UDC 517.5

O. P. Dovhopiatyi, N. S. Ilkevych, E. O. Sevost'yanov, A. L. Targonskii

ON PRIME END DISTORTION ESTIMATES OF MAPPINGS WITH THE POLETSKY CONDITION IN DOMAINS WITH THE POINCARÉ INEQUALITY

O. P. Dovhopiatyi, N. S. Ilkevych, E. O. Sevost'yanov, A. L. Targonskii. *On prime end distortion estimates of mappings with the Poletsky condition in domains with the Poincaré inequality*, Mat. Stud. **61** (2024), 148–159.

This article is devoted to the study of mappings with bounded and finite distortion defined in some domain of the Euclidean space. We consider mappings that satisfy some upper estimates for the distortion of the modulus of families of paths, where the order of the modulus equals to p, $n-1<p\leqslant n$. The main problem studied in the manuscript is the investigation of the boundary behavior of such mappings, more precisely, the distortion of the distance under mappings near boundary points. The publication is primarily devoted to definition domains with "bad boundaries", in which the mappings not even have a continuous extension to the boundary in the Euclidean sense. However, we introduce the concept of a quasiconformal regular domain in which the specified continuous extension is valid and the corresponding distance distortion estimates are satisfied; however, both must be understood in the sense of the so-called prime ends. More precisely, such estimates hold in the case when the mapping acts from a quasiconformal regular domain to an Ahlfors regular domain with the Poincaré inequality. The consideration of domains that are Ahlfors regular and satisfy the Poincaré inequality is due to the fact that, lower estimates for the modulus of families of paths through the diameter of the corresponding sets hold in these domains. (There are the so-called Loewner-type estimates). We consider homeomorphisms and mappings with branching separately. The main analytical condition under which the results of the paper were obtained is the finiteness of the integral averages of some majorant involved in the defining modulus inequality under infinitesimal balls. This condition includes the situation of quasiconformal and quasiregular mappings, because for them the specified majorant is itself bounded in a definition domain. Also, the results of the article are valid for more general classes for which Poletsky-type upper moduli inequalities are satisfied, for example, for mappings with finite length distortion.

1. Introduction. As known, quasiconformal mappings are locally Hölder continuous. In particular, the problem of the Hölder continuity of quasiconformal maps and some of their generalizations was previously studied by various authors, see, e.g., [1–5]. We also have studied this problem for Sobolev and Orlicz-Sobolev classes, see e.g. [6–8]. Now, our goal is to consider mappings that satisfy the Poletsky inequality with respect to p-modulus. At the same time, we are mainly interested in the case when maps act in domains with the $(1;p)$-Poincaré inequality. Here we consider two natural cases: when the mappings are homeomorphisms and when the mappings have branch points.

2020 *Mathematics Subject Classification:* 30C65, 31A15, 30C62.
Keywords: quasiconformal mappings, mappings with bounded and finite distortion, equicontinuity, moduli of families of paths.
doi:10.30970/ms.61.2.148-159

Below we use the following notations:

$$B(x_0,r)=\{x\in\mathbb{R}^n\colon |x-x_0|<r\},\quad \mathbb{B}^n=B(0,1),$$
$$S(x_0,r)=\{x\in\mathbb{R}^n\colon |x-x_0|=r\},\quad \mathbb{S}^{n-1}=S(0,1),$$

Let m be a Lebesgue measure in $\mathbb{R}^n$, let $\mathcal{H}^{n-1}$ be $(n-1)$-measured Hausdorff measure,

$$\Omega_n=m(\mathbb{B}^n),\quad \omega_{n-1}=\mathcal{H}^{n-1}(\mathbb{S}^{n-1}),$$
$$A(x_0,r_1,r_2):=\{x\in\mathbb{R}^n\colon r_1<|x-x_0|<r_2\}\,.$$

Below $M_p(\Gamma)$ denotes *p-modulus of* Γ (see [10]). Given sets E, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ denote by $\Gamma(E,F,D)$ the family of all paths $\gamma\colon[a,b]\to\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in D$ for $t\in(a,b)$. Given a family Γ of paths $\gamma\colon[a,b]\to D$ in D, by $f(\Gamma)$ we denote the family of paths $\{(f\circ\gamma)\colon[a,b]\to f(D),\gamma\in\Gamma\}$. (If $\gamma\colon[a,b]\to\overline{D}$, $\gamma(t)\in D$ if $a<t<b$, then under $f(\gamma)$ we understand the path $f(\gamma(t))$, $t\in(a,b)$. Similarly we are reasoning for the corresponding families Γ of paths γ, whose end points do not belong to D, but belong to the closure of D).

Let $Q\colon\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function equal to zero outside D. Consider the following concept, see [4, Section 7.6]. We say that a mapping $f\colon D\to\overline{\mathbb{R}^n}$ is a *ring Q-mapping at the point $x_0\in\overline{D}$ with respect to p-modulus,* $x_0\neq\infty$, $p\geqslant 1$, if for some $r_0=r(x_0)>0$ and arbitrary $0<r_1<r_2<r_0$ the condition

$$M_p(f(\Gamma(S(x_0,r_1),S(x_0,r_2),D)))\leqslant\int\limits_A Q(x)\cdot\eta^p(|x-x_0|)\,dm(x),\tag{1}$$

is fulfilled, where $\eta\colon(r_1,r_2)\to[0,\infty]$ is arbitrary Lebesgue measurable function that satisfies inequality

$$\int\limits_{r_1}^{r_2}\eta(r)dr\geqslant 1.\tag{2}$$

According to [11, Section 7.22], we say that a Borel function $\rho\colon D\to[0,\infty]$ is an *upper gradient* of the function $u\colon D\to\mathbb{R}$, if the relation $|u(x)-u(y)|\leqslant\int\limits_\gamma\rho|dx|$ holds for all rectifiable paths γ, joining the points x and $y\in D$. Let $u_B:=\frac{1}{m(B)}\int\limits_B u(x)dm(x)$. Given $p\geqslant 1$, we say that D supports the $(1;p)$-*Poincaré inequality,* if there are constants $C\geqslant 1$ and $\tau>0$ such that, the relation

$$\frac{1}{m(B)}\int\limits_B|u(x)-u_B|dm(x)\leqslant C\cdot(\mathrm{diam}B)\left(\frac{1}{m(\tau B)}\int\limits_{\tau B}\rho^p(x)dm(x)\right)^{1/p}$$

holds for any ball $B\subset D$, an arbitrary bounded continuous function $u\colon D\to\mathbb{R}$ and any upper gradient ρ of u. A domain D is called *Ahlfors regular,* if there is $C\geqslant 1$ such that the inequality

$$\frac{1}{C}R^n\leqslant m(B(x_0,R)\cap D)\leqslant CR^n$$

holds for any $x_0\in D$ and an arbitrary $R<\mathrm{diam}D$.

Given sets $A, B\subset {\mathbb R}^n$, we put, as usual,

$$\mathrm{diam} A=\sup\limits_{x,y\in A}|x-y|\,,\quad \mathrm{dist}(A, B)=\inf\limits_{x\in A, y\in B}|x-y|\,.$$

Sometimes instead of $\mathrm{diam} A$ and $\mathrm{dist}(A, B)$ we also write $d(A)$ and $d(A, B)$, respectively.

Recall that, a mapping $f\colon D\rightarrow \overline{{\mathbb R}^n}$ between domains $D\subset {\mathbb R}^n$ and $D^{\,\prime}\subset \overline{{\mathbb R}^n}$ is called *closed* if $C(f, \partial D)\subset \partial D^{\,\prime}$, where, as usual, $C(f, \partial D)$ is the cluster set of f at ∂D.

Later, in the extended Euclidean space $\overline{{\mathbb R}^n}={\mathbb R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is the stereographic projection of $\overline{{\mathbb R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in ${\mathbb R}^{n+1}$, $e_{n+1}=(\underbrace{0,0,\ldots, 0,1}_{n+1})$. Namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \quad x\ne \infty\ne y \tag{3}$$

(see, e.g., [10, Definition 12.1]). Further, for the sets $A, B\subset \overline{{\mathbb R}^n}$ we set

$$h(A, B)=\inf\limits_{x\in A, y\in B}h(x, y)\,,\quad h(A)=\sup\limits_{x,y\in A}h(x,y)\,,$$

where h is the chordal distance defined in (3).

The definition of a prime end used below may be found in [12]. In particular, we say that the end of K is *prime* if K contains a chain of cuts $\{\sigma_m\}$ such that

$$\lim\limits_{m\rightarrow\infty}M(\Gamma(C, \sigma_m, D))=0 \tag{4}$$

for any continuum C in D, where $M(\Gamma(C, \sigma_m, D))$ is the modulus of family $\Gamma(C, \sigma_m, D)$. Here and further $\overline{D}_P$ denotes replenishment of the domain D with its prime ends, and $E_D=\overline{D}_P\setminus D$ is the set of all prime ends in D. We say that, a bounded domain D in ${\mathbb R}^n$ *is regular (in the quasiconformal sense)*, if D may be quasiconformally mapped onto a domain with a locally quasiconformal boundary, the closure of which is a compact set in ${\mathbb R}^n$, in addition, every prime end $P\subset E_D$ is regular. Note that, the closure $\overline{D}_P$ of the regular domain D is *metrizable*; namely, if $g\colon D_0\rightarrow D$ is a quasiconformal mapping of D_0 onto D, where D_0 is a domain with a locally quasiconformal boundary, then for $x, y\in \overline{D}_P$ we put

$$\rho(x, y):=|g^{\,-1}(x)-g^{\,-1}(y)|\,.$$

Here, for $x\in E_D$, the element $g^{\,-1}(x)$ is understood as some (single) point of the boundary D_0, which is well-defined due to [13, Theorem 4.1].

The impression of the prime end $P_0\in E_D$ is defined as the following way:

$$I(P_0)=\bigcap\limits_{m=1}^{\infty}\overline{d_m}\,,$$

where d_m, $m=1,2,\ldots,$ is some decreasing sequence of domains formed by cuts corresponding to P_0. It may be shown that, $I(P_0)$ is well-defined (in other words, $I(P_0)$ does not depend on the selected one sequence d_m, $m=1,2,\ldots,$) in addition, $I(P_0)\subset \partial D$ (see, e.g., [12, Proposition 1].

Given $\delta > 0$ and $p \geqslant 1$, domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, $P_0 \in E_D$, a continuum $A \subset D$ and a Lebesgue-measurable function $Q\colon D \to [0, \infty]$ we denote by $\mathfrak{F}^{p,P_0}_{Q,A,\delta}(D, D')$ the family of all homeomorphisms f of D onto D' that satisfy the relations (1)–(2) for any $x_0 \in I(P_0)$ (here $I(P_0)$ denotes the impression of the prime end of P_0) such that $\mathrm{diam}(f(A)) \geqslant \delta$. The following statement holds.

Theorem 1. *Let $P_0 \in E_D := \overline{D}_P \setminus D$, let D be a regular domain (in the quasiconformal sense), and let D' be an Ahlfors regular bounded domain supporting $(1;p)$-Poincaré inequality, $n-1 < p \leqslant n$. Suppose that, the following conditions hold: 1) for any $y_0 \in \partial D$ there exists $r_0' = r_0'(y_0) > 0$ such that the set $B(y_0, r) \cap D$ is finitely connected for all $0 < r < r_0'$, moreover, for any component K of the set $B(y_0, r) \cap D$ the following condition is fulfilled: any $x, y \in K$ may be joined by some path $\gamma\colon [a, b] \to \mathbb{R}^n$ such that $|\gamma| \in K \cap \overline{B(y_0, \max\{|x - y_0|, |y - y_0|\})}$,*

$$|\gamma| = \{y \in \mathbb{R}^n \colon \exists t \in [a, b] \colon \gamma(t) = y\};$$

2) for each $y_0 \in I(P_0)$ there exists $C = C(y_0) \in (0, +\infty)$ such that

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(y_0,\varepsilon)\cap D} Q(x)\,dm(x) \leqslant C.$$

Then for each $P \in E_D = \overline{D}_P \setminus D$ there exists $y_0 \in \partial D$ such that $I(P) = \{y_0\}$, where $I(P)$ denotes the impression of a prime end P. In addition, there exist a neighborhood U of P_0, $\varepsilon_0 = \varepsilon_0(P_0, D', n, p, \delta, A) > 0$ and a number $\widetilde{C} = \widetilde{C}(p, n, C, D') > 0$ such that the inequality

$$|f(x) - f(y)| \leqslant \widetilde{C} \cdot \max\left\{\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x - x_0|}}, \frac{1}{\log^{n-1}\frac{\varepsilon_0}{|y - x_0|}}\right\} \tag{5}$$

holds for all $f \in \mathfrak{F}^{p,P_0}_{Q,A,\delta}(D, D')$ and any $x, y \in U \cap D$, where $x_0 := I(P_0)$.

Corollary 1. *Under the conditions and notations of Theorem 1, f has a continuous extension to $P_0 \in E_D$, and*

$$|f(x) - f(P_0)| \leqslant \frac{\widetilde{C}}{\log^{n-1}\frac{\varepsilon_0}{|x - x_0|}}. \tag{6}$$

Remark 1. In particular, condition 1) of Theorem 1 is fulfilled if, for each $x_0 \in \partial D$ there exists $r_0' = r_0'(x_0) > 0$ such that the set $B(x_0, r) \cap D$ is finitely connected for all $0 < r < r_0'$, and each component K of $B(x_0, r) \cap D$ is convex. Indeed, let $x, y \in K$. Join x and y by a segment γ inside K. Let $|x - x_0| \geqslant |y - x_0|$. Due to the fact that the ball $\overline{B(x, |x - x_0|)}$ is convex, the segment γ entirely belongs to $\overline{B(x, |x - x_0|)}$. Then γ is the desired path, as required.

Given numbers $p \geqslant 1$ and $\delta > 0$, domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, $P_0 \in E_D$, and a Lebesgue measurable function $Q\colon D \to [0, \infty]$ we denote by $\mathfrak{R}^{p,P_0}_{Q,\delta}(D, D')$ a family of all open, discrete and closed mappings f of D onto D' that satisfy the conditions (1)–(2) at any point $x_0 \in I(P_0)$ (where $I(P_0)$ denotes the impression of a prime end P_0) such that there is a continuum $K_f \subset D'$, for which $\mathrm{diam}(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$. The following theorem holds.

Theorem 2. *Let D be a regular domain, and D' be an Ahlfors regular bounded domain with $(1;p)$-Poincaré inequality, $n-1<p\leqslant n$. Assume that the following conditions are fulfilled: 1) for each $x_0\in\partial D$ there exists $r_0'=r_0'(x_0)>0$ such that the set $B(x_0,r)\cap D$ is finitely connected for all $0<r<r_0'$, moreover, any component K of $B(x_0,r)\cap D$ satisfies the following condition: any $x,y\in K$ may be joined by a path $\gamma\colon[a,b]\to\mathbb{R}^n$ such that $|\gamma|\in K\cap\overline{B(x_0,\max\{|x-x_0|,|y-x_0|\})}$,*

$$|\gamma|=\{x\in\mathbb{R}^n\colon\exists t\in[a,b]\colon\gamma(t)=x\};$$

2) for any $x_0\in I(P_0)$ there is $0<C=C(x_0)<\infty$ such that

$$\limsup_{\varepsilon\to 0}\frac{1}{\Omega_n\cdot\varepsilon^n}\int\limits_{B(x_0,\varepsilon)\cap D}Q(x)dm(x)\leqslant C,$$

where $I(P_0)$ denotes the impression of P_0. Then for each $P\in E_D:=\overline{D}_P\setminus D$ there exists $y_0\in\partial D$ such that $I(P)=\{y_0\}$. In addition, there exist a neighborhood U of P_0, $\varepsilon_0=\varepsilon_0(P_0,D',n,p,\delta)>0$ and $\widetilde{C}=\widetilde{C}(p,n,C,D')>0$ such that

$$|f(x)-f(y)|\leqslant\widetilde{C}\cdot\max\left\{\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}},\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|y-x_0|}}\right\}\qquad(7)$$

for all $f\in\mathfrak{R}^{p,P_0}_{Q,\delta}(D,D')$ and $x,y\in U\cap D$, where $x_0:=I(P_0)$.

Corollary 2. *Under the conditions and notations of Theorem 2, f has a continuous extension to $P_0\in E_D$ and*

$$|f(x)-f(P_0)|\leqslant\frac{\widetilde{C}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}.$$

Remark 2. In particular, all bounded convex domains D' are Ahlfors regular domains with $(1;p)$-Poincaré inequality for any $n-1<p\leqslant n$.

Indeed, bounded convex domains are John domains (see [14, Remark 2.4]) and hence uniform domains in the sense of Martio-Sarvas (see [14, note 2.13(c)]). Such domains are also quasiextremal distance (QED) domains by Gering-Martio. In other words, there exists a constant $1\leqslant A_0^*<\infty$ such that

$$M(\Gamma(E,F,\mathbb{R}^n))\leqslant A_0^*\cdot M(\Gamma(E,F,D))\qquad(8)$$

(see [15, Lemma 2.18]). Since $\mathbb{R}^n$ is a Loewner space ([11, Theorem 8.1]), due to (8) bounded convex domains are also Loewner as metric spaces. But then by [11, Proposition 8.19] bounded convex domains are also Ahlfors regular. Finally, by Theorem 10.5 in [9] bounded convex domains satisfy the $(1;1)$-Poincaré inequality. Therefore, by the Hölder inequality, they also satisfy the $(1;p)$-Poincaré inequality for any $p>1$.

2. Preliminaries. The following statement may be found in [16, Proposition 4.7].

Proposition 1. *Let D be Ahlfors regular domain in which $(1;p)$-Poincaré inequality holds, $n-1<p\leqslant n$. Let $x\in D$, $R>0$ and let E and F are continua in $B(x,R)$. Then there is $M>0$ such that the inequality*

$$M_p(\Gamma(E,F,D))\geqslant\frac{1}{M}\cdot\frac{\min\{\mathrm{diam}E,\mathrm{diam}F\}}{R^{1+p-n}}$$

holds.

Let $a>0$ and let $\varphi\colon [a,\infty)\rightarrow [0,\infty)$ be a nondecreasing function such that, for some constants $\gamma>0$, $T>0$ and all $t\geqslant T$, the inequality

$$\varphi(2t)\leqslant \gamma\cdot\varphi(t) \tag{9}$$

is fulfilled. We will call such functions *functions that satisfy the doubling condition.*

Let $\varphi\colon [a,\infty)\rightarrow [0,\infty)$ be a function with the doubling condition, then the function $\widetilde{\varphi}(t):=\varphi(1/t)$ does not increase and is defined on a half-interval $(0,1/a]$. The following statement is proved in [6, Lemma 3.1].

Proposition 2. *Let $a>0$, let $\varphi\colon [a,\infty)\rightarrow [0,\infty)$ be a nondecreasing function with a doubling condition (9), let $x_0\in \mathbb{R}^n$, $n\geqslant 2$, and let $Q\colon \mathbb{R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function for which there exists $0<C<\infty$ such that*

$$\limsup_{\varepsilon\rightarrow 0}\frac{\varphi(1/\varepsilon)}{\Omega_n\cdot\varepsilon^n}\int\limits_{B(x_0,\varepsilon)}Q(x)dm(x)\leqslant C.$$

Then there exists $\varepsilon_0^{\prime}>0$ such that

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0^{\prime}}\frac{\varphi(1/|x-x_0|)Q(x)dm(x)}{|x-x_0|^n}\leqslant C_1\cdot\left(\log\frac{1}{\varepsilon}\right),\quad \varepsilon\rightarrow 0,$$

where $C_1:=\gamma C\Omega_n 2^n/\log 2$.

Let $D\subset \mathbb{R}^n$, $f\colon D\rightarrow \mathbb{R}^n$ be an open discrete mapping, let $\beta\colon [a,b)\rightarrow \mathbb{R}^n$ be a path and let $x\in f^{-1}(\beta(a))$. A path $\alpha\colon [a,c)\rightarrow D$ is called a *maximal f-lifting* of β with the origin at the point x, if (1) $\alpha(a)=x$; (2) $f\circ\alpha=\beta|_{[a,c)}$; (3) for every $c<c^{\prime}\leqslant b$, there is no a path $\alpha^{\prime}\colon [a,c^{\prime})\rightarrow D$ such that $\alpha=\alpha^{\prime}|_{[a,c)}$ and $f\circ\alpha^{\prime}=\beta|_{[a,c^{\prime})}$. The following statement is true, see [1, Lemma 3.12], cf. [17, Lemma 3.7].

Proposition 3. *Let $f\colon D\rightarrow \mathbb{R}^n$, $n\geqslant 2$, be open discrete mapping, let $x_0\in D$, and let $\beta\colon [a,b)\rightarrow \mathbb{R}^n$ be a path such that $\beta(a)=f(x_0)$ and either $\lim\limits_{t\rightarrow b}\beta(t)$ exists, or $\beta(t)\rightarrow\partial f(D)$ as $t\rightarrow b$. Then β has a maximal f-lifting of $\alpha\colon [a,c)\rightarrow D$ starting at the point x_0. If $\alpha(t)\rightarrow x_1\in D$ as $t\rightarrow c$, then $c=b$ and $f(x_1)=\lim\limits_{t\rightarrow b}\beta(t)$. Otherwise, $\alpha(t)\rightarrow\partial D$ as $t\rightarrow c$.*

3. Proof of Theorem 1. For convenience, let us divide the proof into separate items.

I. Let $f\in\mathfrak{F}_{Q,A,\delta}^{p,P_0}(D,D^{\prime})$. Since the set $B(y_0,r)\cap D$ is finitely connected for all $y_0\in\partial D$ and $0<r<r_0^{\prime}(y_0)$, the domain D is finitely connected at its boundary. Therefore, the domain D is uniform (see [19, Theorem 3.2]). In other words, for any $r>0$ there exists a number $\delta>0$ such that the inequality $M(\Gamma(F^*,F,D))\geqslant\delta$ holds for all continua F, $F^*\subset D$ such that $h(F)\geqslant r$ and $h(F^*)\geqslant r$.

II. Let us to prove that, for any $P\in E_D$ there exists $y_0\in\partial D$ such that $I(P)=\{y_0\}$. We will prove this statement from the opposite, namely, suppose that there exists a prime end $P\in E_D$, which contains two points $x,y\in\partial D$, $x\neq y$. In this case, there are at least two sequences $x_m,y_m\in d_m$, $m=1,2,\ldots$, which converge to x and y as $m\rightarrow\infty$, respectively (here d_m denotes a decreasing sequence of domains formed by some sequence of cuts σ_m, corresponding to the prime end P). Let us join the points x_m and y_m with the path γ_m in the domain d_m. Since $x \neq y$, there exists $m_0 \in \mathbb{N}$ such that $h(\gamma_m) \geqslant d(x,y)/2$, $m > m_0$. Choose any nondegenerate continuum $C \subset D \setminus d_1$. Then, due to the uniformity of the domain D

$$M(\Gamma(|\gamma_m|, C, D)) \geqslant \delta_0 > 0 \tag{10}$$

for some $\delta_0 > 0$ and all $m > m_0$. The inequality (10) contradicts the definition of a prime end P. Indeed, by the definition of the cut σ_m we will have: $\Gamma(|\gamma_m|, C, D) > \Gamma(\sigma_m, C, D)$. Then by (4) we have that $M(\Gamma(|\gamma_m|, C, D)) \leqslant M(\Gamma(\sigma_m, C, D)) \to 0$ as $m \to \infty$. The last relation contradicts (10). Therefore, $I(P) = \{y_0\}$ for some $y_0 \in \partial D$.

III. It remains to prove the relation (5). Let $x_0 := I(P_0)$ and $\varepsilon_0 = \min\{\varepsilon_0'(x_0), r_0', \mathrm{dist}(x_0, A), 1\}$, where $\varepsilon_0' > 0$ is a number from Proposition 2, $\varphi \equiv 1$, r_0' is a number from the conditions of the theorem. By the definition of $\varepsilon_0 > 0$,

$$A \subset D \setminus B(x_0, \varepsilon_0). \tag{11}$$

Since $I(P_0) = \{x_0\}$, we may find a neighborhood U of P_0 in $\overline{D}_P$ such that $U \cap D \subset B(x_0, \widetilde{\varepsilon_0})$, $\widetilde{\varepsilon_0} = \varepsilon_0^2$. By the definition of a regular domain, we may consider that $U \cap D$ is connected. Now $U \cap D$ belongs to one and only one component K of $B(x_0, \widetilde{\varepsilon_0}) \cap D$. Let $x, y \in U \cap D$ and $f \in \mathfrak{F}^{p,P_0}_{Q,A,\delta}(D, D')$. It is possible assume that $|x - x_0| \geqslant |y - x_0|$. By definition r_0' points x and y may be connected by a path K, which contained in the ball $\overline{B(x_0, |x - x_0|)}$. Let $z, w \in f(A) \subset D'$ and $u, v \in A$ be such that $\mathrm{diam} f(A) = |z - w| = |f(u) - f(v)| \geqslant \delta$, see Figure 1.

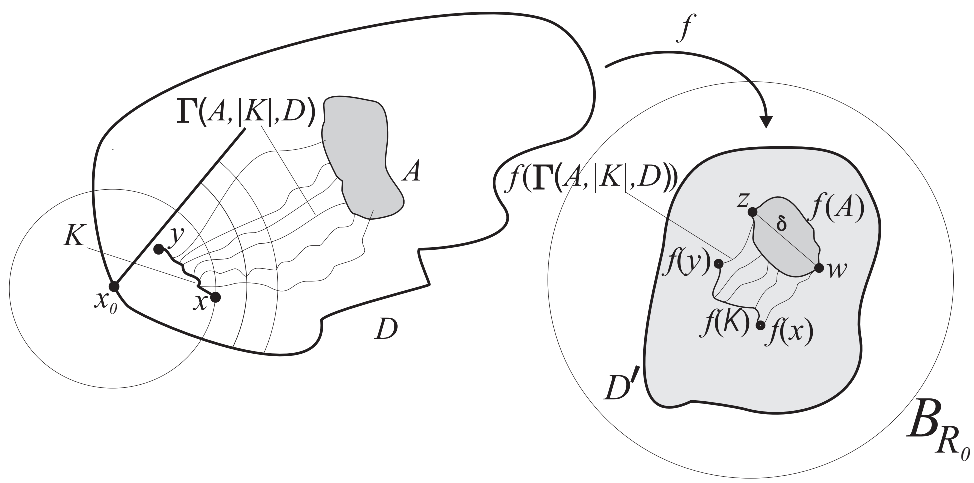

Figure 1: To the proof of Theorem 1

Note that, $f(K)$ and $f(A)$ are continua as a continuous image of the continua K and A under the mapping f, respectively. We fix arbitrarily $y_0 \in D'$. Since D' is bounded, there exists $R_0 > 0$ such that $D' \subset B(y_0, R_0) := B_{R_0}$. In this case, by Proposition 1

$$M_p(\Gamma(f(K), f(A), D')) \geqslant \frac{1}{M} \cdot \frac{\min\{\mathrm{diam} f(K), \mathrm{diam} f(A)\}}{R_0^{1+p-n}}. \tag{12}$$

Observe that,

$$\Gamma(|K|, A, D) > \Gamma(S(x_0, |x - x_0|), S(x_0, \varepsilon_0), D). \tag{13}$$

Indeed, let $\gamma \in \Gamma(|K|, A, D)$, $\gamma\colon [0,1] \to D$, $\gamma(0) \in |K|$ and $\gamma(1) \in A$. Since $|K| \subset B(x_0, |x - x_0|)$, due to (11), we obtain that $A \subset D \setminus B(x_0, |x - x_0|)$. Then $|\gamma| \cap B(x_0, |x - x_0|) \neq$

$\varnothing \neq |\gamma| \cap (D \setminus B(x_0, |x-x_0|))$. By [18, Theorem 1.I.5.46] there exists $t_1 \in (0,1)$ such that $\gamma(t_1) \in S(x_0, |x-x_0|)$. Consider the path $\gamma_1 := \gamma|_{[t_1,1]}$. Recall that $|K| \subset B(x_0, \varepsilon_0)$, in addition, due to (11) $A \subset D \setminus B(x_0, \varepsilon_0)$. Then $\gamma_1 \cap B(x_0, \varepsilon_0) \neq \varnothing \neq |\gamma| \cap (D \setminus B(x_0, \varepsilon_0))$. By [18, theorem 1.I.5.46] there exists $t_2 \in (t_1, 1)$ such that $\gamma_1(t_2) = \gamma(t_2) \in S(x_0, \varepsilon_0)$. Put $\gamma_2 := \gamma|_{[t_1,t_2]}$. Then γ_2 is a subpath of γ and $\gamma_2 \in \Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D)$. This proves (13). In in this case, by the minorization principle of the modulus of families of paths (see, e.g., [10, Theorem 6.4]), by (1) and (13) we obtain that

$$M_p(\Gamma(f(|K|), f(A), D')) = M_p(f(\Gamma(|K|, A, D))) \leqslant$$

$$\leqslant M_p(f(\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D))) \leqslant \int\limits_A Q(z) \cdot \eta^p(|z-x_0|) dm(z), \tag{14}$$

where η is an arbitrary Lebesgue measurable function satisfying (2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$. Set

$$\eta(t) := \begin{cases} \frac{1}{\left(\log \frac{\varepsilon_0}{|x-x_0|}\right)^{n/p} t^{n/p}}, & t \in (|x-x_0|, \varepsilon_0); \\ 0, & t \notin (|x-x_0|, \varepsilon_0). \end{cases}$$

Observe that,

$$\int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{t \log \frac{\varepsilon_0}{|x-x_0|}} = \frac{1}{\log \frac{\varepsilon_0}{|x-x_0|}} \cdot \int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{t} = 1.$$

Then, by Hölder inequality, we obtain that

$$1 = \int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{t \log \frac{\varepsilon_0}{|x-x_0|}} \leqslant \left(\int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{t^{\frac{n}{p}} \cdot \left(\log \frac{\varepsilon_0}{|x-x_0|}\right)^{\frac{n}{p}}} \right)^{\frac{p}{n}} \cdot (\varepsilon_0 - |x-x_0|)^{\frac{n-p}{n}} \leqslant$$

$$\leqslant \left(\int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{t^{\frac{n}{p}} \cdot \left(\log \frac{\varepsilon_0}{|x-x_0|}\right)^{\frac{n}{p}}} \right)^{\frac{p}{n}} = \left(\int\limits_{|x-x_0|}^{\varepsilon_0} \eta(t) dt \right)^{\frac{p}{n}}. \tag{15}$$

By (15), the function η satisfies the condition (2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$. In this case, by (14) we obtain that

$$M_p(\Gamma(f(|K|), f(A), D')) \leqslant \frac{1}{\log^n \frac{\varepsilon_0}{|x-x_0|}} \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} \frac{Q(z)}{|z-x_0|^n} dm(z). \tag{16}$$

Since $|x-x_0| < \varepsilon_0^2$,

$$\log \frac{1}{|x-x_0|} < 2 \log \frac{\varepsilon_0}{|x-x_0|}. \tag{17}$$

By the choice ε_0, by Proposition 2 and by (16) and (17) we obtain that

$$M_p(\Gamma(f(|K|), f(A), D')) \leqslant \frac{2}{\log^{n-1} \frac{\varepsilon_0}{|x-x_0|}} \frac{C\Omega_n 2^n}{\log 2}. \tag{18}$$

Uniting (12) and (18), we have

$$\frac{1}{M}\cdot\frac{\min\{{\rm diam} f(K), {\rm diam} f(A)\}}{R_0^{1+p-n}}\leqslant \frac{2^{n+1}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\frac{C\Omega_n}{\log 2}.$$

Since $\omega_{n-1}=n\Omega_n$, the latter relation may be rewritten as

$$\frac{1}{M}\cdot\frac{\min\{{\rm diam} f(K), {\rm diam} f(A)\}}{R_0^{1+p-n}}\leqslant \frac{\omega_{n-1}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\frac{C2^{n+1}}{n\log 2}. \qquad (19)$$

By the definition of $\mathfrak{F}^{p,P_0}_{Q,A,\delta}(D,D^{\prime})$,

$$\min\{{\rm diam} f(K), {\rm diam} f(A)\}\geqslant \min\{{\rm diam} f(K), \delta\}.$$

Then, from (19) we obtain that

$$\min\{{\rm diam} f(K), \delta\}\leqslant \frac{\omega_{n-1}MR_0^{1+p-n}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\cdot\frac{C2^{n+1}}{n\log 2}. \qquad (20)$$

Observe that, $\frac{\omega_{n-1}MR_0^{1+p-n}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\cdot\frac{C2^{n+1}}{n\log 2}\rightarrow 0$ as $x\rightarrow x_0$. Then there is $0<\sigma=\sigma(x_0, M, R_0, n, p, \delta)$ such that

$$\frac{\omega_{n-1}MR_0^{1+p-n}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\cdot\frac{C2^{n+1}}{n\log 2}<\delta,\quad \forall x\in B(x_0, \sigma). \qquad (21)$$

Let $|x-x_0|<\sigma$, by (20) and (21), we obtain that

$$|f(x)-f(y)|\leqslant {\rm diam} f(K)\leqslant \frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\cdot\frac{\omega_{n-1}MR_0^{1+p-n}C2^{n+1}}{n\log 2}.$$

Observe that, by the triangle inequality $|f(x)-f(y)|\leqslant 2R_0$, thus for $|x-x_0|\geqslant\sigma$ we have that

$$\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\geqslant \frac{1}{\log^{n-1}\frac{\varepsilon_0}{\sigma}}:=\widetilde{P}_0. \qquad (22)$$

Therefore,

$$|f(x)-f(y)|\leqslant 2R_0\leqslant \frac{2R_0}{\widetilde{P}_0}\cdot\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}. \qquad (23)$$

Set $\widetilde{C}:=\max\left\{\frac{\omega_{n-1}MR_0^{1+p-n}C2^{n+1}}{n\log 2}, \frac{2R_0}{\widetilde{P}_0}\right\}$. A constant $\widetilde{C}$ depends only on p, n, C and $D^{\prime}$, because M and R_0 are defined by $D^{\prime}$.

Proof of Corollary 1. By Theorem 1 $I(P_0)=\{x_0\}$. Then the conditions $x\rightarrow P_0$ and $y\rightarrow P_0$ imply that $x\rightarrow x_0$ and $y\rightarrow x_0$. If f would not have a limit as $x\rightarrow P_0$, then we would constructed at least two sequences $x_m\rightarrow P_0$ and $y_m\rightarrow P_0$ as $m\rightarrow\infty$, such that $|f(x_m)-f(y_m)|\geqslant\delta>0$ for some positive $\delta>0$ and all $m=1,2,\ldots$. But this contradicts to (5). Therefore, the limit of f at $x\rightarrow P_0$ exist. To prove (6) it remains to pass in (5) to the limit as $y\rightarrow P_0$. □

4. Proof of Theorem 2. Due to the fact that the proof of this theorem is very similar to all the previous ones, let us limit ourselves to the scheme of the proof. The proof of the fact that, for any $P\in E_D$ there exists $y_0\in\partial D$ such that $I(P)=\{y_0\}$, may be carried out similarly to the proof of Theorem 1.

Let us to prove the ratio (7). As proved above, there is $x_0\in\partial D$ such that $I(P_0)=\{x_0\}$. Let $\varepsilon_0=\min\{\varepsilon_0^{\prime}(x_0), r_0^{\prime}, 1, \delta\}$, where $\varepsilon_0^{\prime}>0$ is the number from Proposition 2, $\varphi\equiv 1$, $r_0^{\prime}$ is a number from the conditions of the theorem. It follows that there is a neighborhood U of P_0 in $\overline{D}_P$ such that $U\cap D$ is connected and $U\cap D\subset B(x_0, \widetilde{\varepsilon_0})$, $\widetilde{\varepsilon_0}=\varepsilon_0^2$. Now $U\cap D$ belongs to one and only one component K of $B(x_0, \widetilde{\varepsilon_0})\cap D$. Let $x, y\in U\cap D$ and $f\in\mathfrak{R}^{p,P_0}_{Q,\delta}(D, D^{\prime})$. We may assume that $|x-x_0|\geqslant|y-x_0|$. By the definition of $r_0^{\prime}$, x and y may be joined by the path in K, contained in the ball $\overline{B(x_0, |x-x_0|)}$.

Let $K_f\subset D^{\prime}$ be a continuum such that $\mathrm{diam}(K_f)\geqslant\delta$ and $h(f^{-1}(K_f), \partial D)\geqslant\delta>0$ (it exists by definition of the class $\mathfrak{R}^{p,P_0}_{Q,\delta}(D, D^{\prime})$). Also let $z, w\in K_f\subset D^{\prime}$ be such that

$$\mathrm{diam}K_f=|z-w|\geqslant\delta.$$

Note that $f(K)$ is a continuum as a continuous image of the continuum K under the mapping f. Fix $y_0\in D^{\prime}$. Then, since $D^{\prime}$ is bounded, there exists $R_0>0$ such that $D^{\prime}\subset B(y_0, R_0):=B_{R_0}$. In this case, by Proposition 1

$$M_p(\Gamma(f(K), K_f, D^{\prime}))\geqslant\frac{1}{M}\cdot\frac{\min\{\mathrm{diam}f(K), \mathrm{diam}K_f\}}{R_0^{1+p-n}}. \tag{24}$$

Let Γ^* be a family $\gamma\colon [0,1)\to D$ of all of maximal f-lifting of paths $\gamma^{\prime}\colon [0,1]\to D^{\prime}$ in $\Gamma=\Gamma(|f(K)|, K_f, D^{\prime})$ starting in $|K|$. Such liftings exist due to Proposition 3. By the same Proposition, due to the closeness of the mapping f, any path $\gamma\in\Gamma^*$ has a continuous extension $\gamma\colon [0,1]\to D$ to the point $b=1$. Then $\gamma(1)\in f^{-1}(K_f)$, that is, $\Gamma^*\subset\Gamma(|K|, f^{-1}(K_f), D)$.

Reasoning similarly to the proof of the relation (13), it is possible prove that

$$\Gamma(|K|, f^{-1}(K_f), D)>\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D). \tag{25}$$

Note that, $f(\Gamma^*)=\Gamma=\Gamma(|f(K)|, K_f, D^{\prime})$. In this case, by the minorization principle of the modulus (see, e.g., [10, Theorem 6.4]), due to (25) and (1) we obtain that

$$M_p(f(\Gamma^*))=M_p(\Gamma(|f(K)|, K_f, D^{\prime}))\leqslant$$

$$\leqslant M_p(f(\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D)))\leqslant\int\limits_A Q(z)\cdot\eta^p(|z-x_0|)\,dm(z), \tag{26}$$

where η is an arbitrary Lebesgue measurable function that satisfies ratio (2) for $r_1=|x-x_0|$, $r_2=\varepsilon_0$. Let us put

$$\eta(t):=\begin{cases}\frac{1}{\left(\log\frac{\varepsilon_0}{|x-x_0|}\right)^{n/p}t^{n/p}}, & t\in(|x-x_0|, \varepsilon_0);\\ 0, & t\notin(|x-x_0|, \varepsilon_0).\end{cases}$$

By (15), η satisfies condition (2) when $r_1=|x-x_0|$ and $r_2=\varepsilon_0$. Then from (26) it follows that

$$M_p(\Gamma(|f(K)|, K_f, D^{\prime}))\leqslant\frac{1}{\log^n\frac{\varepsilon_0}{|x-x_0|}}\int\limits_{A(x_0, |x-x_0|, \varepsilon_0)}\frac{Q(z)}{|z-x_0|^p}dm(z). \tag{27}$$

Since $|x-x_0|<\varepsilon_0^2$, (17) holds. By choosing ε_0, from Proposition 2 and by (27) we obtain that

$$M_p(\Gamma(f(|K|),K_f,D'))\leqslant\frac{2}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\frac{C\Omega_n2^n}{\log 2}. \tag{28}$$

Combining (24) and (28), we obtain that

$$\frac{1}{M}\cdot\frac{\min\{\mathrm{diam} f(K),\mathrm{diam} K_f\}}{R_0^{1+p-n}}\leqslant\frac{2}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\frac{C\Omega_n2^n}{\log 2}.$$

Since $\omega_{n-1}=n\Omega_n$, the last relation may be rewritten as

$$\frac{1}{M}\cdot\frac{\min\{\mathrm{diam} f(K),\mathrm{diam} K_f\}}{R_0^{1+p-n}}\leqslant\frac{\omega_{n-1}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\frac{C2^{n+1}}{n\log 2}. \tag{29}$$

By (), $\min\{\mathrm{diam} f(K),\mathrm{diam} K_f\}\geqslant\min\{\mathrm{diam} f(K),\delta\}$. Then by (29) we obtain that

$$\min\{\mathrm{diam} f(K),\delta\}\leqslant\frac{\omega_{n-1}MR_0^{1+p-n}}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}\cdot\frac{C2^{n+1}}{n\log 2}. \tag{30}$$

By (21) and (23), due to (30) we obtain that

$$|f(x)-f(y)|\leqslant\mathrm{diam} f(K)\leqslant\widetilde{C}\cdot\frac{1}{\log^{n-1}\frac{\varepsilon_0}{|x-x_0|}}, \tag{31}$$

where $\widetilde{C}:=\max\left\{\frac{\omega_{n-1}MR_0^{1+p-n}C2^{n+1}}{n\log 2},\frac{2R_0}{P_0}\right\}$. The constant $\widetilde{C}$ depends only on p, n, C and D', because M and R_0 are totally defined by D'. The proof is complete. □

Proof of Corollary 2. Proof of Corollary 2 is totally similar to the proof of Corollary 1. □

For some other studies of mappings with conditions on the distortion of the modulus of families of paths, see, for example, in [20–26].

REFERENCES

1. O. Martio, S. Rickman, J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A1, **465** (1970), 1–13.
2. O. Martio, R. Näkki, *Boundary Hölder continuity and quasiconformal mappings*, J. London Math. Soc. (2), **44** (1991), 339–350.
3. V.I. Ryazanov, E.A. Sevost'yanov, *Toward the theory of ring Q-homeomorphisms*, Israel J. Math., **68** (2008), 101–118.
4. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, Moduli in modern mapping theory, Springer Monographs in Mathematics, Springer, New York etc., 2009.
5. M. Arsenović, M. Mateljević, *On the Hölder continuity of ring Q-homeomorphisms*, Georgian Math. J., **29** (2022), №6, 805—811.
6. V. Ryazanov, R. Salimov, E. Sevost'yanov, *On the Hölder property of mappings in domains and on boundaries*, Journal of Mathematical Sciences, **246** (2020), №1, 60–74.

7. M. Mateljević, R. Salimov, E. Sevost'yanov, *Hölder and Lipschitz continuity in Orlicz-Sobolev classes, distortion and harmonic mappings,* Filomat, **36** (2022), №16, 5359–5390.
8. M. Mateljević, E. Sevost'yanov, *On the behavior of Orlicz-Sobolev mappings with branching on the unit sphere*, Journal of Mathematical Sciences, **270** (2023), №3, 467–499.
9. P. Hajlasz, P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc., **145** (2000), №688, 1–101.
10. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., V.229, Springer–Verlag, Berlin etc., 1971.
11. J. Heinonen, Lectures on Analysis on metric spaces, Springer Science+Business Media, New York, 2001.
12. D.A. Kovtonyuk, V.I. Ryazanov, *On the theory of prime ends for space mappings*, Ukrainain Mathematical Journal, **67** (2015), №4, 528—541.
13. R. Näkki, *Prime ends and quasiconformal mappings*, Journal d'Analyse Mathématique, **35** (1979), 13–40.
14. O. Martio, J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A1 Math., **4** (1978/1979), 384–401.
15. F.W. Gehring, O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings*, Journal d'Analyse Mathématique, **45** (1985), 181–206.
16. T. Adamowicz, N. Shanmugalingam, *Non-conformal Loewner type estimates for modulus of curve families*, Ann. Acad. Sci. Fenn. Math., **35** (2010), 609—626.
17. M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in $n-$space,* Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertattiones, **11** (1976), 1–44.
18. K. Kuratowski, Topology, V.2, Academic Press, New York–London, 1968.
19. R. Näkki, *Continuous boundary extension of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I. Math., **511** (1972), 1–10.
20. R.R. Salimov, E.A. Sevost'yanov, *ACL and differentiability of open discrete ring (p, Q)-mappings*, Mat. Stud., **35** (2011), №1, 28–36.
21. V.I. Ryazanov, E.A. Sevost'yanov, *On the convergence of spatial homeomorphisms*, Mat. Stud., **39** (2013), №1, 34–44.
22. E.A. Sevost'yanov, S.A. Skvortsov, N.S. Ilkevych, *On boundary behavior of mappings with two normalized conditions*, Mat. Stud., **49** (2018), №2, 150–157.
23. E.A. Sevost'yanov, S.A. Skvortsov, N.S. Ilkevych, *On removable singularities of mappings in uniform spaces*, Mat. Stud., **52** (2019), №1, 24–31.
24. E.A. Sevost'yanov, S.A. Skvortsov, I.A. Sverchevska, *On boundary extension of one class of mappings in terms of prime ends*, Mat. Stud., **53** (2020), №1, 29–40.
25. O.P. Dovhopiatyi, E.A. Sevost'yanov, *On compact classes of solutions of Dirichlet problem in simply connected domains*, Mat. Stud., **58** (2022), №2, 159–173.
26. R.R. Salimov, E.O. Sevost'yanov, V.A. Targonskii, *On modulus inequality of the order p for the inner dilatation*, Mat. Stud., **59** (2023), №2, 141–155.

Zhytomyr Ivan Franko State University
Zhytomyr, Ukraine
alexdov1111111@gmail.com
ilkevych1980@gmail.com
esevostyanov2009@gmail.com
targonsk@zu.edu.ua

Institute of Applied Mathematics and Mechanics of NAS of Ukraine
Sloviansk, Ukraine
esevostyanov2009@gmail.com

Received 20.02.2024
Revised 12.05.2024