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REFLECTIONLESS SCHRÖDINGER OPERATORS AND MARCHENKO PARAMETRIZATION

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Let $T_q = -d^2/dx^2 + q$ be a Schrödinger operator in the space $L_2(\mathbb{R})$. A potential q is called reflectionless if the operator T_q is reflectionless. Let $\mathcal{Q}$ be the set of all reflectionless potentials of the Schrödinger operator, and let $\mathcal{M}$ be the set of nonnegative Borel measures on $\mathbb{R}$ with compact support. As shown by Marchenko, each potential $q \in \mathcal{Q}$ can be associated with a unique measure $\mu \in \mathcal{M}$. As a result, we get the bijection $\Theta\colon \mathcal{Q} \to \mathcal{M}$. In this paper, we show that one can define topologies on $\mathcal{Q}$ and $\mathcal{M}$, under which the mapping Θ is a homeomorphism.

1. Introduction. In the Hilbert space $L_2(\mathbb{R})$, we consider the self-adjoint and bounded below Schrödinger operator T_q generated by the differential expression

$$\mathfrak{t}_q(f) = -f'' + qf$$

with a locally integrable real-valued potential $q \in L_{1,\mathrm{loc}}(\mathbb{R})$. For an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$, the equation $\mathfrak{t}_q(f) = zf$ has unique solutions $f_\pm(\cdot, z, q)$ that are square integrable on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, and satisfy the condition $f_\pm(0, z, q) = 1$. The formula

$$m_\pm(z) := m_\pm(z, q) := f'_\pm(0, z, q)$$

defines the Weyl–Titchmarsh m-functions on the half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. It is known (see [1]) that the pair (m_+, m_-) uniquely determines the potential q. We call the operator T_q (a potential q) *reflectionless* (see [2]) if the function

$$n_q(\lambda) := \begin{cases} m_+(\lambda^2), & \operatorname{Im}\lambda > 0,\ \operatorname{Re}\lambda \neq 0; \\ m_-(\lambda^2), & \operatorname{Im}\lambda < 0,\ \operatorname{Re}\lambda \neq 0 \end{cases} \tag{1}$$

has an analytic continuation to the domain $\mathbb{C} \setminus i\mathbb{R}$. We can also suggest (see [3] and [4]) an equivalent definition of the reflectionless potential q in terms of the limiting values of the functions $m_\pm$ on $(0, \infty)$.

Denote by $\mathcal{Q}$ the set of all reflectionless potentials q, and by $\mathcal{M}$ the set of nonnegative Borel measures on $\mathbb{R}$ with compact support. If $q \in \mathcal{Q}$, there exists a unique measure $\nu_q \in \mathcal{M}$ such that, for λ with $\operatorname{Im}\lambda \cdot \operatorname{Re}\lambda \neq 0$,

$$n_q(\lambda) = i\lambda + \int \frac{d\nu_q(t)}{t - i\lambda}.$$

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As shown in [2], the mapping

$$\mathcal{Q} \ni q \overset{\Theta}{\mapsto} \nu_q \in \mathcal{M} \tag{2}$$

is bijective, so that the set $\mathcal{Q}$ is parameterized by elements of the set $\mathcal{M}$. This parameterization is called the Marchenko parametrization.

In the present paper, we show that one can define topologies on $\mathcal{Q}$ and $\mathcal{M}$, under which the mapping (2) is a homeomorphism. To formulate the main result of the paper, let us introduce some notations.

For a measure $\mu \in \mathcal{M}$, we define the numerical characteristics

$$\alpha(\mu) := \sup\{|\lambda| \mid \lambda \in \operatorname{supp}\mu\}, \qquad \gamma(\mu) := \alpha^2(\mu) + \mu(\mathbb{R})$$

and for an arbitrary $n \in \mathbb{N}$, we put

$$\mathcal{Q}(n) := \{q \in \mathcal{Q} \mid \|q\|_\infty \le n\}, \qquad \mathcal{M}(n) := \{\mu \in \mathcal{M} \mid \gamma(\mu) \le n\}.$$

We define the topology of uniform convergence on compact subsets of $\mathbb{R}$ on the set $\mathcal{Q}(n)$ and the topology of weak convergence on the set $\mathcal{M}(n)$. It turns out that the topological spaces $\mathcal{Q}(n)$ and $\mathcal{M}(n)$ are metric compacts (see Section 2). Observe that if $q \in \mathcal{Q}$, then $q \le 0$.

We denote by φ_n the embedding $\mathcal{Q}(n)$ in $\mathcal{Q}$ and equip $\mathcal{Q}$ with the inductive topology with respect to the family $\{(\mathcal{Q}(n), \varphi_n)\}_{n\in\mathbb{N}}$. Analogously, we denote by ψ_n the embedding $\mathcal{M}(n)$ in $\mathcal{M}$ and equip $\mathcal{M}$ with the inductive topology with respect to $\{(\mathcal{M}(n), \psi_n)\}_{n\in\mathbb{N}}$.

The main result of this paper is:

Theorem 1. *The mapping Θ is a homeomorphism from $\mathcal{Q}$ to $\mathcal{M}$.*

Note that there are similar but different results in the papers [2, 3].

2. Preliminaries. For an arbitrary real-valued potential $q \in L_\infty(\mathbb{R})$, we put

$$\beta(q) := -\inf\{\lambda \mid \lambda \in \sigma(T_q)\},$$

where $\sigma(T_q)$ is the spectrum of the operator T_q. The results of [2] (see Theorem 2.1 and Lemma 1.4) imply that the following theorem holds.

Theorem 2. *Let $q \in \mathcal{Q}$ and $\mu = \Theta(q)$. Then*

$$\beta(q) \le \gamma(\mu) \le 2\beta(q), \quad \|q\|_\infty \le 2\beta(q). \tag{3}$$

Corollary 1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} &\Theta(\mathcal{Q}(n)) \subset \mathcal{M}(2n), \quad \Theta^{-1}(\mathcal{M}(n)) \subset \mathcal{Q}(2n),\\ &\mathcal{M}(n) \subset \Theta(\mathcal{Q}(2n)), \quad \mathcal{Q}(n) \subset \Theta^{-1}(\mathcal{M}(2n)). \end{aligned} \tag{4}$$

Proof. Take $q \in \mathcal{Q}$; then, obviously, $\beta(q) \le \|q\|_\infty$, and in view of (3), we have that

$$\gamma(\mu) \le 2\|q\|_\infty, \quad \|q\|_\infty \le 2\gamma(\mu).$$

Inclusions (4) follow from these inequalities. □

Recall that the set $\mathcal{Q}(n)$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}$. In view of [2], for an arbitrary $n \in \mathbb{N}$ the space $\mathcal{Q}(n)$ is countably compact. Since the topology on $\mathcal{Q}(n)$ is generated by the metric

$$d(q_1, q_2) := \max_{x\in\mathbb{R}}(1 + x^2)^{-1}|q_1(x) - q_2(x)|, \qquad q_1, q_2 \in \mathcal{Q}(n), \tag{5}$$

for an arbitrary $n \in \mathbb{N}$ the space $\mathcal{Q}(n)$ is a metric compact.

Denote by $C_0(\mathbb{R})$ the space of all complex-valued continuous functions on $\mathbb{R}$ with compact support. For each measure $\mu \in \mathcal{M}$ and each function $f \in C_0(\mathbb{R})$, we put

$$(\mu, f) := \int_{\mathbb{R}} f\, d\mu.$$

We equip the set $\mathcal{M}(n)$ with the topology of weak convergence, i.e., a sequence $(\mu_j)_{j\in\mathbb{N}}$ in $\mathcal{M}(n)$ is convergent to $\mu \in \mathcal{M}(n)$ if and only if $\lim_{j\to\infty}(\mu_j, f) = (\mu, f)$ for all $f \in C_0(\mathbb{R})$.

By Helly's theorems (see [5]), for every $n \in \mathbb{N}$ the space $\mathcal{M}(n)$ is countably compact. Note that the topology of the space $\mathcal{M}(n)$ is metrizable. Indeed, there exists a countable set $\{\varphi_k\}_{k\in\mathbb{N}}$ in $C_0(\mathbb{R})$ with the topology of uniform convergence such that its linear span $\operatorname{lin}\{\varphi_k\}_{k\in\mathbb{N}}$ is everywhere dense in $C_0(\mathbb{R})$ and $\|\varphi_k\|_\infty = 1$ for all $k \in \mathbb{N}$. It is easy to see that the metric

$$d(\mu, \nu) := \sum_{k\in\mathbb{N}} 2^{-k}|(\mu - \nu, \varphi_k)|, \qquad \mu, \nu \in \mathcal{M},$$

generates the topology of weak convergence on $\mathcal{M}(n)$, and hence $\mathcal{M}(n)$ is a metric compact.

Let $q \in \mathcal{Q}$. Denote by $s(\cdot, z, q)$ and $c(\cdot, z, q)$ the solutions of the equation

$$-f'' + q(x)f = zf, \qquad x \in \mathbb{R},$$

which satisfy the initial data

$$c(0, z, q) = s'(0, z, q) = 1, \qquad c'(0, z, q) = s(0, z, q) = 0.$$

Then $f_+(x, z, q) = c(x, z, q) + m_+(z, q)s(x, z, q)$ for an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$.

According to the classical Weyl theorem (see [6]) the equality

$$\int_0^{+\infty} |f_+(x, z, q)|^2\, dx = \frac{\operatorname{Im} m_+(z, q)}{\operatorname{Im} z}$$

holds. If $\lambda \in \Omega := \{\zeta \in \mathbb{C} \mid 0 < \arg\zeta < \pi/2\}$, then (see (1))

$$f_+(x, \lambda^2, q) = c(x, \lambda^2, q) + n_q(\lambda)s(x, \lambda^2, q), \tag{6}$$

and thus

$$\int_0^{+\infty} |f(x, \lambda^2, q)|^2\, dx = \frac{\operatorname{Im} n_q(\lambda)}{2 \operatorname{Im}\lambda \operatorname{Re}\lambda}. \tag{7}$$

3. Proof of Theorem 1.

Lemma 1. *Let $n \in \mathbb{N}$. The mapping Θ^{-1} acts continuously from the space $\mathcal{M}(n)$ into the space $\mathcal{Q}(2n)$.*

Proof. Since the spaces $\mathcal{M}(n)$ and $\mathcal{Q}(2n)$ are metric compacts, it suffices to prove that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ converges to $\mu \in \mathcal{M}(n)$ in $\mathcal{M}(n)$, then the sequence $q_k = \Theta^{-1}(\mu_k)$, $k \in \mathbb{N}$, converges to $q = \Theta^{-1}(\mu)$ in $\mathcal{Q}(2n)$. The proof is divided into two parts.

Part 1: Let a sequence $(\mu_k)_{k\in\mathbb{N}}$ converge to $\mu \in \mathcal{M}(n)$ in the space $\mathcal{M}(n)$ and $q_k := \Theta^{-1}(\mu_k)$, $k \in \mathbb{N}$. In view of (4), the sequence $(q_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{Q}(2n)$. Since $\mathcal{Q}(2n)$ is a countable compact, from the sequence $(q_k)_{k\in\mathbb{N}}$ one can choose a subsequence $(\widetilde{q}_k)_{k\in\mathbb{N}}$, which converges to a function $\widetilde{q}$ in the space $\mathcal{Q}(2n)$. Let us show that $\widetilde{q} = \Theta^{-1}(\mu)$.

Fix an arbitrary $\lambda \in \Omega$. Since $\sup_{k\in\mathbb{N}} \|\widetilde{q}_k\|_\infty \le 2n$, it is easy to check that the sets

$$\{c(\cdot, \lambda^2, \widetilde{q}_k)\}_{k\in\mathbb{N}}, \qquad \{s(\cdot, \lambda^2, \widetilde{q}_k)\}_{k\in\mathbb{N}}$$

are relatively compact subsets in each space $C^2[-m, m]$ $(m \in \mathbb{N})$. Thus, using Cantor's diagonal process, from the sequence $(\widetilde{q}_k)_{k\in\mathbb{N}}$ one can choose a subsequence $(\widehat{q}_k)_{k\in\mathbb{N}}$ such that the sequences

$$(c(\cdot, \lambda^2, \widehat{q}_k))_{k\in\mathbb{N}}, \qquad (s(\cdot, \lambda^2, \widehat{q}_k))_{k\in\mathbb{N}}$$

converge in each space $C^2[-m, m]$ $(m \in \mathbb{N})$. It is obvious that those sequences converge to the functions $c(\cdot, \lambda^2, \widetilde{q})$ and $s(\cdot, \lambda^2, \widetilde{q})$, respectively.

It follows from (6) and (7) that for all $k \in \mathbb{N}$

$$f_+(x, \lambda^2, \widehat{q}_k) = c(x, \lambda^2, \widehat{q}_k) + n_{\widehat{q}_k}(\lambda) s(x, \lambda^2, \widehat{q}_k), \qquad x \in \mathbb{R}, \tag{8}$$

and

$$\int_0^{+\infty} |f(x, \lambda^2, \widehat{q}_k)|^2 \, dx = \frac{\operatorname{Im} n_{\widehat{q}_k}(\lambda)}{2 \operatorname{Im} \lambda \, \operatorname{Re} \lambda}, \tag{9}$$

where

$$n_{\widehat{q}_k}(\lambda) = i\lambda + \int \frac{d\widehat{\mu}_k(t)}{t - i\lambda}, \qquad \widehat{\mu}_k = \Theta(\widehat{q}_k).$$

Since the sequence $(\widehat{\mu}_k)_{k\in\mathbb{N}}$ converges weakly to μ, then $\lim\limits_{k\to\infty} n_{\widehat{q}_k}(\lambda) = n_q(\lambda)$, where $q = \Theta^{-1}(\mu)$. Taking into account (8) and (9), we obtain that the sequence $(f_+(\cdot, \lambda^2, \widehat{q}_k))_{k\in\mathbb{N}}$ converges to the solution

$$y(x) = c(x, \lambda^2, \widetilde{q}) + n_q(\lambda) s(x, \lambda^2, \widetilde{q}), \qquad x \in \mathbb{R},$$

of the equation $\mathfrak{t}_q(f) = \lambda^2 f$ uniformly on compacts, moreover, $y \in L_2(\mathbb{R}_+)$ and $y(0) = 1$. Uniqueness of the right Weyl–Titchmarsh solution (see [2]) implies that $y = f_+(\cdot, \lambda^2, \widetilde{q})$, and, hence,

$$n_{\widetilde{q}}(\lambda) = f'_+(0, \lambda^2, \widetilde{q}) = y'(0) = n_q(\lambda).$$

Since $\lambda \in \Omega$ is arbitrary, we conclude that $n_{\widetilde{q}} = n_q$. It means that $\widetilde{q} = q = \Theta^{-1}(\mu)$.

Part 2: Let a sequence $(\mu_k)_{k\in\mathbb{N}}$ converge to $\mu \in \mathcal{M}(n)$ in $\mathcal{M}(n)$ and $q_k := \Theta^{-1}(\mu_k)$, $k \in \mathbb{N}$. Assume the sequence $(q_k)_{k\in\mathbb{N}}$ does not converge in $\mathcal{Q}(2n)$. Since the space $\mathcal{Q}(2n)$ is metric compact, the set of accumulation points of the set $\{q_k\}_{k\in\mathbb{N}}$ contains at least two points. Thus from the sequence $(q_k)_{k\in\mathbb{N}}$ one can choose two subsequences, which converge to some functions u_1 and u_2, respectively, moreover, $u_1 \ne u_2$. But it follows from Part 1 that $u_1 = \Theta^{-1}(\mu) = u_2$. We have got a contradiction. The proof is complete. □

Lemma 2. *Let $n \in \mathbb{N}$. The mapping Θ acts continuously from the space $\mathcal{Q}(n)$ into the space $\mathcal{M}(2n)$.*

Proof. Let $n \in \mathbb{N}$. By Lemma 1 the mapping Θ^{-1} acts continuously and injective from the metric compact $\mathcal{M}(2n)$ into the metric compact $\mathcal{Q}(4n)$. Thus the image $\Theta^{-1}(\mathcal{M}(2n))$ is a metric compact with the metric (5). Therefore the mapping

$$\Theta^{-1}\colon \mathcal{M}(2n) \to \Theta^{-1}(\mathcal{M}(2n))$$

is a homeomorphism. Hence the mapping

$$\Theta\colon \Theta^{-1}(\mathcal{M}(2n)) \to \mathcal{M}(2n)$$

is continuous. In view of (4), we have $\mathcal{Q}(n) \subset \Theta^{-1}(\mathcal{M}(2n))$. Thus the mapping $\Theta\colon \mathcal{Q}(n) \to \mathcal{M}(2n)$ is continuous too. □

Proof of Theorem 1. The closure of a set F (the set G) in $\mathcal{Q}$ ($\mathcal{M}$) is equivalent to the fact that for all $n \in \mathbb{N}$ the set $F \cap \mathcal{Q}(n)$ ($G \cap \mathcal{M}(n)$) is closed in the space $\mathcal{Q}(n)$ (in the space $\mathcal{M}(n)$). Note that if the set F (G) belongs to $\mathcal{Q}(n)$ ($\mathcal{M}(n)$) and is closed in $\mathcal{Q}(m)$ ($\mathcal{M}(m)$), $n \le m$, then it is closed in $\mathcal{Q}(n)$ ($\mathcal{M}(n)$). Since the sets $\mathcal{Q}(n)$ ($\mathcal{M}(n)$) are compacts, in view of Lemma 1 and 2 their images $\Theta(\mathcal{Q}(n))$ ($\Theta^{-1}(\mathcal{M}(n))$) are compacts in $\mathcal{M}(2n)$ ($\mathcal{Q}(2n)$), and, hence, are closed in $\mathcal{M}(2n)$ ($\mathcal{Q}(2n)$).

Let G be a closed set in the space $\mathcal{M}$. Let us show that the set $\Theta^{-1}(G)$ is closed in $\mathcal{Q}$. Indeed, for an arbitrary $n \in \mathbb{N}$

$$\Theta^{-1}(G) \cap \mathcal{Q}(n) = \Theta^{-1}[G \cap \Theta(\mathcal{Q}(n))].$$

Since the set $\Theta(\mathcal{Q}(n))$ is closed in $\mathcal{M}(2n)$, then $G \cap \Theta(\mathcal{Q}(n))$ is closed in $\mathcal{M}(2n)$. It follows from Lemma 2 that the set $\Theta^{-1}[G \cap \Theta(\mathcal{Q}(n))]$ is closed in $\mathcal{Q}(n)$ as the preimage of a closed set under a continuous map. Therefore, the set $\Theta^{-1}(G) \cap \mathcal{Q}(n)$ is closed in $\mathcal{Q}(n)$ for arbitrary $n \in \mathbb{N}$, and, hence, the set $\Theta^{-1}(G)$ is closed in $\mathcal{Q}$.

Analogously, we prove that if F is a closed set in $\mathcal{Q}$, then the set $\Theta(F)$ is closed in $\mathcal{M}$. It follows from the above that Θ is a homeomorphism between $\mathcal{Q}$ and $\mathcal{M}$. □

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