D. Keerthana, V. Visalakshi

ISOTROPY GROUP ON SOME TOPOLOGICAL TRANSFORMATION GROUP STRUCTURE

D. Keerthana, V. Visalakshi. *Isotropy group on some topological transformation group structures*, Mat. Stud. **62** (2024), 93–101.

This paper explores the topological properties of irresolute topological groups, their quotient maps, and the role of topology in normal subgroups. It provides a detailed analysis using examples and counterexamples. The study focuses on the essential features of irresolute topological groups and their quotient groups, for understanding the topological aspects of isotropy groups. For a transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ and a point $y \in \mathsf{Y}$, the set

$$\mathsf{H}_y = \{h \in \mathsf{H} \colon hy = y\}$$

consisting of elements of H that fix y, is called the isotropy group at y.

The paper highlights the distinct topological characteristics of isotropy groups in transformation group structure. It demonstrates that if $(\mathsf{H}, \mathsf{Y}, \psi)$ is an Irr*-topological transformation group, then $(\mathsf{H}/\mathrm{Ker}\ \psi, \mathsf{Y}, \overline{\psi})$ forms an effective Irr*-topological transformation group. By investigating both irresolute topological groups and isotropy groups, the study provides a clear understanding of their topological features. This research improves our understanding of these groups by offering clear examples and counterexamples, leading to a thorough conclusion about their different topological features.

1. Introduction. Investigating continuous algebraic operations emerges as a natural approach for the study of topological group. The examination of sets characterized by both algebraic and topological structures prompts a thorough investigation into the interplay between these two domains. Noteworthy contributions by mathematicians like Andrew Gleason, Deane Montgomery, and Leo Zippin have significantly advanced our understanding of the structure and properties of topological groups. In the framework of topological group structures, algebraic operations endowed with weaker forms of continuity have led to the exploration of semi-topological groups, paratopological groups, and quasi-topological groups between 1930s and 1950s [1, 14, 18]. Diverse topological groups, including S-topological groups [2], irresolute topological groups [11], almost topological groups [17], and p-topological groups [19], have also been studied.

The mid-20th century witnessed the formalization and systematic development of the theory of topological transformation groups. This field continues to evolve with ongoing advancements in mathematics, as researchers examines deep connections with other areas such as geometry, algebraic topology, and functional analysis. Gleason's introduction of topological transformation groups [7] serves as a bridge between algebraic and topological structures. When a topological group acts continuously on a topological space, it gives rise to a topological transformation group.

2020 *Mathematics Subject Classification:* 54H15.
Keywords: topological groups; transformation groups; irresolute topological groups; quotient groups; isotropy groups.
doi:10.30970/ms.62.1.93-101

The concept of semi open sets, a generalization of open sets in topology, emerged as part of the broader development of topology in the early 20th century. In 1963, Levine [9] popularized the term “semi open"by incorporating closure and interior operators. Properties related to semi open sets have been studied by researchers such as Sterling Gene Crossley [6], C. Dorsett [5], Maheswari [10], among others. C. Rajapandiyan et al., [15] have contributed significantly to the establishment of a structure for S-topological transformation groups through the exploration of semi open sets in the context of topological transformation groups. Additionally, the definition of fixed points within S-topological transformation groups and their fundamental properties are explored [16]. Similarly, Keerthana Dhanasekar et al. [8] introduced a novel structure for topological transformation groups, which includes irresolute actions on Irr-topological groups and irresolute topological groups. Isotropy groups have proven the importance in the study of homogeneous spaces. The isotropy group at a point comprises all transformations that fixes the point. Isotropy group has gained prominence in differential geometry and topology.

The paper is structured as follows, section 2 provides the necessary preliminaries essential for the development of the primary outcomes. In Section 3, an in-depth exploration is conducted on topological characteristics associated with an irresolute topological groups and its quotient groups. Section 4 is dedicated to topological aspects of isotropy groups. The section further investigates the properties by relevant examples and counterexamples.

2. Preliminaries. This section provides the essential definition for constructing the theoretical framework pertaining to irresolute functions and exploring the topological aspects of transformation groups. The presented definition serves as a foundational element, enabling the development of a comprehensive theory that examines the irresolute functions and the transformation groups within the topological context. $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ denotes the topological space. For $\mathsf{B} \subseteq \mathsf{Y}$, B^c denotes the complement of B, $Cl(\mathsf{B})$ denotes the closure of B and $Int(\mathsf{B})$ denotes the interior of B.

Definition 1 ([9])**.** A set $\mathsf{B} \subseteq \mathsf{Y}$ is *semi open* $\iff$ $\mathsf{B} \subseteq Cl(Int(\mathsf{B}))$. The class of all semi open sets in Y is denoted by $SO(\mathsf{Y})$.

Definition 2 ([6])**.** A set $\mathsf{B} \subseteq \mathsf{Y}$ is *semi closed set* $\iff$ $Int(Cl(\mathsf{B})) \subseteq \mathsf{B}$. The class of all semi closed sets in Y is denoted by $SC(\mathsf{Y})$.

Definition 3 ([6])**.** A function $g\colon \mathsf{Y} \to \mathsf{Z}$ is said to be *irresolute* $\iff$ for any $D_1 \in SO(\mathsf{Z})$, $g^{-1}(D_1) \in SO(\mathsf{Y})$.

Definition 4 ([10])**.** Let Y be a topological space then
1. Y is *semi-T_0* if $\forall y_1 \neq y_2$ of Y, there exist $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{Y})$ containing y_1 and y_2, containing one but not the other respectively.
2. Y is *semi-T_1* if $\forall y_1 \neq y_2$ of Y, there exist $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{Y})$ containing y_1 but not y_2, and y_2 but not y_1 respectively.
3. Y is *semi-T_2* if $\forall y_1 \neq y_2$ of Y, there exist $\mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{Y})$ and $\mathsf{D}_1 \cap \mathsf{D}_2 = \varnothing$ containing y_1 and y_2, respectively.

Definition 5 ([21])**.** A space Y is called *extremally disconnected* if the closure of each open subset of Y is open.

Definition 6 ([6])**.** A function $g\colon \mathsf{Y} \to \mathsf{Z}$ is said to be *pre semi open* $\iff$ for all $\mathsf{B} \in SO(\mathsf{Y}),\ g(\mathsf{B}) \in SO(\mathsf{Z})$.

Definition 7 ([9])**.** Let $g\colon \mathsf{Y} \to \mathsf{Z}$ be a function such that $\forall\ \mathsf{D}_1 \in \mathcal{T}_{\mathsf{Z}}$, $g^{-1}(\mathsf{D}_1) \in SO(\mathsf{Y})$, then g is said to be *semi-continuous.*

Definition 8 ([5])**.** A space Y is called semi compact if every semi open cover of Y has a finite subcover. A subset $\mathsf{B} \subseteq \mathsf{Y}$ is said to be *semi compact* if it retains this property when viewed as a subspace.

Definition 9 ([4])**.** Two non empty subsets B, C of a topological space Y are said to be *semi separated* $\iff$ $\mathsf{B} \cap sCl(\mathsf{C}) = sCl(\mathsf{B}) \cap \mathsf{C} = \varnothing$.

Definition 10 ([4])**.** In topological space Y, a set B is said to be a *semi connected set* if it cannot be expressed as the union of two semi separated sets.

Definition 11 ([20])**.** Let Y be a topological space and $y \in \mathsf{Y}$. The semi component of y is the union of all semi connected subsets of Y containing y. Further if $\mathsf{E} \subset \mathsf{Y}$ and if $y \in \mathsf{E}$, then the union of all semi connected sets containing y and contained in E is called the *semi component of* Y *corresponding to* y.

Definition 12 ([6])**.** Let Y and Z be a topological spaces. Y and Z are said to be *semi homeomorphic* $\iff$ there exists a function $g\colon \mathsf{Y} \to \mathsf{Z}$, such that g is irresolute, pre semi open, one to one and onto. Such an function g is called a semi homeomorphism.

Definition 13 ([3])**.** Let $(\mathsf{H}, \circledast)$ be a group and, $\mathcal{T}_{\mathsf{H}}$ be a topology on H. Then $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ is said to be a *topological group* if both the multiplication map $\mathfrak{m}\colon \mathsf{H} \times \mathsf{H} \to \mathsf{H}$ and the inverse map $\mathfrak{i}\colon \mathsf{H} \to \mathsf{H}$ are continuous.

Definition 14 ([11])**.** Let $(\mathsf{H}, \circledast)$ be a group and, $\mathcal{T}_{\mathsf{H}}$ be a topology on H. Then $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ is said to be an *Irr-topological group* if both the multiplication map $\mathfrak{m}\colon \mathsf{H} \times \mathsf{H} \to \mathsf{H}$ and the inverse map $\mathfrak{i}\colon \mathsf{H} \to \mathsf{H}$ are irresolute.

Definition 15 ([11])**.** Let $(\mathsf{H}, \circledast)$ be a group and, $\mathcal{T}_{\mathsf{H}}$ be a topology on H. Then $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ is said to be an *irresolute topological group* if for each $h_1, h_2 \in \mathsf{H}$ and each $\mathsf{D}_3 \in SO(\mathsf{H})$ containing $h_1 \circledast h_2^{-1}$ there exist $\mathsf{D}_1 \in SO(\mathsf{H})$ containing h_1 and $\mathsf{D}_2 \in SO(\mathsf{H})$ containing h_2 such that $\mathsf{D}_1 \circledast \mathsf{D}_2^{-1} \subset \mathsf{D}_3$.

Definition 16 ([3])**.** Let $(\mathsf{H}, \circledast)$ be a group and Y be a set. Then a map $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ such that:

1. $\psi(e, y) = y$, $\forall\ y \in \mathsf{Y}$, where e is the identity of H;
2. $\psi(h_2, \psi(h_1, y)) = \psi(h_2 h_1, y)$ for all $h_1, h_2 \in \mathsf{H}$ and $y \in \mathsf{Y}$.

The triple $(\mathsf{H}, \mathsf{Y}, \psi)$ is called a *transformation group or* H*-action on* Y, and Y is called a H*-set.*

Definition 17 ([3])**.** A triplet $(\mathsf{H}, \mathsf{Y}, \psi)$ is called a *topological transformation group (TTG)* in which H is a topological group, Y is a topological space, and $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ is a continuous map satisfying the following conditions,

1. $\psi(e, y) = y$, for all $y \in \mathsf{Y}$, where e is the identity element of H.
2. $\psi(h_2, (h_1, y)) = \psi(h_2 h_1, y)$, for every $h_1, h_2 \in \mathsf{H}$ and $y \in \mathsf{Y}$ The space Y, along with a given action ψ of H, is called a H-space.

Theorem 1 ([12])**.** *Let* $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ *be an extremally disconnected irresolute topological group,* K *its normal subgroup. Then* $(\mathsf{H}/\mathsf{K}, \overline{\circledast}, s\tau_{\mathbf{Q}})$ *is an irresolute topological group.*

Lemma 1 ([11])**.** *Let* $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ *be an irresolute topological group,* $\mathsf{A}, \mathsf{B} \subseteq \mathsf{H}$. *If* $\mathsf{A} \in SO(\mathsf{H})$ *then* $\mathsf{A} \circledast \mathsf{B},\ \mathsf{B} \circledast \mathsf{A} \in SO(\mathsf{H})$.

Lemma 2 ([3])**.** Ker ψ *is a normal subgroup of* H.

Definition 18 ([3])**.** A transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ is said to be effective if $\bigcap_{y \in \mathsf{Y}} \mathsf{H}_y = \{e\}$.

Theorem 2 ([13])**.** *Let* $(\mathsf{H}, \circledast, \mathcal{T}_{\mathsf{H}})$ *be an irresolute topological group and* K *be a semi connected component of* H, *then* K *is semi closed, and a normal subgroup of* H.

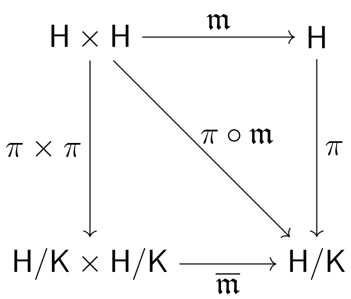
3. Topological properties of irresolute topological groups and their quotient groups. In this section, a comprehensive examination of the topological characteristics inherent in an irresolute topological groups, as well as the corresponding properties of their quotient groups are investigated. This investigation helps to study the topological aspects of isotropy group in detail.

Lemma 3. *Let* H *be an irresolute topological group and* K *be its semi closed subgroup, then the quotient space* H/K *is semi-*T_2 *and the projection map* $\mathfrak{p}\colon \mathsf{H} \to \mathsf{H}/\mathsf{K}$ *is irresolute and pre semi open.*

Proof. The projection map $\mathfrak{p}$ is irresolute in nature. Since $\mathfrak{p}^{-1}(\mathfrak{p}(\mathsf{D}))$ is semi open if D is semi open. Hence $\mathfrak{p}(\mathsf{D})$ is semi open in H/K by definition. To prove H/K is semi-T_2, suppose that $h_1\mathsf{K} \neq h_2\mathsf{K}$ for $h_1, h_2 \in \mathsf{H}$. Then $h_1^{-1}h_2 \notin \mathsf{K}$. Since K is semi closed, there exist $\mathsf{D}_1 \in SO(\mathsf{H})$ containing $h_1^{-1}h_2$ with $\mathsf{D}_1 \cap \mathsf{K} = \varnothing$. The map $\mathfrak{g}\colon \mathsf{H} \times \mathsf{H} \times \mathsf{H} \to \mathsf{H}$ defined by $\mathfrak{g}(h_1, h_2, h_3) = h_1h_2h_3$, inverse image any semi open set D_4 of H is product of semi open sets $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ of H, since $\mathfrak{g}(h_1, h_2, h_3) = \mathfrak{m}(h_1, \mathfrak{m}(h_2, h_3))$. From the map $\mathfrak{g}$ at $(e, h_1^{-1}h_2, e)$, there exists $\mathsf{D}_2 \in SO(\mathsf{H})$ containing e with $\mathsf{D}_2h_1^{-1}h_2\mathsf{D}_2 \subset \mathsf{D}_1$, there is a symmetric semi open set D_3 containing e with $\mathsf{D}_3 \subset \mathsf{D}_2$. Since $\mathsf{D}_3h_1^{-1}h_2\mathsf{D}_3 \subset \mathsf{D}_1$, $\mathsf{D}_3h_1^{-1}h_2\mathsf{D}_3 \cap \mathsf{K} = \varnothing$. Thus $h_1^{-1}h_2\mathsf{D}_3\mathsf{K} \cap \mathsf{D}_3\mathsf{K} = \varnothing$ which implies that $h_2\mathsf{D}_3\mathsf{K} \cap h_1\mathsf{D}_3\mathsf{K} = \varnothing$. Since K is a subgroup of H, $h_2\mathsf{D}_3\mathsf{K} \cap h_1\mathsf{D}_3\mathsf{K} = \varnothing$. Thus there exist disjoint semi open sets $h_1\mathsf{D}_3\mathsf{K}$ and $h_2\mathsf{D}_3\mathsf{K}$ in H/K containing $h_1\mathsf{K}$ and $h_2\mathsf{K}$, respectively. □

Proposition 1. H/K *is an irresolute topological group for any semi-closed normal subgroup* K *of an irresolute topological group* H.

Proof. H/K is a group, since K is a normal subgroup of H. Let the map $\overline{\mathfrak{m}}\colon \mathsf{H}/\mathsf{K} \times \mathsf{H}/\mathsf{K} \to \mathsf{H}/\mathsf{K}$ be defined by $\overline{\mathfrak{m}}(h_1\mathsf{K},\ h_2\mathsf{K}) = h_1 \circledast h_2^{-1}\mathsf{K}$. To prove H/K is an irresolute topological group, its enough to prove that for each $h_1\mathsf{K},\ h_2\mathsf{K} \in \mathsf{H}/\mathsf{K}$ and each $\mathsf{D}_3 \in SO(\mathsf{H}/\mathsf{K})$ containing $h_1 \circledast h_2^{-1}\mathsf{K}$, there exist $\mathsf{D}_1 \in SO(\mathsf{H}/\mathsf{K})$ containing $h_1\mathsf{K}$ and $\mathsf{D}_2 \in SO(\mathsf{H}/\mathsf{K})$ containing $h_2\mathsf{K}$ such that $\mathsf{D}_1 \circledast \mathsf{D}_2^{-1} \subset \mathsf{D}_3$. Consider the commutative diagram,



since π is surjective, $\pi\times\pi$ is surjective. Since π is pre semi open, $\forall\ \mathsf{D}_1 \in SO(\mathsf{H}/\mathsf{K})$, $\overline{\mathfrak{m}}^{-1}(\mathsf{D}_1) = (\pi\times\pi)\circ(\pi\times\pi)^{-1}\circ(\overline{\mathfrak{m}}^{-1}(\mathsf{D}_1)) = (\pi\times\pi)\circ((\pi\circ\mathfrak{m})^{-1}(\mathsf{D}_1))$. Since inverse image of any semi open of H/K is product of semi open sets of H under $\pi\circ\mathfrak{m}$ and $\forall \mathsf{D}_1, \mathsf{D}_2 \in SO(\mathsf{H}), \exists \mathsf{E}_1, \mathsf{E}_2 \in SO(\mathsf{H}/\mathsf{K})$ such that $(\pi\times\pi)(\mathsf{D}_1\times\mathsf{D}_2) \subseteq \mathsf{E}_1\times\mathsf{E}_2$. Therefore $\overline{\mathfrak{m}}^{-1}(\mathsf{D}_1)$ is the product of semi open subsets of H/K which implies that H/K is an irresolute topological group. □

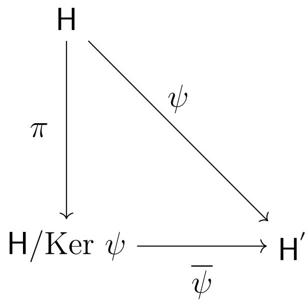
Corollary 1. *Let H_0 be the semi component of an irresolute topological group H containing identity then H/H_0 is an irresolute topological group.*

Proof. The proof follows from Proposition 1 and Theorem 2 □

Let H and H' be an irresolute topological groups. Then a map $\psi\colon \mathsf{H} \to \mathsf{H}'$ is called a homomorphism of an irresolute topological groups when ψ is irresolute as well as a homomorphism of groups. If ψ is a semi homeomorphism as well as an isomorphism of groups, ψ is called an isomorphism of an irresolute topological groups.

Proposition 2. *If $\psi\colon \mathsf{H} \to \mathsf{H}'$ is a homomorphism of an semi-T_2 irresolute topological groups, then* Ker $\psi = \{h \in \mathsf{H}\colon \psi(h) = e' :$ *the identity of* $\mathsf{H}'\}$ *is a semi closed normal subgroup of* H. *In particular the induced map $\overline{\psi}\colon \mathsf{H}/\mathrm{Ker}\ \psi \to \mathsf{H}'$ is irresolute as well as an isomorphism of groups if ψ is surjective. Likewise $\overline{\psi}$ is an isomorphism of an irresolute topological groups, if H is semi compact or ψ is pre semi open.*

Proof. Ker ψ is a normal subgroup of H. Since the identity e' of H' is semi closed, it follows from irresoluteness of that Ker $\psi = \psi^{-1}(e')$ is semi closed. Consider the following commutative diagram



where π is the natural projection. Since $\mathsf{H}/\mathrm{Ker}\ \psi$ is endowed with the quotient topology and ψ is irresolute then $\forall\ \mathsf{D}_1 \in SO(\mathsf{H}')$, the set $\pi^{-1}(\overline{\psi}^{-1}(\mathsf{D}_1)) = \psi^{-1}(\mathsf{D}_1)$ is semi open. Thus $\overline{\psi}^{-1}(\mathsf{D}_1)$ is semi open from the definition of the quotient topology. Thus $\overline{\psi}$ is irresolute. If ψ is surjective, then $\overline{\psi}$ is an isomorphism of groups. Suppose H is semi compact. Then its image $\mathsf{H}/\mathrm{Ker}\ \psi$ is also semi compact. This, together with the fact that H' is semi-T_2, implies that $\overline{\psi}$ is a semi closed map. Hence $\overline{\psi}$ is a semi homeomorphism, and this gives an isomorphism of an irresolute topological groups. Similarly if ψ is a pre semi open map, $\overline{\psi}$ is also a pre semi open map. Hence $\overline{\psi}$ is a semi homeomorphism and this gives an isomorphism of an irresolute topological groups. □

Remark 1. If H, H' is an irresolute topological groups, then Proposition 2 need not be true.

Example 1. Let $\mathsf{H} = \{0, 1, 2, 3\}$ be the group and $\mathcal{T}_{\mathsf{H}} = \{\varnothing, \mathsf{H}, \{0, 2\}, \{1, 3\}\}$ be the topolgy on H. $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ forms an irresolute topological group but $\{0\}$ is not semi closed.

4. Topological properties of isotropy groups within transformation groups. In this section, the topological characteristics exhibited by isotropy groups within transformation groups are explored. Essential examples and counterexamples are provided for indepth understanding of the topological properties.

Definition 19 ([8])**.** A transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ on Y is said to be an *Irr-topological transformation group (Irr-TTG)* if H is an Irr-topological group, Y is a topological space, and the map $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ is irresolute.

Example 2. Let $\mathsf{H} = \{1, 3, 5, 7\}$ be a group under multiplication modulo 8, and the topology on H be $\mathcal{T}_{\mathsf{H}} = \{\varnothing, \mathsf{H}, \{1\}, \{1, 3, 5\}\}$, H acting on itself forms an Irr-topological transformation group.

Definition 20. A transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ on Y is said to be an *Irr*-topological transformation group (Irr*-TTG)* if H is an Irr-topological group, Y is a topological space, and the map $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ such that $\forall\ h \in \mathsf{H}, y \in \mathsf{Y}, \forall$ semi open set D_3 containing $gy \in \mathsf{Y}$, there exist semi open sets D_1 and D_2 containing h and y respectively such that $\mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$.

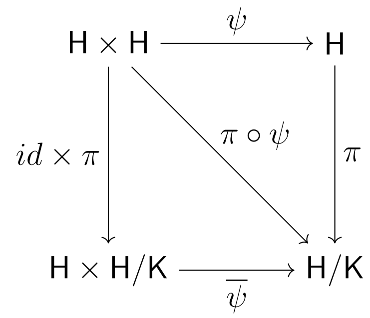
Example 3. Let $\mathsf{H} = \{I, A, B, C\}$ be a group under matrix multiplication, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and the topology on H be $\mathcal{T}_{\mathsf{H}} = \{\varnothing, \mathsf{H}, \{I\}, \{I, A, B\}\}$ and $\mathsf{Y} = \{a, b, c, d\}$, where $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathcal{T}_{\mathsf{Y}} = \{\varnothing, \mathsf{Y}\}$ such that $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ forms an Irr-topological group and $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ forms a topological space and $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ such that $\psi(M, y) = My$ forms an Irr*-topological transformation group.

Definition 21. A transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ on Y is said to be a *I*-topological transformation group (I*-TTG)* if H is a irresolute topological group, Y is a topological space, and the map $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ is irresolute.

Example 4. Let $\mathsf{H} = \{e, (12)(34), (13)(24), (14)(23)\}$ be a group under composition, and the topology on H be
$\mathcal{T}_{\mathsf{H}} = \{\varnothing, \mathsf{H}, \{e\}, \{e, (12)(34), (13)(24)\}\}$ and $\mathsf{Y} = \{1, 2, 3, 4\}$, $\mathcal{T}_{\mathsf{Y}} = \{\varnothing, \mathsf{Y}, \{1, 2\}, \{3, 4\}\}$ such that $(\mathsf{H}, \mathcal{T}_{\mathsf{H}})$ forms an Irr-topological group and $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ forms a topological space and $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ such that $\psi(\sigma, y) = \sigma(y)$ forms a I*-TTG.

Definition 22. A transformation group $(\mathsf{H}, \mathsf{Y}, \psi)$ on Y is said to be an *I-topological transformation group (I-TTG)* if H is an irresolute topological group, Y is a topological space, and the map $\psi\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{Y}$ such that $\forall\ h \in \mathsf{H}, y \in \mathsf{Y}, \forall$ semi open set D_3 containing $hy \in \mathsf{Y}$, there exist a semi open sets D_1 and D_2 containing h and y respectively such that $\mathsf{D}_1\mathsf{D}_2 \subseteq \mathsf{D}_3$.

Example 5. Let H be an irresolute topological group and K be a subgroup of H. Let the map $\psi\colon \mathsf{H} \times \mathsf{H}/\mathsf{K} \to \mathsf{H}/\mathsf{K}$ defines a G-action $\psi(h', h\mathsf{K}) = h'h\mathsf{K}$, for any topology, H acts on H/K



Hence π is surjective and pre semi open. $id \times \pi$ is surjective and pre-semi open. From the above commutative diagram inverse image of any semi open set D_1 of H/K is the product of semi open sets D_2 and D_3 of H and H/K respectively under $\overline{\psi}$.

Definition 23. Let $(\mathsf{H}, \mathsf{Y}, \psi)$ be a transformation group then for $y \in \mathsf{Y}$, the set $\mathsf{H}_y = \{h \in \mathsf{H} \colon hy = y\}$ of elements of H leaving y fixed is called the *isotropy group at* y.

Example 6. Let $\mathsf{H} = \{e, i, j, ij \colon i^2 = e, j^2 = e, ij = ji\}$ be a group, H acting on $\mathsf{H}/\mathsf{K} = \{e\mathsf{K}, j\mathsf{K}\}, \mathsf{K} = \{e, i\}$ be the cosets, forms a transformation group. The isotropy group of $e\mathsf{K}$ and $j\mathsf{K}$ be $\mathsf{H}_{e\mathsf{K}} = \mathsf{H}_{j\mathsf{K}} = \{e, i\}$.

Lemma 4. *Ker* $\psi = \bigcap_{y \in \mathsf{Y}} \mathsf{H}_y$.

Indeed,

$$h \in \mathrm{Ker}\,\psi \iff \psi(h, y) = hy = y \ (\forall y \in \mathsf{Y}) \iff h \in \mathsf{H}_y \ \forall y \in \mathsf{Y} \iff h \in \bigcap_{y \in \mathsf{Y}} \mathsf{H}_y.$$

Lemma 5. *Let* $(\mathsf{H}, \mathsf{Y}, \psi)$ *be an Irr*-topological transformation group, fix* $y \in \mathsf{Y}$, *a map* $\mathfrak{i} \colon \mathsf{H} \to \mathsf{H} \times \mathsf{Y}$ *such that* $\mathfrak{i}(h) = (h, y)$, *then* $\forall h \in \mathsf{H}, y \in \mathsf{Y}, (h, y) \in \mathsf{D}_1 \times \mathsf{D}_2, \mathsf{D}_1 \in SO(\mathsf{H}), \mathsf{D}_2 \in SO(\mathsf{Y})$, $\mathfrak{i}^{-1}(\mathsf{D}_1 \times \mathsf{D}_2) \subseteq \mathsf{D}_1 \in SO(\mathsf{H})$ *and* $\mathsf{H} - \mathsf{H}_y = \mathfrak{i}^{-1}(\psi^{-1}(\mathsf{Y} - \{y\}))$.

Proof. Let $h \in \mathsf{H} - \mathsf{H}_y$ which implies $\psi \circ \mathfrak{i}(h) = \psi(h, y) \neq y$. Let $h \in i^{-1}(\psi^{-1}(\mathsf{Y} - \{y\}))$. Thus $\mathsf{H} - \mathsf{H}_y \subset i^{-1}(\psi^{-1}(\mathsf{Y} - \{y\}))$, $\psi \circ i(h) = \psi(h, y) \in \mathsf{Y} - \{y\}$. Hence $\psi(h, y) \neq y$, which implies $h \in \mathsf{H} - \mathsf{H}_y$. Thus $\mathsf{H} - \mathsf{H}_y \supset i^{-1}(\psi^{-1}(\mathsf{Y} - \{y\}))$. □

Theorem 3. *The isotropy group* H_y *at any point* y *of* Y *is semi-closed if* Y *is semi-*T_1.

Proof. By assumption, $\mathsf{Y} - \{y\}$ is open. It follows from Lemma 5 that $\mathsf{H} - \mathsf{H}_y$ is semi open, and hence H_y is semi closed. □

Example 7. Let $\mathsf{H} = \{e, i, j, ij \colon i^2 = e, j^2 = e, ij = ji\}$ be the group, $\mathcal{T}_\mathsf{H} = \{\varnothing, \mathsf{H}, \{e, i\}, \{j, ij\}\}$; $(\mathsf{H}, \mathcal{T}_\mathsf{H})$ forms an irresolute topological group and H acting $\mathsf{H}/\mathsf{K} = \{e\mathsf{K}, j\mathsf{K}\}, \mathsf{K} = \{e, i\}$ be the cosets, forms an I-TTG. The isotropy group of $e\mathsf{K}$ and $j\mathsf{K}$ be $\mathsf{H}_{e\mathsf{K}} = \mathsf{H}_{j\mathsf{K}} = \{e, i\}$ which is semi closed, since H/K is semi-T_1.

Example 8. In Example 7, if H acting on itself. The istropy group of any element y be $\mathsf{H}_y = \{e\}$ which is not semi closed, since H is not semi-T_1.

Corollary 2. Ker ψ *is a closed subgroup of* H *if* Y *is semi-*T_1.

Proof. Corollary 2 follows from Theorem 3 Lemma 4. □

Corollary 3. $\mathsf{H}/$Ker ψ *is an irresolute topological group if* Y *is semi-*T_1.

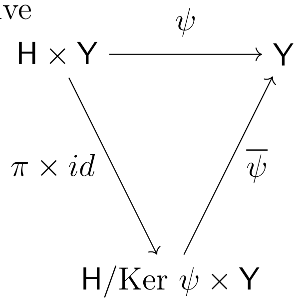
Proof. From Lemma 2 and Corollary 2, Ker ψ is a semi closed normal subgroup of H . Hence Proposition 1 yields Corollary 3. □

Theorem 4. *If* H *is a extremely disconnected space then* $\mathsf{H}/$Ker ψ *is an irresolute topological group.*

Proof. Since Ker ψ is a normal subgroup of H. From Theorem 1, $\mathsf{H}/$Ker ψ is an irresolute topological group. □

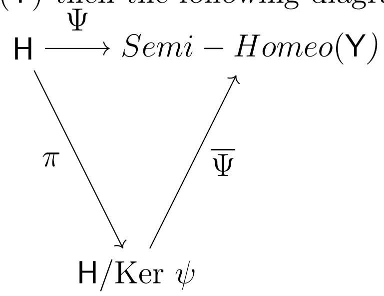
Theorem 5. *An Irr*-topological transformation group* $(\mathsf{H}, \mathsf{Y}, \psi)$ *induces an effective Irr*-topological transformation group* $(\mathsf{H}/\mathrm{Ker}\ \psi, \mathsf{Y}, \overline{\psi})$ *if* Y *is semi-*T_1.

Proof. Define $\overline{\psi}\colon \mathsf{H}/\text{Ker } \psi \times \mathsf{Y} \to \mathsf{Y}$ by $\overline{\psi}(h\text{Ker } \psi,\ y) = \psi(h, y)$. It is easily verified that $\overline{\psi}$ is well defined and satisfies the conditions (1) and (2) of group action (Definition 16). Hence it is enough to check inverse image of any semi open set of Y is product of semi open sets of $\mathsf{H}/\text{Ker } \psi$ and Y under $\overline{\psi}$. The natural projection $\pi\colon \mathsf{H} \to \mathsf{H}/\text{Ker } \psi$ is pre semi open, $\pi \times id\colon \mathsf{H} \times \mathsf{Y} \to \mathsf{H}/\text{Ker } \psi \times \mathsf{Y}$ is also pre semi open. From commutative diagram, the inverse image of any semi open of Y is product of semi open sets of $\mathsf{H}/\text{Ker } \psi$ and Y under $\overline{\psi}$. Evidently the action $\overline{\psi}$ is effective



□

Corollary 4. *If* Y *is semi-*T_1*, then there exists a semi continuous injective group homomorphism* $\Psi\colon G \to Semi - Homeo(\mathsf{Y})$ *then the following diagram commutes*



□

Proof. The proof follows from Theorem 5.

Let H be an irresolute topological group and K a semi closed subgroup of H. Let

$$\mathsf{K}_0 = \bigcap_{h \in \mathsf{H}} h\mathsf{K}h^{-1},$$

then K_0 is a semi closed normal subgroup of H. Moreover K_0 is the maximum subgroup among the normal subgroups of H contained in K.

Proposition 3. *Let* $(\mathsf{H}, \mathsf{H}/\mathsf{K}, \psi)$ *be the* H*-action defined on Example* 5*, then the equality* Ker $\psi = \mathsf{K}_0$ *holds.*

Proof. For any $k \in \mathsf{K}_0$ and $h \in \mathsf{H}$, $kh\mathsf{K} = h(h^{-1}kh)\mathsf{K} = h\mathsf{K}$. Hence K_0 acts trivially on H/K. That is, $\mathsf{K}_0 \subset \text{Ker } \psi$. Conversely, suppose that $h_0 h\mathsf{K} = h\mathsf{K}$ for all $h \in \mathsf{H}$. Then $h_0 \in h\mathsf{K}h^{-1}$ for all $h \in \mathsf{H}$. That is, $h_0 \in \mathsf{K}_0$, which implies that $\mathsf{K}_0 \supset \text{Ker } \psi$. □

Corollary 5. H/K_0 *acts effectively on* H/K.

□

Proof. Proof follows from Theorem 5 and Proposition 3.

5. Conclusion. In this paper the topological properties associated with an irresolute topological groups and their quotient groups are investigated. Additionally, the investigation extended to the topological properties of isotropy groups within transformation groups, supported by pertinent examples and counterexamples. Through these explorations, the valuable insights into the topological characterization of an irresolute topological groups and isotropy groups are gained. The illustrative examples and counterexamples has enriched our understanding, contributing to a comprehensive conclusion regarding the diverse topological aspects present in both irresolute topological groups and isotropy groups within transformation groups.

REFERENCES

1. E. Bohn, *Semi-topological groups*, American Mathematical Monthly, **72** (1965), №9, 996–998. https://doi.org/10.2307/2313342.
2. M.S. Bosan, Moiz ud Din Khan, Ljubiša D.R. Kočinac, *On s-topological groups*, Mathematica Moravica, **18** (2014), №2, 35–44. https://doi.org/10.5937/MatMor1402035B.
3. G.E. Bredon, Introduction to compact transformation groups, Academic press, 1972.
4. P. Das, *Note on semi connectedness*, Indian J. Mech. Math., **12** (1974), 31–34.
5. C. Dorsett, *Semi compactness, semi separation axioms, and product spaces*, Bull. Malaysian Math. Soc. (2), **4** (1981), №1, 21–28.
6. S. Gene Crossley, S.K. Hildebrand, *Semi-topological properties*, Fundamenta Mathematicae, **74** (1972), №3, 233–254.
7. A.M. Gleason, R.S. Palais, *On a class of transformation groups*, American Journal of Mathematics, **79** (1957), №3, 631–648, https://doi.org/10.2307/2372567.
8. K. Dhanasekar, V. Visalakshi, *On some topological structures of transformation groups,* IAENG International Journal of Applied Mathematics, **54** (2024), №5, 887–893. https://www.iaeng.org/IJAM/issues_v54/issue_5/IJAM_54_5_11.pdf.
9. N. Levine, *Semi open sets and semi-continuity in topological spaces*, The American mathematical monthly, **70** (1963), №1, 36–41. https://doi.org/10.2307/2312781.
10. S.N. Maheshwari, *Some new separations axioms*, Ann. Soc. Sci. Bruxelles, Ser. I, **89** (1975), 395–402.
11. Moiz ud Din Khan, Afra Siab, Ljubiša D.R. Kočinac, *Irresolute-topological groups*, Mathematica Moravica, **19** (2015), №1, 73–80. https://doi.org/10.5937/MATMOR1501073K.
12. Moiz ud Din Khan, R. Noreen, M.S. Bosan, *Semi-quotient mappings and spaces,* Open Mathematics, **14** (2016), №1, 1014–1022. https://doi.org/10.1515/ math-2016-0093.
13. R. Noreen, M.S. Bosan, M.D. Khan, *Semi connectedness in irresolute topological groups*, Science International, **27** (2015), №6.
14. Piyu Li, Lei Mou, *On quasitopological groups*, Topology and its Applications, **161** (2014), 243–247. https://doi.org/10.1016/j.topol.2013.10.022.
15. C. Rajapandiyan, V. Visalakshi, S. Jafari, *On a new type of topological transformation group*, Asia Pacific Journal of Mathematics, **11** (2024), №5. https://doi.org/10.28924/APJM/11-5.
16. C. Rajapandiyan, V. Visalakshi, *Fixed point set and equivariant map of a S-topological transformation group,* International Journal of Analysis and Applications, **22** (2024). http://dx.doi.org/10.28924/2291-8639-22-2024-20.
17. M. Ram, *On almost topological groups*, Mathematica Moravica, **23** (2019), №1, 97–106. https://doi.org/10.5937/MatMor1901097R.
18. O.V. Ravsky, *Paratopological groups I*, Mat. Stud., **16**(2001), №1, 37–48.
19. S. Jafari, P. Gnanachandra, A.M. Kumar, *On p-topological groups*, Mathematica Moravica, **25** (2021), №2, 13–27. https://doi.org/10.5937/MatMor2102013J.
20. J.P. Sarker, H. Dasgupta, *Locally semi-connectedness in topological spaces*, Indian J. Pure Appl. Math., **16** (1985), №12, 1488–1494.
21. M.H. Stone, *Algebraic characterizations of special Boolean rings*, Fundamenta Mathematicae, **29** (1937), №1, 223–303.

Department of Mathematics, College of Engineering and Technology
SRM Institute of Science and Technology
SRM Nagar, Kattankulathur-603203, Tamil Nadu, India
kd1002@srmist.edu.in
visalakv@srmist.edu.in (corresponding author)

Received 23.01.2024
Revised 23.07.2024