(r,\infty;h) + S(r,h) \leq \frac{7}{4}T(r,h) + S(r,h),$$

which is a contradiction.

Hence $\frac{kc}{\hat{c}} = 1$. We assume that h is non constant. Therefore, (5) transforms into

$$\frac{g^2}{3}\left(h^3 - 1\right) + \chi_1 (h-1) \equiv 0, \tag{6}$$

i.e, $g^2 \equiv -\frac{3\chi_1}{(h^2+h+1)}$. Now $g^2 + \chi_1 = \frac{\chi_1 (h-1)(h+2)}{h^2+h+1}$.

As, f and g share $(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\}, 0)$, the possible values of h are ± 1. Hence, by the above equation, it is evident that

$$\overline{N}(r, i\sqrt{\chi_1}; g) + \overline{N}(r, -i\sqrt{\chi_1}; g) = \overline{N}(r,1;h).$$

It follows that -2 is Picard's exceptional value of h. Let the zeros of the polynomial z^2+z+1 be γ_i, $i \in \{1,2\}$. By the Second Fundamental Theorem we get,

$$2T(r,h) \leq \overline{N}(r,\gamma_1;h) + \overline{N}(r,\gamma_2;h) + \overline{N}(r,-2;h) + \overline{N}(r,\infty;h) + S(r,h) \leq \frac{3}{2}T(r,h) + S(r,h),$$

a contradiction. Hence, h is constant and (6) gives $h^3 = h = 1$, which implies $h = 1$, i.e. $f \equiv g$.

□

Lemma 8. *Under the same assumptions of Theorem 1 (i) let*

$$\left(\frac{f^3}{3} + \frac{\chi_2 f^2}{2}\right) \equiv \frac{kc}{\hat{c}}\left(\frac{g^3}{3} + \frac{\chi_2 g^2}{2}\right),$$

then we obtain $f \equiv g$.

Proof. Proceeding in the similar manner as done in the case of Lemma 7, we can obtain an analogous equation of (6),

$$\frac{g}{3}\left(h^3 - 1\right) + \frac{\chi_2}{2}(h-1) \equiv 0.$$

Clearly, f and g share $(\{0\}, \infty)$. Next observe that,

$$f + \chi_2 \equiv \frac{\chi_2}{2}\frac{(h-1)(h+2)}{h^2+h+1} \quad \text{and} \quad g + \chi_2 \equiv -\frac{\chi_2}{2}\frac{(h-1)(2h+1)}{h^2+h+1}.$$

From the above two expressions, we see -2 and $-\frac{1}{2}$ are Picard's exceptional values of h. Hence, h omits four values 0, ∞, -2 and $-\frac{1}{2}$. It is a contradiction to Nevanlinna four value theorem. Therefore, $f \equiv g$. □

Lemma 9 ([4]). *Let $\phi(z) = a^2(z^{n-m} - A)^2 - 4b(z^{n-2m} - A)(z^n - A)$, where $A, a, b \in \mathbb{C}^*$, $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, $gcd(m, n) = 1$, $n > 2m$. If ω^l is the mth root of unity for $l = 0, 1, \ldots, m-1$, then*
(i) $\phi(z)$ has no multiple zero, when $A \neq \omega^l$.
(ii) $\phi(z)$ has exactly one multiple zero, when $A = \omega^l$ and that is of multiplicity 4. In particular, when $A = 1$, then the multiple zeros is 1.

4. Proof of the theorem.

Proof of Theorem 1. (ii) $\chi_2 = 0$.

Let F and G be given by (2). Since f, g share $(S, \widehat{S}; 2)$, from (2) it follows that F and G share $(1, 2)$.

Suppose $H \not\equiv 0$. First we wish to prove $\Phi \not\equiv 0$. On the contrary, let $\Phi \equiv 0$. By (4), we have for a non-zero constant c' that

$$F - 1 = c'(G - 1). \tag{7}$$

Next, using (7) and the definition of H we get, $H \equiv 0$, a contradiction. Hence $\Phi \not\equiv 0$.

Using Lemma 2 for $m = 2$, Lemma 1, Lemma 3, Lemma 4, Lemma 6 (i) for $p = 0$, Lemma 5 for $m = 2$ and the Second Fundamental Theorem we get

$$\begin{aligned} 4(T(r,f) + T(r,g)) \leq \\ \leq \overline{N}(r, i\sqrt{\chi_1}; f) + \overline{N}(r, -i\sqrt{\chi_1}; f) + \overline{N}(r, 1; F) + \overline{N}(r, \infty; f) + \overline{N}(r, i\sqrt{\chi_1}; g) + \\ + \overline{N}(r, -i\sqrt{\chi_1}; g) + \overline{N}(r, 1; G) + \overline{N}(r, \infty; g) - N_0(r, 0; f') - N_0(r, 0; g') + S(r,f) + S(r,g) \leq \\ \leq 3(\overline{N}(r, i\sqrt{\chi_1}; f) + \overline{N}(r, -i\sqrt{\chi_1}; f)) + 2(T(r,f) + T(r,g)) - \\ - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r,f) + S(r,g) \leq \\ \leq \left(2 + \frac{5}{6} + \frac{5}{12}\right)(T(r,f) + T(r,g)) + S(r,f) + S(r,g), \end{aligned}$$

which is a contradiction.

Hence, $H \equiv 0$. Then for two constants $A (\neq 0)$, B we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B \tag{8}$$

and

$$T(r,f) = T(r,g) + S(r,g). \tag{9}$$

I. Claim $B = 0$.
Suppose $B \neq 0$.
Case 1. ∞ is Picard's exceptional value of both f and g. By (8) we can have

$$F - 1 \equiv \frac{G-1}{B\left((G-1) + \frac{A}{B}\right)}. \tag{10}$$

Subcase 1.1. Consider $A \neq B$. Take the polynomial

$$\phi(z) = k\left(\frac{g^3}{3} + \chi_1 g\right) - \widehat{c}\left(1 - \frac{A}{B}\right).$$

First note that, if $i\sqrt{\chi_1}$ is a zero of the polynomial $\phi(z)$, then it will of multiplicity 2 and the other simple zero is α. Here,

$$\overline{N}(r, i\sqrt{\chi_1}; g) + \overline{N}(r, \alpha; g) = \overline{N}(r, \infty; f).$$

As, pole is Picard's exceptional value of both f and g, $i\sqrt{\chi_1}$ and α are also Picard's exceptional values of g, which is not possible.

By the similar argument, we can say that $-i\sqrt{\chi_1}$ is not a zero of the polynomial $\phi(z)$. Hence, all the zeros of $\phi(z)$ are simple and they are Picard's exceptional values of g, a contradiction.

Subcase 1.2. Take $A = B$. Therefore, (10) changes into

$$F - 1 \equiv \frac{G-1}{AG}.$$

Note that factors of G are g and $(g^2+3\chi_1)$. Clearly, 0, $i\sqrt{3\chi_1}$, $-i\sqrt{3\chi_1}$ are Picard's exceptional values of g, which is not possible.

Case 2. Suppose ∞ is not Picard's exceptional value of both f and g, i.e., there exists a complex number z_0 such that $f(z_0) = g(z_0) = \infty$, which contradicts (8).

Hence, our statement is proved. Therefore, (8) reduces to

$$G - 1 \equiv A(F - 1). \tag{11}$$

II. Claim $A = 1$.

Suppose $A \neq 1$ and $Q(z) = \frac{z^3}{3} + \chi_1 z$. From (11), by a simple calculation we can obtain

$$Q(g) \equiv \frac{A\widehat{c}}{kc}\Big\{Q(f) - c\Big(1 - \frac{1}{A}\Big)\Big\}. \tag{12}$$

Suppose that, the polynomial $\psi(z)$ is defined as

$$\psi(z) = Q(z) - c\Big(1 - \frac{1}{A}\Big).$$

First suppose $i\sqrt{\chi_1}$ is a zero of $\psi(z)$, then clearly it is of multiplicity 2 and other zero is simple, namely, β. As f, g share $(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\}, 0)$, it follows that $i\sqrt{\chi_1}$ is Picard's exceptional value of f. By the Second Fundamental Theorem and (12), we have

$$\begin{aligned} 2T(r, g) &\leq \overline{N}(r, 0; g) + \overline{N}(r, i\sqrt{3\chi_1}, g) + \overline{N}(r, -i\sqrt{3\chi_1}; g) + \overline{N}(r, \infty; g) + S(r, g) \leq \\ &\leq \overline{N}(r, \beta; f) + \overline{N}(r, \infty; g) + S(r, g) \leq \frac{3}{2}T(r, g) + S(r, g), \end{aligned}$$

which is a contradiction. Similar contradiction arises, if $-i\sqrt{\chi_1}$ is a zero of the polynomial $\psi(z)$. Hence, all the zeros of $\psi(z)$ are simple. As f, g share $(\{i\sqrt{\chi_1}, -i\sqrt{\chi_1}\}, 0)$, we have four possibilities as follows:

(i) there exists some complex number z_1 such that $f(z_1) = g(z_1) = i\sqrt{\chi_1}$,

(ii) for some complex number z_2, $f(z_2) = i\sqrt{\chi_1}$, $g(z_2) = -i\sqrt{\chi_1}$,

(iii) a complex number z_3 exists such that $f(z_3) = -i\sqrt{\chi_1}$, $g(z_3) = i\sqrt{\chi_1}$,

(iv) there exists a complex number z_4 such that $f(z_4) = g(z_4) = -i\sqrt{\chi_1}$. Taken into account all the combinations from (12), a simple calculation yields,

$$A = \frac{c(2ki\chi_1\sqrt{\chi_1} - 3\widehat{c})}{\widehat{c}(2i\chi_1\sqrt{\chi_1} - 3c)} = \frac{-c(2ki\chi_1\sqrt{\chi_1} + 3\widehat{c})}{\widehat{c}(2i\chi_1\sqrt{\chi_1} - 3c)} = \frac{-c(2ki\chi_1\sqrt{\chi_1} - 3\widehat{c})}{\widehat{c}(2i\chi_1\sqrt{\chi_1} + 3c)} = \frac{c(2ki\chi_1\sqrt{\chi_1} + 3\widehat{c})}{\widehat{c}(2i\chi_1\sqrt{\chi_1} + 3c)}.$$

Taking two combinations at a time from all possible combinations of the above equations we have either $\chi_1 = 0$ or $c = 0$ or $\widehat{c} = 0$ or $\frac{\widehat{c}}{c} = \pm k$. According to the hypothesis of the theorem, the only viable option is $\widehat{c} = ck$ and it is tackled by Lemma 7. So, our claim is proved and we have $F \equiv G$. Then in view of Lemma 7 one has $f \equiv g$.

(i) $\chi_1 = 0$

Following the similar arguments and Lemma 6 (ii) that was used in the proof of Theorem 1 to tackle the situation of $H \not\equiv 0$, we can get $H \equiv 0$. Next by a simple calculation as done in the case of $\chi_2 = 0$, we can again proceed to show that $A = 1$. As usual under the suggestion $A \neq 1$ we can obtain a similar equation of (12) as follows

$$\frac{g^3}{3} + \frac{\chi_2 g^2}{2} \equiv \frac{A\widehat{c}}{kc}\Big\{\frac{f^3}{3} + \frac{\chi_2 f^2}{2} - c\Big(1 - \frac{1}{A}\Big)\Big\}. \tag{13}$$

Suppose

$$\psi_1(z) = \frac{z^3}{3} + \frac{\chi_2 z^2}{2} - c\Big(1 - \frac{1}{A}\Big).$$

Claim. *All the factors of $\psi_1(f)$ are simple.*

From the construction of $\psi_1(z)$ it is evident that 0 can not be a multiple zero of $\psi_1(z)$. Suppose that $-\chi_2$ is a multiple zero of $\psi_1(z)$. Therefore, we have

$$c\Big(1 - \frac{1}{A}\Big) = \frac{\chi_2^3}{6}. \tag{14}$$

From (13), a simple calculation yields

$$\frac{g^3}{3} + \frac{\chi_2 g^2}{2} \equiv \frac{A\widehat{c}}{6kc}\left\{(f + \chi_2)^2(2f - \chi_2)\right\}. \tag{15}$$

By the supposition of the theorem, from (15), we get $E_f(\{-\chi_2\}) = E_g(\{0\})$. Hence, there exist a complex number z_0 such that $f(z_0) = 0$ and $g(z_0) = -\chi_2$. Using it from (15) we have $A = -\frac{kc}{\widehat{c}}$. Putting the expression of A in (14) we obtain $kc + \widehat{c} = \frac{k\chi_2^3}{6}$, which is a contradiction to the hypothesis of Theorem 1 (i) it follows that, 0 and $-\chi_2$ are Picard's exceptional values of f and g respectively. Therefore, $E_f\big(\{\frac{\chi_2}{2}\}\big) = E_g\big(\{-\frac{3\chi_2}{2}\}\big)$.

Case 1. Let $\frac{\chi_2}{2}$ and $-\frac{3\chi_2}{2}$ are not Picard's exceptional values of f and g respectively.

We note that there exists an entire function $\sigma_1(z)$ such that,

$$\frac{g}{f + \chi_2} = e^{\sigma_1(z)}. \tag{16}$$

As $f \not\equiv 0$ and $g \not\equiv -\chi_2$, from (16) we get that, $e^{\sigma_1(z)} \neq -1$, $\forall z$. We know there exists a complex number z_0 with, $f(z_0) = \frac{\chi_2}{2}$ and $g(z_0) = -\frac{3\chi_2}{2}$. From (16) we have,

$$e^{\sigma_1(z_0)} = \frac{g(z_0)}{f(z_0) + \chi_2} = -1,$$

a contradiction.

Case 2. Let $\frac{\chi_2}{2}$ is Picard's exceptional value of f and $-\frac{3\chi_2}{2}$ is Picard's exceptional value of g. Calculating by the similar manner of Case 1, we will get a contradiction.

Hence our claim is established and all the factors of $\psi_1(f)$ are simple say $(f - \alpha_i)$, $i \in \{1, 2, 3\}$. By the Second Fundamental Theorem we have

$$2T(r, f) \leq \sum_{i=1}^{3} \overline{N}(r, \alpha_i; f) + \overline{N}(r, \infty; f) + S(r, f) \leq$$

$$\leq \overline{N}\Big(r, -\frac{3\chi_2}{2}; g\Big) + \overline{N}(r, \infty; f) + S(r, f) \leq \frac{3}{2}T(r, f) + S(r, f),$$

which is a contradiction. Hence, $A = 1$ and $F \equiv G$. Next by Lemma 8, we get $f \equiv g$. □

5. Application. Let us recall the definition of strong uniqueness polynomials for meromorphic (entire) functions (SUPM (SUPE) in short), which was first introduced in [20].

Definition 8 ([20])**.** Let P be a polynomial in $\mathbb{C}$ and c be any arbitrary non zero constant. Suppose for two non constant meromorphic (entire) functions f and g, $P(f) \equiv cP(g)$ implies $f \equiv g$, then P is called SUPM (SUPE).

The first SUPM was exhibited by Yi. In 1996, Yi ([21, p. 78, (24)]), proved that for $m \geq 2$ and $n > 2m + 8$, the polynomial $z^n + az^{n-m} + b$ is a SUPM.

Recently, in [6], we have extended and generalized the definition of SUPM, namely SUPMWS as follows:

Definition 9 ([6])**.** Let $P(z)$ and $Q(z)$ be two polynomials in $\mathbb{C}$ and c be any arbitrary non zero constant. Suppose for two non constant meromorphic (entire) functions f and g, $P(f) \equiv cQ(g)$ implies $f \equiv g$, then the pair $P(z)$ and $Q(z)$ is called strong uniqueness polynomial for meromorphic (entire) functions in the wider sense, SUPMWS (SUPEWS) in brief.

First we will show that SUPMWS exists.

5.1. Existence of SUPMWS. Let us consider two polynomials

$$P_1(z) = Q_1(z) - c_1 = \frac{z^8}{8} - \frac{2az^7}{7} + \frac{a^2z^6}{6} - c_1, \quad \widehat{P}_1(z) = k_1Q_1(z) - \widehat{c}_1,$$

where k_1, c_1, $\widehat{c}_1$ are non zero complex numbers such that $P_1(z)$ and $\widehat{P}_1(z)$ do not have multiple zeros. The zeros of $Q_1(z)$ are 0 and θ_1, θ_2 say. We wish to show that, $P_1(z)$ and $\widehat{P}_1(z)$ are SUPMWS. Suppose A is a non-zero complex number such that $P_1(f) \equiv A\widehat{P}_1(g)$, i.e.,

$$Q_1(f) \equiv A\left(k_1Q_1(g) - \left(\widehat{c}_1 - \frac{c_1}{A}\right)\right). \tag{17}$$

We assume that, $A \neq \frac{c_1}{\widehat{c}_1}$. Now, by the Second Fundamental Theorem we obtain,

$$\begin{aligned} 8T(r,g) &= T(r, Q_1(g)) + O(1) \leq \\ &\leq \overline{N}(r,\infty;Q_1(g)) + \overline{N}(r,0;Q_1(g)) + \overline{N}\left(r, \left(\widehat{c}_1 - \frac{c_1}{A}\right); Q_1(g)\right) + S(r, Q_1(g)) \leq \\ &\leq \overline{N}(r,\infty;g) + \overline{N}(r,0,g) + \overline{N}(r,\theta_1,g) + \overline{N}(r,\theta_2,g) + \overline{N}(r,0,f) + \overline{N}(r,\theta_1,f) + \\ &+\overline{N}(r,\theta_2,f) + S(r,f) + S(r,g) \leq 7T(r,g) + S(r,f) + S(r,g), \end{aligned}$$

a contradiction. Hence, $A = \frac{c_1}{\widehat{c}_1}$. Therefore, (17) yields, $Q_1(f) \equiv \frac{k_1c_1}{\widehat{c}_1}Q_1(g)$. Taking $f \equiv gh$, a simple calculation yields

$$\frac{g^2}{8}\left(h^8 - \frac{k_1c_1}{\widehat{c}_1}\right) - \frac{2ag}{7}\left(h^7 - \frac{k_1c_1}{\widehat{c}_1}\right) + \frac{a^2}{6}\left(h^6 - \frac{k_1c_1}{\widehat{c}_1}\right) \equiv 0.$$

First suppose h is constant, then we have $h^8 = h^7 = h^6 = \frac{k_1c_1}{\widehat{c}_1}$, which implies $h = 1$.

Next, if h is non constant then calculating from the above equation we obtain

$$\left(g - \frac{8a(h^7 - \frac{k_1c_1}{\widehat{c}_1})}{7(h^8 - \frac{k_1c_1}{\widehat{c}_1})}\right)^2 \equiv \frac{a^2\eta(h)}{147(h^8 - \frac{k_1c_1}{\widehat{c}_1})^2},$$

$$\eta(z) = 192\Big(z^7 - \frac{k_1c_1}{\widehat{c}_1}\Big)^2 - 196\Big(z^6 - \frac{k_1c_1}{\widehat{c}_1}\Big)\Big(z^8 - \frac{kc_1}{\widehat{c}_1}\Big).$$

If $k_1c_1 \neq \widehat{c}_1$, by Lemma 9, we can say $\eta(h)$ has no multiple factors in h. Then $\eta(z)$ has 14 distinct simple zeros say π_i for $i \in \{1, 2, \ldots, 14\}$ and each π_i point of h is of multiplicity 2. Then applying the Second Fundamental Theorem we can write

$$12T(r,h) \leq \sum_{i=1}^{14} \overline{N}(r,\pi_i;h) + S(r,h) \leq 7T(r,h) + S(r,h),$$

which is a contradiction. Thus we get, $k_1c_1 = \widehat{c}_1$. So, $Q_1(f) \equiv Q_1(g)$, i.e., $P_1(f) \equiv P_1(g)$ and hence by Theorem 1.4 of [22], we can conclude that $f \equiv g$.

Now, we can see from Lemma 7 that $P(z)$ and $\widehat{P}(z)$ defined by (1), are SUPMWS of order $(1,2)$ where as their counterparts in Lemma 8 are of order $(2,1)$. So the pair of polynomials $P(z)$ $(\widehat{P}(z))$ in Theorem 1 (i) and Theorem 1 (ii) are symmetric polynomials. We have also observed that initially the presence of $\widehat{P}(z)$ is required for the notion of wider sense. We conjecture that, when the degree of the polynomial is small, the SUPMWS exists under certain constraints like $n = 3$. Therefore, the following question is inevitable.

Question 2. *What is the minimum degree of the polynomial to be an* SUPMWS *without any additional assumptions?*

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