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ON THE UPFAMILY EXTENSION OF A DOPPELSEMIGROUP

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A family $\mathcal{U}$ of non-empty subsets of a set D is called an *upfamily* if for each set $U \in \mathcal{U}$ any set $F \supset U$ belongs to $\mathcal{U}$. The upfamily extension $\upsilon(D)$ of D consists of all upfamilies on D. Any associative binary operation $*\colon D \times D \to D$ can be extended to an associative binary operation

$$*: \upsilon(D) \times \upsilon(D) \to \upsilon(D), \quad \mathcal{U} * \mathcal{V} = \left\langle \bigcup_{a\in U} a * V_a : U \in \mathcal{U},\ \{V_a\}_{a\in U} \subset \mathcal{V}\right\rangle.$$

In the paper, we show that the upfamily extension $(\upsilon(D), \dashv, \vdash)$ of a (strong) doppelsemigroup $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup as well and study some properties of this extension. Also we introduce the upfamily functor in the category **DSG** whose objects are doppelsemigroups and morphisms are doppelsemigroup homomorphisms. We prove that the automorphism group of the upfamily extension of a doppelsemigroup $(D, \dashv, \vdash)$ of cardinality $|D| \geq 2$ contains a subgroup, isomorphic to $C_2 \times \mathrm{Aut}(D, \dashv, \vdash)$. Also we describe the structure of upfamily extensions of all two-element doppelsemigroups and their automorphism groups.

1. Introduction. Given a semigroup $(S, \dashv)$, consider a semigroup $(S, \vdash)$ defined on the same set. We say that the semigroups $(S, \vdash)$ and $(S, \dashv)$ are *interassociative* provided

$$(x \dashv y) \vdash z = x \dashv (y \vdash z) \quad \text{and} \quad (x \vdash y) \dashv z = x \vdash (y \dashv z)$$

for all $x, y, z \in S$. When this occurs, $(S, \vdash)$ is said to be an *interassociate* of $(S, \dashv)$, or that the semigroups are interassociates of each other. If the semigroups $(S, \vdash)$ and $(S, \dashv)$ are interassociative, then rearranging the parentheses in an expression that contains only operations $\vdash$, $\dashv$ and elements of S will not change the result. In 1971, Zupnik [40] coined the term interassociativity in a general groupoid setting. However, he required only one of the two defining equations to hold. The present concept of interassociativity for semigroups originated in 1986 in Drouzy [11], where it is noted that every group is isomorphic to each of its interassociates. In 1983, Gould and Richardson [22] introduced *strong interassociativity*, defined by the above equations along with $x \dashv (y \vdash z) = x \vdash (y \dashv z)$. J. B. Hickey in 1983 [23] and 1986 [24] dealt with the special case of interassociativity in which the operation $\vdash$ is defined by specifying $a \in S$ and stipulating that $x \vdash y = x \dashv a \dashv y$ for all $x, y \in S$. Clearly $(S, \vdash)$, which Hickey calls a *variant* of $(S, \dashv)$, is a semigroup that is an interassociate of $(S, \dashv)$. It is easy to show that if $(S, \dashv)$ is a monoid, every interassociate $(S, \vdash)$ must satisfy the condition $x \vdash y = x \dashv a \dashv y$ for some fixed element $a \in S$ and for all $x, y \in S$, that is $(S, \vdash)$ is a variant of $(S, \dashv)$. Methods of constructing interassociates were developed, for semigroups in general and for specific classes of semigroups, in 1997 by Boyd, Gould and Nelson [9]. The description of all interassociates of finite monogenic semigroups was presented by Gould, Linton and Nelson in 2004, see [21].

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This paper is devoted to study of doppelsemigroups which are sets with two associative binary operations satisfying axioms of interassociativity. More accurately, a *doppelsemigroup* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a non-empty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(D_1) \quad (x \dashv y) \vdash z = x \dashv (y \vdash z),$$

$$(D_2) \quad (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

Thus, we can see that in any doppelsemigroup $(D, \dashv, \vdash)$, $(D, \vdash)$ is an interassociate of $(D, \dashv)$, and conversely, if a semigroup $(D, \vdash)$ is an interassociate of a semigroup $(D, \dashv)$, then $(D, \dashv, \vdash)$ is a doppelsemigroup. A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* [34] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A doppelsemigroup $(D, \dashv, \vdash)$ is said to be *strong* [36] if it satisfies the axiom $x \dashv (y \vdash z) = x \vdash (y \dashv z)$.

The study of doppelsemigroups was initiated by A. Zhuchok. The idea of doppelsemigroups bases on the study of dimonoids in the sense of Loday [27]. Doppelalgebras introduced by Richter [29] in the context of algebraic K-theory are linear analogs of doppelsemigroups and commutative dimonoids are examples of doppelsemigroups. Consequently, doppelsemigroup theory has connections to doppelalgebra theory and dimonoid theory. A doppelsemigroup can also be determined by using the notion of a duplex [28]. Free duplexes were constructed in [28]. Doppelsemigroups are closely related to bisemigroups considered in the work of Schein [30]. The latter algebras have applications in the theory of binary relations [31]. If operations of a doppelsemigroup coincide, we obtain the notion of a semigroup.

Many classes of doppelsemigroups were studied by A. Zhuchok and his coauthors. The free product of doppelsemigroups, the free (strong) doppelsemigroup, the free commutative (strong) doppelsemigroup, the free n-nilpotent (strong) doppelsemigroup and the free rectangular doppelsemigroup were constructed in [34, 36, 39]. Relatively free doppelsemigroups were studied in [37]. The free n-dinilpotent (strong) doppelsemigroup was constructed in [33, 36]. In [35], A. Zhuchok described the free left n-dinilpotent doppelsemigroup. Representations of ordered doppelsemigroups by binary relations were studied by Yu. Zhuchok and J. Koppitz (see [38]).

In [19], the task of describing all pairwise non-isomorphic (strong) doppelsemigroups with at most three elements has been solved. We proved that there exist 8 pairwise non-isomorphic two-element doppelsemigroups among which 6 doppelsemigroups are commutative. All two-element doppelsemigroups are strong. It was proved that there exist 75 pairwise non-isomorphic three-element doppelsemigroups among which 41 doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into 17 pairs of dual doppelsemigroups. Also up to isomorphism there are 65 strong doppelsemigroups of order 3, and all non-strong doppelsemigroups are not commutative. In [20], we studied cyclic doppelsemigroups. A doppelsemigroup $(G, \dashv, \vdash)$ is called *a group doppelsemigroup* if $(G, \dashv)$ is a group. A group doppelsemigroup $(G, \dashv, \vdash)$ is said to be *cyclic* if $(G, \dashv)$ is a cyclic group. It was proved that up to isomorphism there exist $\tau(n)$ finite cyclic (strong) doppelsemigroups of order n, where τ is the number of divisors function. There exist infinite many pairwise non-isomorphic infinite cyclic (strong) doppelsemigroups.

In this paper, we investigate the extension $(\upsilon(D), \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$. The thorough study of various extensions of semigroups was started in [13] and continued in [1]–[8], [14]–[18]. The largest among these extensions is the semigroup $\upsilon(S)$ of all upfamilies on a semigroup S. The extension $\upsilon(S)$ is called *the upfamily extension* of S. A family $\mathcal{U}$ of non-empty subsets of a set X is called an *upfamily* if for each set $U \in \mathcal{U}$ any subset $F \supset U$ belongs to $\mathcal{U}$ (In [13] instead of the notion "upfamily" it was used the notion "inclusion hyperspace"). Each family $\mathcal{A}$ of non-empty subsets of X generates the upfamily $\langle A \subset X : A \in \mathcal{A} \rangle = \{U \subset X : \exists A \in \mathcal{A}\ (A \subset U)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{B}$ is called an *ultrafilter* if $\mathcal{B} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{B}$. The family $\beta(X)$ of all ultrafilters on a set X is called the Stone-Čech compactification of X, see [25], [32]. An ultrafilter, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle \{x\} \rangle$ generated by the singleton $\{x\}$, and hence we consider $X \subset \beta(X) \subset \upsilon(X)$. It was shown in [13] that any associative binary operation $*\colon S \times S \to S$ can be extended to an associative binary operation $*\colon \upsilon(S) \times \upsilon(S) \to \upsilon(S)$ by the formula

$$\mathcal{U} * \mathcal{V} = \Big\langle \bigcup_{a \in U} a * V_a : U \in \mathcal{U},\ \ \{V_a\}_{a \in U} \subset \mathcal{V} \Big\rangle$$

for upfamilies $\mathcal{U}, \mathcal{V} \in \upsilon(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$.

The problem of calculation the cardinality $\upsilon(n)$ of $\upsilon(X)$ for a set X of cardinality n is not trivial and is tightly connected with the classical (and still unsolved) problem of Dedekind [10] who suggested to determine the number $M(n)$ of all monotone Boolean functions of n Boolean variable. The function $M(n)$ growths very quickly. Its exact values are known only for $n \leq 9$ and are given in Table 1 taken from On-Line Encyclopedia of Integer Sequences (https://oeis.org/A000372).

n	$\lvert M(n) \rvert$
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788
9	286386577668298411128469151667598498812366

Tab. 1: The values of the function $M(n)$ for $n \leq 9$

Observe that for each upfamily $\mathcal{U} \in \upsilon(X)$ the characteristic function $\chi_{\mathcal{U}}\colon \mathcal{P}(X) \to \{0,1\}$ of $\mathcal{U}$ is monotone with respect to the inclusion relation on the power-set $\mathcal{P}(X)$ of X. Moreover, $\chi_{\mathcal{U}}(\varnothing) = 0$ and $\chi_{\mathcal{U}}(X) = 1$. Conversely, each monotone function $f\colon \mathcal{P}(X) \to \{0,1\}$ with $f(\varnothing) = 0$ and $f(X) = 1$ determines an upfamily $f^{-1}(1)$. This observation implies that for a finite set X of size n the size $\upsilon(n)$ of the set $\upsilon(X)$ equals $M(n) - 2$.

Each map $f\colon X \to Y$ induces the map

$$\upsilon(f)\colon \upsilon(X) \to \upsilon(Y), \quad \upsilon(f)\colon \mathcal{U} \mapsto \big\langle f(U) : U \in \mathcal{U} \big\rangle, \ \text{ see [12].}$$

If $\varphi\colon S \to S'$ is a semigroup homomorphism, then $\upsilon(\varphi)\colon \upsilon(S) \to \upsilon(S')$ is a semigroup homomorphism as well, see [13].

In this paper, we show that the upfamily extension $(\upsilon(D), \dashv, \vdash)$ of a (strong) doppelsemigroup $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup as well and study some properties of this extension. Also we introduce the upfamily functor υ in the category **DSG** of doppelsemigroups and their homomorphisms and show that this functor preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups. On the other hand, the functor υ does not preserve commutative doppelsemigroups and group doppelsemigroups. We prove that the automorphism group of the upfamily extension of a doppelsemigroup $(D, \dashv, \vdash)$ of cardinality $|D| \geq 2$ contains a subgroup, isomorphic to $C_2 \times \mathrm{Aut}\,(D, \dashv, \vdash)$. Also we describe the structure of upfamily extensions of all two-element doppelsemigroups and their automorphism groups.

2. Extending operations from a doppelsemigroup to its upfamily extension. In this section, we show that the upfamily extension of a (strong) doppelsemigroup is a (strong) doppelsemigroup as well.

Proposition 1. *If $(D, \dashv, \vdash)$ is a doppelsemigroup, then $(\upsilon(D), \dashv, \vdash)$ is a doppelsemigroup as well.*

Proof. It is necessary to show that
$$(\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W} = \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W}) \quad \text{and} \quad (\mathcal{U} \vdash \mathcal{V}) \dashv \mathcal{W} = \mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W})$$
for any upfamilies $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \upsilon(D)$.

Let us prove that $(\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W} = \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$ for any upfamilies $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \upsilon(D)$.

First, we prove the inclusion $(\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W} \subset \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$. Take any subset $A \in (\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W}$ and choose a set $B \in \mathcal{U} \dashv \mathcal{V}$ such that $A \supset \bigcup_{z \in B} z \vdash W_z$ for some family $\{W_z\}_{z \in B} \subset \mathcal{W}$. Next, for the set $B \in \mathcal{U} \dashv \mathcal{V}$ choose a set $U \in \mathcal{U}$ such that $B \supset \bigcup_{x \in U} x \dashv V_x$ for some family $\{V_x\}_{x \in U} \subset \mathcal{V}$. It is clear that for each $x \in U$ and $y \in V_x$ the product $x \dashv y$ is in B and hence $W_{x \dashv y}$ is defined. Consequently, $\bigcup_{y \in V_x} y \vdash W_{x \dashv y} \in \mathcal{V} \vdash \mathcal{W}$ for all $x \in U$ and hence $\bigcup_{x \in U} x \dashv (\bigcup_{y \in V_x} y \vdash W_{x \dashv y}) \in \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$. Since
$$\bigcup_{x \in U} \bigcup_{y \in V_x} x \dashv (y \vdash W_{x \dashv y}) = \bigcup_{x \in U} \bigcup_{y \in V_x} (x \dashv y) \vdash W_{x \dashv y} \subset A,$$
we get $A \in \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$. This proves the inclusion $(\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W} \subset \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$.

To prove the reverse inclusion, fix a set $A \in \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$ and choose a set $U \in \mathcal{U}$ such that $A \supset \bigcup_{x \in U} x \dashv B_x$ for some family $\{B_x\}_{x \in U} \subset \mathcal{V} \vdash \mathcal{W}$. Next, for each $x \in U$ find a set $V_x \in \mathcal{V}$ such that $B_x \supset \bigcup_{y \in V_x} y \vdash W_{x,y}$ for some family $\{W_{x,y}\}_{y \in V_x} \subset \mathcal{W}$. Let $Z = \bigcup_{x \in U} x \dashv V_x$. For each $z \in Z$ we can find $x \in U$ and $y \in V_x$ such that $z = x \dashv y$ and put $W_z = W_{x,y}$. Then $Z \in \mathcal{U} \dashv \mathcal{V}$ and $\bigcup_{z \in Z} z \vdash W_z \in (\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W}$. Taking into account
$$\bigcup_{z \in Z} z \vdash W_z \subset \bigcup_{x \in U} \bigcup_{y \in V_x} (x \dashv y) \vdash W_{x,y} = \bigcup_{x \in U} \bigcup_{y \in V_x} x \dashv (y \vdash W_{x,y}) \subset A,$$
we conclude $A \in (\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W}$.

In the same way one can check that $(\mathcal{U} \vdash \mathcal{V}) \dashv \mathcal{W} = \mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W})$ for any upfamilies $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \upsilon(D)$.

□

Proposition 2. *If $(D, \dashv, \vdash)$ is a strong doppelsemigroup, then $(\upsilon(D), \dashv, \vdash)$ is a strong doppelsemigroup as well.*

Proof. It is necessary to show that $\mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W}) = \mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W})$ for any upfamilies $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \upsilon(D)$.

To prove the inclusion $\mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W}) \subset \mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W})$, fix a set $A \in \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$ and choose a set $U \in \mathcal{U}$ such that $A \supset \bigcup_{x\in U} x \dashv B_x$ for some family $\{B_x\}_{x\in U} \subset \mathcal{V} \vdash \mathcal{W}$. Next, for each $x \in U$ find a set $V_x \in \mathcal{V}$ such that $B_x \supset \bigcup_{y\in V_x} y \vdash W_{x,y}$ for some family $\{W_{x,y}\}_{y\in V_x} \subset \mathcal{W}$. Taking into account

$$A \supset \bigcup_{x\in U} x \dashv B_x \supset \bigcup_{x\in U} x \dashv (\bigcup_{y\in V_x} y \vdash W_{x,y}) = \bigcup_{x\in U}\bigcup_{y\in V_x} x \dashv (y \vdash W_{x,y}) =$$
$$= \bigcup_{x\in U}\bigcup_{y\in V_x} x \vdash (y \dashv W_{x,y}) = \bigcup_{x\in U} x \vdash (\bigcup_{y\in V_x} y \dashv W_{x,y}) \in \mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W}),$$

we conclude $A \in (\mathcal{U} \vdash (\mathcal{V} \dashv \mathcal{W})$.

In the same way one can check that $(\mathcal{U} \dashv \mathcal{V}) \vdash \mathcal{W} \supset \mathcal{U} \dashv (\mathcal{V} \vdash \mathcal{W})$ for any upfamilies $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \upsilon(D)$. □

3. Some properties of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$. By definition, the *center* of a doppelsemigroup $(D, \dashv, \vdash)$ is the set

$$C(D, \dashv, \vdash) = \{a \in D\colon a \dashv x = x \dashv a \text{ and } a \vdash x = x \vdash a \text{ for all } x \in D\}.$$

It follows from this definition that $C(D, \dashv, \vdash) = C(D, \dashv) \cap C(D, \vdash)$.

Proposition 3. *If the center $C(D, \dashv, \vdash)$ of the doppelsemigroup $(D, \dashv, \vdash)$ is non-empty, then it is a subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$.*

Proof. Let $a, b \in C(D, \dashv, \vdash)$. Taking into account that the centers of the semigroups $(D, \dashv)$ and $(D, \vdash)$ are subsemigroups of the semigroups $(D, \dashv)$ and $(D, \vdash)$ respectively, and

$$(a \dashv b) \vdash x = (b \dashv a) \vdash x = b \dashv (a \vdash x) = b \dashv (x \vdash a) = (x \vdash a) \dashv b = x \vdash (a \dashv b),$$
$$(a \vdash b) \dashv x = (b \vdash a) \dashv x = b \vdash (a \dashv x) = b \vdash (x \dashv a) = (x \dashv a) \vdash b = x \dashv (a \vdash b),$$

we conclude that $a \dashv b, a \vdash b \in C(D, \dashv, \vdash)$ and we are done. □

Proposition 4. *The center of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$ contains the center of $(D, \dashv, \vdash)$. If $(D, \dashv, \vdash)$ is a group doppelsemigroup, then $C(\upsilon(D), \dashv, \vdash) = C(D, \dashv) \cap C(D, \vdash)$.*

Proof. Let $a \in C(D, \dashv, \vdash)$. Then for every upfamily $\mathcal{U} \in \upsilon(D)$ we get

$$a \dashv \mathcal{U} = \{a \dashv U\colon U \in \mathcal{U}\} = \{U \dashv a\colon U \in \mathcal{U}\} = \mathcal{U} \dashv a,$$
$$a \vdash \mathcal{U} = \{a \vdash U\colon U \in \mathcal{U}\} = \{U \vdash a\colon U \in \mathcal{U}\} = \mathcal{U} \vdash a,$$

which means that (the principal ultrafilter generated by) a belongs to the center of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.

If $(D, \dashv, \vdash)$ is a group doppelsemigroup, then applying [13, Theorem 2] we obtain $C(\upsilon(D), \dashv) = C(D, \dashv)$ and $C(\upsilon(D), \vdash) = C(D, \vdash)$, and hence

$$C(\upsilon(D), \dashv, \vdash) = C(D, \dashv) \cap C(D, \vdash).$$

□

An element z of a doppelsemigroup $(D, \dashv, \vdash)$ is called a *zero* (resp. a *left zero*, a *right zero*) if $a \dashv z = z \dashv a = a \vdash z = z \vdash a = z$ (resp. $z \dashv a = z \vdash a = z$, $a \dashv z = a \vdash z = z$) for any $a \in D$.

Proposition 5. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. For an element $z \in D \subset \upsilon(D)$ the following conditions are equivalent: 1) z is a left (right) zero of a semigroup $(D, \dashv)$; 2) z is a left (right) zero of a doppelsemigroup $(D, \dashv, \vdash)$; 3) z is a left (right) zero of a semigroup $(D, \vdash)$; 4) z is a left (right) zero of a semigroup $(\upsilon(D), \vdash)$; 5) z is a left (right) zero of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$; 6) z is a left (right) zero of a semigroup $(\upsilon(D), \dashv)$.*

Proof. The implications (2) ⇒ (3), (5) ⇒ (6) and (6) ⇒ (1) are trivial.

(1) ⇒ (2) Assume that z is a left zero of a semigroup $(D, \dashv)$. Taking into account that for any $a \in D$, $z \vdash a = (z \dashv a) \vdash a = z \dashv (a \vdash a) = z$, we conclude that z is also a left zero of a semigroup $(D, \vdash)$, and hence z is a left zero of a doppelsemigroup $(D, \dashv, \vdash)$.

(3) ⇒ (4) Let z be a left zero of a semigroup $(D, \vdash)$. Then for every upfamily $\mathcal{U} \in \upsilon(D)$ we get $z \vdash \mathcal{U} = \langle z \vdash U \colon U \in \mathcal{U} \rangle = \langle \{z\} \colon U \in \mathcal{U} \rangle = z$, which means that (the principal ultrafilter generated by) z is a left zero of a semigroup $(\upsilon(D), \vdash)$.

(4) ⇒ (5) Assume that z is a left zero of a semigroup $(\upsilon(D), \vdash)$. Taking into account that for any $\mathcal{U} \in \upsilon(D)$, $z \dashv \mathcal{U} = (z \vdash \mathcal{U}) \dashv \mathcal{U} = z \vdash (\mathcal{U} \dashv \mathcal{U}) = z$, we conclude that z is also a left zero of a semigroup $(\upsilon(D), \dashv)$, and hence z is a left zero of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.

In the same way one can check the right zero case. □

A doppelsemigroup $(D, \dashv, \vdash)$ is said to be a *left (right) zero doppelsemigroup* if each of its elements is a left (right) zero. By LO_X (RO_X) we denote the left (right) zero doppelsemigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of LO_X and RO_X we use LO_n and RO_n respectively.

Let us note that for a subdoppelsemigroup $(T, \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$ the homomorphism $i \colon \upsilon(T) \to \upsilon(D)$, $i \colon \mathcal{U} \to \langle \mathcal{U} \rangle_D$ is injective, and thus we can identify the doppelsemigroup $(\upsilon(T), \dashv, \vdash)$ with the doppelsubsemigroup $i((\upsilon(T), \dashv, \vdash)) \subset (\upsilon(D), \dashv, \vdash)$. Therefore, for each family $\mathcal{B}$ of nonempty subsets of T we identify the upfamilies $\langle \mathcal{B} \rangle_T = \{U \in T \mid \exists B \in \mathcal{B}(B \subset U)\} \in \upsilon(T)$ and $\langle \mathcal{B} \rangle_D = \{U \in D \mid \exists B \in \mathcal{B}(B \subset U)\} \in \upsilon(D)$.

Proposition 6. *If T is a left (right) zero subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$, then $\upsilon(T)$ is a left (right) zero subdoppelsemigroup of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$ as well.*

Proof. Let T be a left zero subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$. Then for any $\mathcal{U}, \mathcal{V} \in \upsilon(T)$,

$$\mathcal{U} \dashv \mathcal{V} = \Big\langle \bigcup_{a \in U} a \dashv V_a \colon U \in \mathcal{U},\ U \subset T,\ \ V_a \in \mathcal{V},\ V_a \subset T \text{ for all } a \in U \Big\rangle =$$

$$= \Big\langle \bigcup_{a \in U} \{a\} \colon U \in \mathcal{U} \Big\rangle = \mathcal{U}.$$

Therefore, by Proposition 5, $(\upsilon(T), \dashv, \vdash)$ is a left zero subdoppelsemigroup of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$ as well.

For a right zero subdoppelsemigroup the proof is similar. □

Propositions 5 and 6 imply the following corollary.

Corollary 1. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. Then the following conditions are equivalent:*
1) $(D, \dashv)$ is a left (right) zero semigroup; 2) $(D, \dashv, \vdash)$ is a left (right) zero doppelsemigroup;
3) $(D, \vdash)$ is a left (right) semigroup; 4) $(\upsilon(D), \vdash)$ is a left (right) zero semigroup;
5) $(\upsilon(D), \dashv, \vdash)$ is a left (right) zero doppelsemigroup; 6) $(\upsilon(D), \dashv)$ is a left (right) zero semigroup.

An element e of a doppelsemigroup $(D, \dashv, \vdash)$ is called an *identity* (resp. a *left identity*, a *right identity*) if $a \dashv e = e \dashv a = a \vdash e = e \vdash a = a$ (resp. $e \dashv a = e \vdash a = e$, $a \dashv e = a \vdash e = a$) for any $a \in D$.

Proposition 7. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. For an element $e \in D \subset \upsilon(D)$ the following conditions are equivalent: 1) e is a left (right) identity of a doppelsemigroup $(D, \dashv, \vdash)$; 2) e is a left (right) identity of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.*

Proof. The implication (2) ⇒ (1) is trivial.

(1) ⇒ (2) Let e be a left identity of a doppelsemigroup $(D, \dashv, \vdash)$. Then for every upfamily $\mathcal{U} \in \upsilon(D)$ we get
$e \vdash \mathcal{U} = \{e \vdash U : U \in \mathcal{U}\} = \{U : U \in \mathcal{U}\} = \mathcal{U}$, $e \dashv \mathcal{U} = \{e \dashv U : U \in \mathcal{U}\} = \{U : U \in \mathcal{U}\} = \mathcal{U}$, which means that (the principal ultrafilter generated by) e is a left identity of a doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.

In the same way one can check the right identity case. □

Remark 1. In general case, a left (right) identity of a semigroup $(D, \dashv)$ is not a left (right) identity of a doppelsemigroup $(D, \dashv, \vdash)$. For the doppelsemigroup $(\{0, 1\}, \min, *)$, where $a * b = 0$ for any $a, b \in \{0, 1\}$, the semigroup $(\{0, 1\}, \min)$ is a monoid while the semigroup $(\{0, 1\}, *)$ contains no left (right) identity.

A non-empty subset I of a doppelsemigroup $(D, \dashv, \vdash)$ is called an *ideal* (resp. a *left ideal*, a *right ideal*) if $(S \dashv I) \cup (S \vdash I) \cup (I \dashv S) \cup (I \vdash S) \subset I$ (resp. $(S \dashv I) \cup (S \vdash I) \subset I$, $(I \dashv S) \cup (I \vdash S) \subset I$).

Proposition 8. *If I is a left (right) ideal of a doppelsemigroup $(D, \dashv, \vdash)$, then $\upsilon(I)$ is a left (right) ideal of the upfamily extension $(\upsilon(D), \dashv, \vdash)$ as well.*

Proof. Indeed, let $\mathcal{U} \in \upsilon(D)$, $\mathcal{M} \in \upsilon(I)$. Then

$$\mathcal{U} \dashv \mathcal{M} = \Big\langle \bigcup_{a \in U} a \dashv M_a : U \in \mathcal{U},\ \ M_a \in \mathcal{M},\ M_a \subset I \text{ for all } a \in U \Big\rangle =$$
$$= \Big\langle \bigcup_{a \in U} a \dashv M_a : U \in \mathcal{U},\ \ \{M_a\}_{a \in U} \subset \mathcal{M},\ \bigcup_{a \in U} a \dashv M_a \subset I \Big\rangle \in \upsilon(I).$$

By analogy $\mathcal{U} \vdash \mathcal{M} \in \upsilon(I)$, and therefore $\upsilon(I)$ is a left ideal of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.

In the same way one can check the right ideal case. □

By definition, the *minimal ideal* of a doppelsemigroup $(D, \dashv, \vdash)$ is an ideal containing no other ideal of $(D, \dashv, \vdash)$. It is also called the *kernel* of a doppelsemigroup $(D, \dashv, \vdash)$, denoted $K(D)$.

Proposition 9. *If a doppelsemigroup $(D, \dashv, \vdash)$ contains a left (right) zero, then the minimal ideal $K(D)$ of $(D, \dashv, \vdash)$ coincides with the set of all left (right) zeros of $(D, \dashv, \vdash)$.*

Proof. Let Z be the supdoppelsemigroup of all left zeros of $(D, \dashv, \vdash)$. Then for every $d, t \in D$ and every $z \in Z$ we get $(d \dashv z) \dashv t = d \dashv (z \dashv t) = d \dashv z$ and $(d \vdash z) \dashv t = d \vdash (z \dashv t) = d \vdash z$. By Proposition 5, $d \dashv z, d \vdash z \in Z$, that is $D \dashv Z \cup D \vdash Z \subset Z$, and hence Z is a left (right) ideal. It follows from definition of left zeros that $Z \dashv D = Z \vdash D = Z$. This shows that Z is an ideal of $(D, \dashv, \vdash)$. It suffices to check that Z lies in each ideal I of $(D, \dashv, \vdash)$. Indeed, $Z = Z \dashv I \subset D \dashv I \subset I$.

In the same way one can check the right zero case. □

A semigroup $(S,*)$ is said to be *right simple* if $a*S=S$ for any $a\in S$. Taking into account that according to Proposition 18 of [13], for a right simple semigroup $(S,*)$ the semigroup $(\upsilon(S),*)$ contains a right zero, we conclude by Proposition 5 the following proposition.

Proposition 10. *Let $(D,\dashv,\vdash)$ be a doppelsemigroup. If $(D,\dashv)$ or $(D,\vdash)$ is a right simple semigroup, then the doppelsemigroup $(\upsilon(D),\dashv,\vdash)$ contains a right zero, and hence the minimal ideal $K(\upsilon(D))$ of $(\upsilon(D),\dashv,\vdash)$ coincides with the set of all right zeros of $(\upsilon(D),\dashv,\vdash)$.*

Corollary 2. *If $(G,\dashv,\vdash)$ is a group doppelsemigroup, then the doppelsemigroup $(\upsilon(G),\dashv,\vdash)$ contains a right zero, and hence the minimal ideal $K(\upsilon(G))$ of $(\upsilon(G),\dashv,\vdash)$ coincides with the set of all right zeros of $(\upsilon(G),\dashv,\vdash)$.*

4. The upfamily functor in the category DSG. A map $\varphi\colon D_1\to D_2$ is called a *homomorphism of doppelsemigroups* $(D_1,\dashv_1,\vdash_1)$ and $(D_2,\dashv_2,\vdash_2)$ if

$$\varphi(a\dashv_1 b)=\varphi(a)\dashv_2\varphi(b)\quad\text{and}\quad\varphi(a\vdash_1 b)=\varphi(a)\vdash_2\varphi(b)$$

for all $a,b\in D_1$.

A bijective homomorphism is called an *isomorphism of doppelsemigroups*. If there exists an isomorphism between the doppelsemigroups $(D_1,\dashv_1,\vdash_1)$ and $(D_2,\dashv_2,\vdash_2)$, then $(D_1,\dashv_1,\vdash_1)$ and $(D_2,\dashv_2,\vdash_2)$ are said to be *isomorphic*, denoted $(D_1,\dashv_1,\vdash_1)\cong(D_2,\dashv_2,\vdash_2)$. An isomorphism $\psi\colon D\to D$ is called an *automorphism* of a doppelsemigroup $(D,\dashv,\vdash)$. By $\mathrm{Aut}\,(D,\dashv,\vdash)$ we denote the automorphism group of a doppelsemigroup $(D,\dashv,\vdash)$.

According to Proposition 8 of [13], each homomorphism $\varphi\colon D_1\to D_2$ of doppelsemigroups $(D_1,\dashv_1,\vdash_1)$ and $(D_2,\dashv_2,\vdash_2)$ induces the homomorphism

$$\upsilon(\varphi)\colon\upsilon(D_1)\to\upsilon(D_2),\quad \upsilon(\varphi)\colon\mathcal{U}\mapsto\big\langle\varphi(U)\subset D_2\colon U\in\mathcal{U}\big\rangle,$$

of doppelsemigroups $(\upsilon(D_1),\dashv_1,\vdash_1)$ and $(\upsilon(D_2),\dashv_2,\vdash_2)$.

Denote by **DSG** the category of doppelsemigroups whose objects are doppelsemigroups and morphisms are doppelsemigroup homomorphisms.

A *covariant functor* $F\colon\mathcal{C}\to\mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of an object map $F\colon v\mathcal{C}\to v\mathcal{D}$ which assigns to each $a\in v\mathcal{C}$, an object $F(a)\in v\mathcal{D}$ and a morphism map F which assigns to each morphism $f\colon a\to b$ in $\mathcal{C}$, a morphism $F(f)\colon F(a)\to F(b)$ in $\mathcal{D}$ such that

1) $F(\mathrm{id}_a)=\mathrm{id}_{F(a)}$ for each $a\in v\mathcal{C}$;
2) $F(f\circ g)=F(f)\circ F(g)$ for all morphisms $f,g\in\mathcal{C}$ for which the composition $f\circ g$ exists.

Consider $\upsilon\colon\mathbf{DSG}\to\mathbf{DSG}$ which assigns to each doppelsemigroup $(D,\dashv,\vdash)$ the doppelsemgroup $(\upsilon(D),\dashv,\vdash)$ of upfamilies on D, its morphism map assigns to each doppelsemigroup homomorphism $\varphi\colon D_1\to D_2$, the doppelsemigroup homomorphism $\upsilon(\varphi)\colon\upsilon(D_1)\to\upsilon(D_2)$. Taking into account that for any $\mathcal{U}\in\upsilon(D)$,

$$\upsilon(\mathrm{id}_D)(\mathcal{U})=\big\langle\mathrm{id}_D(U)\colon U\in\mathcal{U}\big\rangle=\big\langle U\colon U\in\mathcal{U}\big\rangle=\mathcal{U}=\mathrm{id}_{\upsilon(D)}(\mathcal{U}),$$
$$\upsilon(\varphi)\circ\upsilon(\psi)(\mathcal{U})=\upsilon(\varphi)(\big\langle\psi(U)\colon U\in\mathcal{U}\big\rangle)=\big\langle\varphi\circ\psi(U)\colon U\in\mathcal{U}\big\rangle=\upsilon(\varphi\circ\psi)(\mathcal{U}),$$

we conclude that $\upsilon(\mathrm{id}_D)=\mathrm{id}_{\upsilon(D)}$ and $\upsilon(\varphi\circ\psi)=\upsilon(\varphi)\circ\upsilon(\psi)$, and hence this construction defines the covariant functor $\upsilon\colon\mathbf{DSG}\to\mathbf{DSG}$. This functor is said to be the *upfamily functor* in the category of doppelsemigroups.

Combining Propositions 2, 5 and 7 with Corollary 1, we get the following proposition.

Proposition 11. *The upfamily functor υ in* **DSG** *preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups.*

Recall that every Abelian group $(G, \dashv)$ is isomorphic to each of its interassociates $(G, \vdash)$, and hence a doppelsemigroup $(G, \dashv, \vdash)$ is commutative. The following proposition show that the upfamily functor υ in **DSG** does not preserve commutative doppelsemigroups and group doppelsemigroups.

Proposition 12. *For any group doppelsemigroup $(G, \dashv, \vdash)$ of order $|G| > 1$ the doppelsemigroup $(\upsilon(G), \dashv, \vdash)$ is neither commutative nor a group doppelsemigroup.*

Proof. According to Corollary 3 of [13] for any group $(G, \dashv)$ the center of the semigroup $(\upsilon(G), \dashv)$ coincides with the center of $(G, \dashv)$. Since $|\upsilon(G)| > |G|$ for $|G| > 1$, we conclude that $(\upsilon(G), \dashv)$ is not commutative, and hence the doppelsemigroup $(\upsilon(G), \dashv, \vdash)$ is not commutative as well.

Taking into account that for any group $(G, \dashv)$ of order $|G| > 1$ the semigroup $(\upsilon(G), \dashv)$ contains at least two right zeros: $\{G\}$ and $\langle\{g\} \colon g \in G\rangle$, see [13, Prop. 15, 18], we conclude that $(\upsilon(G), \dashv)$ is not a group, and hence the doppelsemigroup $(\upsilon(G), \dashv, \vdash)$ is not a group doppelsemigroup. □

5. Automorphism groups of the upfamily extension of doppelsemigroups. Taking into account that the construction υ defines a covariant functor in the category of doppelsemigroups, we conclude the following proposition.

Proposition 13. *If $\psi \colon D_1 \to D_2$ is an isomorphism from a doppelsemigroup $(D_1, \dashv_1, \vdash_1)$ to a doppelsemigroup $(D_2, \dashv_2, \vdash_2)$, then $\upsilon(\psi) \colon \upsilon(D_1) \to \upsilon(D_2)$ is an isomorphism as well.*

Corollary 3. *If $\psi \colon D \to D$ is an automorphism of a doppelsemigroup $(D, \dashv, \vdash)$, then $\upsilon(\psi) \colon \upsilon(D) \to \upsilon(D)$ is an automorphism of the upfamily extension $(\upsilon(D), \dashv, \vdash)$.*

The set $\upsilon(D)$ of upfamilies on a set D possesses the unary operation

$$\perp \colon \upsilon(D) \to \upsilon(D), \ \perp \colon \mathcal{U} \mapsto \mathcal{U}^\perp = \{A \subset D \colon \forall U \in \mathcal{U} \ (A \cap U \neq \varnothing)\}$$

called the *transversality map*. This operation is involutive in the sense that $(\mathcal{U}^\perp)^\perp = \mathcal{U}$ for any $\mathcal{U} \in \upsilon(D)$, see [13].

The following proposition was proved in [13].

Proposition 14. *For a semigroup $(S, *)$ the extended associative operation $* \colon \upsilon(S) \to \upsilon(S)$ commutes with the transversality map in the sense that $(\mathcal{U} * \mathcal{V})^\perp = \mathcal{U}^\perp * \mathcal{V}^\perp$ for any upfamilies $\mathcal{U}, \mathcal{V} \in \upsilon(S)$.*

Corollary 4. *The transversality map $\perp \colon \upsilon(D) \to \upsilon(D)$ is an involutive automorphism of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.*

In the following proposition we show that the automorphism group of the upfamily extension of a doppelsemigroup $(D, \dashv, \vdash)$ contains a subgroup, isomorphic to the group $C_2 \times \mathrm{Aut}\,(D, \dashv, \vdash)$.

Proposition 15. *The automorphism group $\mathrm{Aut}\,(\upsilon(D), \dashv, \vdash)$ of the extension $(\upsilon(D), \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$ of cardinality $|D| \geq 2$ contains a subgroup, isomorphic to $C_2 \times \mathrm{Aut}\,(D, \dashv, \vdash)$.*

Proof. By Corollary 3, for any automorphism $\psi\colon D\to D$ of a doppelsemigroup $(D,\dashv,\vdash)$ its upfamily extension $\upsilon(\psi)\colon \upsilon(D)\to\upsilon(D)$ is an automorphism of $(\upsilon(D),\dashv,\vdash)$. Taking into account that the restriction of the transversality automorphism $\perp\colon \upsilon(D)\to\upsilon(D)$ to $D\subset\beta(D)$ is the identity automorphism of D and the extension $\upsilon(\mathrm{id}_D)$ of the identity automorphism id_D of $(D,\dashv,\vdash)$ is the identity automorphism $\mathrm{id}_{\upsilon(D)}$ of $(\upsilon(D),\dashv,\vdash)$, we conclude that $\perp\notin\{\upsilon(\psi)\colon \psi\in\mathrm{Aut}\,(D,\dashv,\vdash)\}\cong\mathrm{Aut}\,(D,\dashv,\vdash)$.

Let us show that $\upsilon(\psi)\circ\perp=\perp\circ\upsilon(\psi)$ for any $\psi\in\mathrm{Aut}\,(D,\dashv,\vdash)$. Indeed, for any $\mathcal{U}\in\upsilon(D)$,

$$\upsilon(\psi)\circ\perp(\mathcal{U})=\upsilon(\psi)(\mathcal{U}^{\perp})=\langle\psi(A)\subset D\colon A\in\mathcal{U}^{\perp}\rangle=$$
$$=\langle\psi(A)\subset D\colon \forall U\in\mathcal{U}\ (A\cap U\neq\varnothing)\rangle=\langle V\subset D\colon \forall U\in\mathcal{U}\ (\psi^{-1}(V)\cap U\neq\varnothing)\rangle=$$
$$=\langle V\subset D\colon \forall U\in\mathcal{U}\ (V\cap\psi(U)\neq\varnothing)\rangle=\langle\psi(U)\colon U\in\mathcal{U}\rangle^{\perp}=\perp\circ\upsilon(\psi)(\mathcal{U}).$$

Thus, $\mathrm{Aut}\,(\upsilon(D),\dashv,\vdash)$ contains the subgroup $\{\perp,\mathrm{id}_{\upsilon(D)}\}\times\{\upsilon(\psi)\colon \psi\in\mathrm{Aut}\,(D,\dashv,\vdash)\}$ which is isomorphic to $C_2\times\mathrm{Aut}\,(D,\dashv,\vdash)$. □

6. Doppelsemigroups of upfamilies over two-element doppelsemigroups. Firstly, recall some useful facts which we shall often use in this section. In fact, each semigroup $(S,\dashv)$ can be consider as a (strong) doppelsemigroup $(S,\dashv,\dashv)$ with the automorphism group $\mathrm{Aut}\,(S,\dashv,\dashv)=\mathrm{Aut}\,(S,\dashv)$, and we denote this *trivial* doppelsemigroup by S. As always, we denote by $(S,\dashv_a)$ a variant of a semigroup $(S,\dashv)$, where $x\dashv_a y=x\dashv a\dashv y$.

A semigroup $(S,\dashv)$ is called a *null semigroup* if there exists an element $z\in S$ such that $x\dashv y=z$ for any $x,y\in S$. In this case the element z is the zero of $(S,\dashv)$. All null semigroups on the same set are isomorphic. By O_X we denote a null semigroup on a set X. If X is finite of cardinality $|X|=n$, then instead of O_X we use O_n.

Following the algebraic tradition, we take for a model of the class of cyclic groups of order n the multiplicative group $C_n=\{z\in\mathbb{C}\colon z^n=1\}$ of n-th roots of 1. For a set X by S_X we denote the group of all bijections of X.

Let $(D_1,\dashv_1,\vdash_1)$ be a doppelsemigroup such that for each doppelsemigroup $(D_2,\dashv_2,\vdash_2)$ the isomorphisms $(D_2,\dashv_2)\cong(D_1,\dashv_1)$ and $(D_2,\vdash_2)\cong(D_1,\vdash_1)$ imply the isomorphism $(D_2,\dashv_2,\vdash_2)\cong(D_1,\dashv_1,\vdash_1)$. If $\mathbb{S}$ and $\mathbb{T}$ are model semigroups of classes of semigroups isomorphic to $(D_1,\dashv_1)$ and $(D_1,\vdash_1)$, respectively, then by $\mathbb{S}\between\mathbb{T}$ we denote a model doppelsemigroup of the class of doppelsemigroups isomorphic to $(D_1,\dashv_1,\vdash_1)$.

It is well-known that there are exactly five pairwise non-isomorphic semigroups having two elements: the multiplicative cyclic group $C_2=\{-1,1\}$, the linear semilattice $L_2=\{0,1\}$ with min-operation, the null semigroup $O_2=\{0,1\}$ with zero 0, the left zero semigroup LO_2 with operation $ab=a$, and the right zero semigroup RO_2 with operation $ab=b$.

It was proved in [19] that there exist 6 pairwise non-isomorphic commutative 2-element doppelsemigroups: C_2, $C_2\between C_2^{-1}=(\{-1,1\},\cdot,\cdot_{-1})$, L_2, O_2, $L_2\between O_2=(\{0,1\},\min,\min_0)$, and $O_2\between L_2=(\{0,1\},\min_0,\min)$, and two non-isomorphic non-commutative doppelsemigroups of order 2: LO_2 and RO_2. All two-element doppelsemigroups are strong.

The following Table 2 of all two-element doppelsemigroups and their automorphism groups is taken from [19].

In the following we describe the structure of upfamily extensions of all two-element doppelsemigroups and their automorphism groups. Since all two-element doppelsemigroups are strong and by Proposition 2 the upfamily extension of a strong doppelsemigroup is a strong doppelsemigroup, we conclude that all the following extensions are strong doppelsemigroups.

D	C_2	$C_2 \between C_2^{-1}$	L_2	O_2	$L_2 \between O_2$	$O_2 \between L_2$	LO_2	RO_2
$\mathrm{Aut}\,(D)$	C_1	C_1	C_1	C_1	C_1	C_1	C_2	C_2

Tab. 2: Two-element doppelsemigroups and their automorphism groups

6.1. The doppelsemigroup $\upsilon(C_2)$. For the cyclic group C_2, the doppelsemigroup $\upsilon(C_2)$ is a non-commutative strong doppelsemigroup of order 4. In addition to two principal ultrafilters, it contains two upfamilies $\mathcal{Z} = \{C_2\}$ and $\mathcal{Z}^\perp = \langle\{g\}: g \in C_2\rangle$ which are right zeros in $\upsilon(C_2)$, see [13]. The subdoppelsemigroup $I = \upsilon(C_2) \setminus C_2 = \{\mathcal{Z}, \mathcal{Z}^\perp\}$ of right zeros of $\upsilon(C_2)$ is the unique proper ideal of $\upsilon(C_2)$. Taking into account that $\mathrm{Aut}\,(C_2) \cong C_1$ and the identity automorphism of C_2 can be extended to an automorphism of $\upsilon(C_2)$ in exactly two ways

$$\mathrm{id}_{\upsilon(C_2)}\colon \mathcal{U} \mapsto \mathcal{U} \text{ and } \perp\colon \mathcal{U} \mapsto \mathcal{U}^\perp,$$

we conclude that $\mathrm{Aut}\,(\upsilon(C_2)) \cong C_2$.

6.2. The doppelsemigroup $\upsilon(C_2 \between C_2^{-1})$. Let us consider non-trivial two-element group doppelsemigroup $C_2 \between C_2^{-1} = (\{-1,1\}, \cdot, \cdot_{-1})$, where $a \cdot_{-1} b = a(-1)b = -ab$. According to Proposition 5, right zeros $\mathcal{Z} = \{C_2\}$ and $\mathcal{Z}^\perp = \langle\{g\}: g \in C_2\rangle$ of the semigroup $(\upsilon(C_2), \cdot)$ are right zeros of the doppelsemigroup $\upsilon(C_2 \between C_2^{-1})$ as well. By Proposition 4, the center of the doppelsemigroup $\upsilon(C_2 \between C_2^{-1})$ coincide with $C(\{-1,1\}, \cdot) \cap C(\{-1,1\}, \cdot_{-1}) = \{-1,1\}$. It follows that $\upsilon(C_2 \between C_2^{-1})$ is a non-commutative strong doppesemigroup.

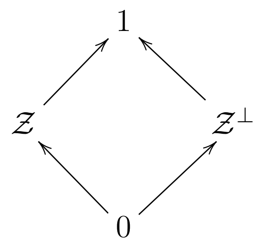
Taking into account that $\mathrm{Aut}\,(C_2 \between C_2^{-1}) \cong C_1$ and the identity automorphism of $C_2 \between C_2^{-1}$ can be extended to an automorphism of $\upsilon(C_2 \between C_2^{-1})$ in exactly two ways

$$\mathrm{id}_{\upsilon(C_2)}\colon \mathcal{U} \mapsto \mathcal{U} \text{ and } \perp\colon \mathcal{U} \mapsto \mathcal{U}^\perp,$$

we conclude that $\mathrm{Aut}\,(\upsilon(C_2 \between C_2^{-1})) \cong C_2$.

6.3. The doppelsemigroup $\upsilon(L_2)$. For the linear semilattice $L_2 = \{0,1\}$ with min-operation the semigroup $\upsilon(L_2)$ of order 4 is a semilattice as well, see [4]. In addition to two principal ultrafilters, it contains two upfamilies $\mathcal{Z} = \{L_2\}$ and $\mathcal{Z}^\perp = \langle\{g\}: g \in L_2\rangle$.

The order structure of the semilattice $\upsilon(L_2)$ is described in the following diagram:



Looking at this diagram we see that the semilattice $\upsilon(L_2)$ is not linear.

Taking into account that $\mathrm{Aut}\,(L_2) \cong C_1$ and the identity automorphism of L_2 can be extended to an automorphism of $\upsilon(L_2)$ in exactly two ways

$$\mathrm{id}_{\upsilon(L_2)}\colon \mathcal{U} \mapsto \mathcal{U} \text{ and } \perp\colon \mathcal{U} \mapsto \mathcal{U}^\perp,$$

we conclude that $\mathrm{Aut}\,(\upsilon(L_2)) \cong C_2$.

6.4. The doppelsemigroup $\upsilon(O_2)$. It was proved in [18] that the upfamily extension of a null semigroup is a null semigroup as well, and $\mathrm{Aut}\,(\upsilon(O_n)) = S_{|\upsilon(O_n)|-1}$ for $n > 1$. We conclude that $\upsilon(O_2) \cong O_4$ and $\mathrm{Aut}\,(\upsilon(O_2)) \cong S_3$.

6.5. The doppelsemigroups $\upsilon(L_2 \between O_2)$ and $\upsilon(O_2 \between L_2)$. For the doppelsemigroups $L_2 \between O_2 = (\{0,1\}, \min, \min_0)$ and $O_2 \between L_2 = (\{0,1\}, \min_0, \min)$ the semigroup $\upsilon(O_2) \cong O_4$ is a variant (generated by 0) of the semilattice $\upsilon(L_2)$.

By Proposition 2.3 of [19], $\mathrm{Aut}\,(\upsilon(L_2 \between O_2)) \cong \mathrm{Aut}\,(\upsilon(O_2 \between L_2)) \cong \mathrm{Aut}\,(\upsilon(L_2)) \cong C_2$.

6.6. The doppelsemigroups $\upsilon(LO_2)$ and $\upsilon(RO_2)$. It is well-known that $\mathrm{Aut}\,(LO_n) \cong S_n$ and $\mathrm{Aut}\,(RO_n) \cong S_n$.

Taking into account that by Corollary 1, the upfamily extension of a left (right) zero doppelsemigroup is a left (right) zero doppelsemigroup as well, we conclude that $\upsilon(LO_2) \cong LO_4$, $\upsilon(RO_2) \cong RO_4$, and hence $\mathrm{Aut}\,(\upsilon(LO_2)) \cong \mathrm{Aut}\,(\upsilon(RO_2)) \cong S_4$.

We summarize the obtained results on the automorphism groups of the upfamily extensions of two-element (strong) doppelsemigroups in the following Table 3.

D	C_2	$C_2 \between C_2^{-1}$	L_2	O_2	$L_2 \between O_2$	$O_2 \between L_2$	LO_2	RO_2
$\mathrm{Aut}\,(\upsilon(D))$	C_2	C_2	C_2	S_3	C_2	C_2	S_4	S_4

Tab. 3: Automorphism groups of the upfamily extensions of two-element doppelsemigroups

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