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ON LOCALLY COMPACT SHIFT CONTINUOUS TOPOLOGIES ON THE SEMIGROUP $\boldsymbol{B}_{[0,\infty)}$ WITH AN ADJOINED COMPACT IDEAL

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Let $[0,\infty)$ be the set of all non-negative real numbers. The set $\boldsymbol{B}_{[0,\infty)}=[0,\infty)\times[0,\infty)$ with the following binary operation $(a,b)(c,d)=(a+c-\min\{b,c\},b+d-\min\{b,c\})$ is a bisimple inverse semigroup. In the paper we study Hausdorff locally compact shift-continuous topologies on the semigroup $\boldsymbol{B}_{[0,\infty)}$ with an adjoined compact ideal of the following tree types. The semigroup $\boldsymbol{B}_{[0,\infty)}$ with the induced usual topology τ_u from $\mathbb{R}^2$, with the topology τ_L which is generated by the natural partial order on the inverse semigroup $\boldsymbol{B}_{[0,\infty)}$, and the discrete topology are denoted by $\boldsymbol{B}^1_{[0,\infty)}$, $\boldsymbol{B}^2_{[0,\infty)}$, and $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$, respectively. We show that if S_1^I (S_2^I) is a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ ($\boldsymbol{B}^2_{[0,\infty)}$) with an adjoined compact ideal I then either I is an open subset of S_1^I (S_2^I) or the topological space S_1^I (S_2^I) is compact. As a corollary we obtain that the topological space of a Hausdorff locally compact shift-continuous topology on $S^1_{\mathbf{0}}=\boldsymbol{B}^1_{[0,\infty)}\cup\{\mathbf{0}\}$ (resp. $S^2_{\mathbf{0}}=\boldsymbol{B}^2_{[0,\infty)}\cup\{\mathbf{0}\}$) with an adjoined zero $\mathbf{0}$ is either homeomorphic to the one-point Alexandroff compactification of the topological space $\boldsymbol{B}^1_{[0,\infty)}$ (resp. $\boldsymbol{B}^2_{[0,\infty)}$) or zero is an isolated point of $S^1_{\mathbf{0}}$ (resp. $S^2_{\mathbf{0}}$). Also, we proved that if $S^I_{\mathfrak{d}}$ is a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ with an adjoined compact ideal I then I is an open subset of $S^I_{\mathfrak{d}}$.

1. Introduction and preliminaries. In this paper we shall follow the terminology of [13, 14, 15, 16, 18, 32, 35].

A semigroup S is called *inverse* if for any element $x\in S$ there exists a unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*. On an inverse semigroup S the semigroup operation determines the following partial order $\preccurlyeq$: $s\preccurlyeq t$ if and only if there exists $e\in E(S)$ such that $s=te$. This partial order is called the natural partial order on S.

Remark 1. For arbitrary elements s,t of an inverse semigroup S the following conditions are equivalent (see [32, Chap. 3]):

$$(\alpha)\ s\preccurlyeq t;\quad (\beta)\ s=ss^{-1}t;\quad (\gamma)\ s=ts^{-1}s.$$

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A topological space X is called *locally compact* if every poin x of X has an open neighbourhood with the compact closure.

A *(semi)topological semigroup* is a topological space with a (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- an *inverse semigroup* topology if (S,τ) is a topological inverse semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [15].

The bicyclic monoid admits only the discrete semigroup Hausdorff topology [17]. Bertman and West in [12] extended this result for the case of Hausdorff semitopological semigroups. If a Hausdorff (semi)topological semigroup T contains the bicyclic monoid $\mathscr{C}(p,q)$ as a dense proper semigroup then $T\setminus\mathscr{C}(p,q)$ is a closed ideal of T [17, 23]. Moreover, the closure of $\mathscr{C}(p,q)$ in a locally compact topological inverse semigroup can be obtained (up to isomorphism) from $\mathscr{C}(p,q)$ by adjoining the additive group of integers in a suitable way [17].

Stable and Γ-compact topological semigroups do not contain the bicyclic monoid [5, 30, 31]. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [6, 7, 11, 29].

In [1] Ahre considered the following semigroup. Let $[0,\infty)$ be the set of all non-negative real numbers. The set $\boldsymbol{B}_{[0,\infty)}=[0,\infty)\times[0,\infty)$ with the following binary operation

$$(a,b)(c,d)=(a+c-\min\{b,c\},b+d-\min\{b,c\})=\begin{cases}(a+c-b,d), & \text{if } b<c;\\ (a,d), & \text{if } b=c;\\ (a,b+d-c), & \text{if } b>c\end{cases}$$

is a bisimple inverse semigroup. The semigroup $\boldsymbol{B}_{[0,\infty)}$ and the bicyclic monoid $\mathscr{C}(p,q)$ are partial cases of bicyclic extensions of linearly ordered groups which are presented in [19, 20, 21, 28]. It is obvious that semigroup $\boldsymbol{B}_{[0,\infty)}$ is isomorphic to the semigroup of partial bijections, namely as the semigroup of shifts of closed rays in the half-line (see [28]). This representation shows the closed relation of the semigroup $\boldsymbol{B}_{[0,\infty)}$ to the bicyclic semigroup, which also has a similar representation by shifts of rays in the set of positive integers.

By $\boldsymbol{B}^1_{[0,\infty)}$ we denote the semigroup $\boldsymbol{B}_{[0,\infty)}$ with the usual topology. It is obvious that $\boldsymbol{B}^1_{[0,\infty)}$ is a locally compact topological inverse semigroup [1]. In [2, 3] it is shown that the closure of $\boldsymbol{B}^1_{[0,\infty)}$ in a locally compact topological inverse semigroup can be obtained (up to isomorphism) from $\boldsymbol{B}^1_{[0,\infty)}$ by adjoining the additive group of reals in a suitable way.

For any non-negative real number α we denote the following subsets in $\boldsymbol{B}_{[0,\infty)}$:

$$L^+_\alpha=\{(x,x+\alpha)\colon x\geqslant 0\}\qquad\text{and}\qquad L^-_\alpha=\{(x+\alpha,x)\colon x\geqslant 0\}.$$

It obvious that $\boldsymbol{B}_{[0,\infty)}=\bigsqcup_{\alpha\geqslant 0}L^+_\alpha\sqcup\bigsqcup_{\alpha>0}L^-_\alpha$ and $L^+_0=L^-_0$. Put τ_L be a topology on $\boldsymbol{B}_{[0,\infty)}$

which is generating by the bases

$$\mathcal{B}(x,x+\alpha)=\{U_\varepsilon(x,x+\alpha)=\{(x+y,x+y+\alpha)\in L^+_\alpha\colon |y|<\varepsilon\}:\varepsilon>0\}$$

and

$$\mathcal{B}(x+\alpha,x)=\{U_\varepsilon(x+\alpha,x)=\{(x+y+\alpha,x+y)\in L^-_\alpha\colon |y|<\varepsilon\}:\varepsilon>0\}$$

at any points $(x,x+\alpha)\in L^+_\alpha$ and $(x+\alpha,x)\in L^-_\alpha$, respectively, for arbitrary $\alpha\in[0,+\infty)$. The semigroup $\boldsymbol{B}_{[0,\infty)}$ with the topology τ_L is denoted by $\boldsymbol{B}^2_{[0,\infty)}$. The definitions of the topology τ_L and the natural partial order on $\boldsymbol{B}_{[0,\infty)}$ imply that τ_L is generated by the natural partial order of $\boldsymbol{B}_{[0,\infty)}$ (see [22]). We observe that $\boldsymbol{B}^2_{[0,\infty)}$ is a Hausdorff locally compact topological inverse semigroup [4]. Moreover for any non-negative real number α, L^+_α and L^-_α are open-and-closed subsets of $\boldsymbol{B}^2_{[0,\infty)}$ which are homeomorphic to $[0,+\infty)$ with the usual topology, i.e.,

$$\boldsymbol{B}^2_{[0,\infty)}=\bigoplus_{\alpha\geqslant 0}L^+_\alpha\oplus\bigoplus_{\alpha>0}L^-_\alpha.$$

The closure of the topological inverse semigroup $\boldsymbol{B}^2_{[0,\infty)}$ in (locally compact) topological semigroups is studied in [4].

By $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ we denote the semigroup $\boldsymbol{B}_{[0,\infty)}$ with the discrete topology. It is obvious that $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ is a locally compact topological inverse semigroup.

In the paper [23] it is proved that every Hausdorff locally compact shift-continuous topology on the bicyclic monoid with adjoined zero is either compact or discrete. This result was extended by Bardyla onto the a polycyclic monoid [8] and graph inverse semigroups [9], and by Mokrytskyi onto the monoid of order isomorphisms between principal filters of $\mathbb{N}^n$ with adjoined zero [34]. In [24] the results of the paper [23] were extended to the monoid $\mathbf{I}\mathbb{N}_\infty$ of all partial cofinite isometries of positive integers with adjoined zero. In [27] the similar dichotomy was proved for so called bicyclic extensions $\boldsymbol{B}^{\mathscr{F}}_\omega$ when a family $\mathscr{F}$ consists of inductive non-empty subsets of ω. Algebraic properties on a group G such that if the discrete group G has these properties then every locally compact shift continuous topology on G with adjoined zero is either compact or discrete studied in [33]. Also, in [26] it is proved that the extended bicyclic semigroup $\mathscr{C}^0_{\mathbb{Z}}$ with adjoined zero admits continuum many shift-continuous topologies, however every Hausdorff locally compact semigroup topology on $\mathscr{C}^0_{\mathbb{Z}}$ is discrete. In [10] Bardyla proved that a Hausdorff locally compact semitopological McAlister semigroup $\mathcal{M}_1$ is either compact or discrete. However, this dichotomy does not hold for the McAlister semigroup $\mathcal{M}_2$ and moreover, $\mathcal{M}_2$ admits continuum many different Hausdorff locally compact inverse semigroup topologies [10].

In this paper we extend the results of paper [23] onto the topological monoids $\boldsymbol{B}^1_{[0,\infty)}$ and $\boldsymbol{B}^2_{[0,\infty)}$. In particular we show that if S^I_1 (S^I_2) is a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ ($\boldsymbol{B}^2_{[0,\infty)}$) with an adjoined compact ideal I then either I is an open subset of S^I_1 (S^I_2) or the semigroup S^I_1 (S^I_2) is compact. Also, we proved that if $S^I_{\mathfrak{d}}$ is a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ with an adjoined compact ideal I then I is an open subset of $S^I_{\mathfrak{d}}$.

2. A locally compact semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined compact ideal. Later in this section by S^I_1 we denote a Hausdorff locally compact semitopological semigroup which is the semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined non-open compact ideal I.

Lemma 1. *Let S be a Hausdorff locally compact semitopological semigroup with a compact ideal I. Then for any open neighbourhood $U(I)$ of the ideal I and any $x\in S$ there exists an open neighbourhood $V(I)$ of I with the compact closure $\overline{V(I)}$ such that $x\cdot V(I)\subseteq U(I)$ and $V(I)\cdot x\subseteq U(I)$.*

Proof. Fix an arbitrary open neighbourhood $U(I)$ of the ideal I and any $x \in S$. Since I is an ideal of S, for any $\alpha \in I$ there exists $\beta \in I$ such that $x \cdot \alpha = \beta$. Since $U(I)$ is an open neighbourhood of β, separate continuity of the semigroup operation in S implies that there exists an open neighbourhood $V(\alpha)$ of α in S such that $x \cdot V(\alpha) \subseteq U(I)$. The local compactness of the space S implies that without loss of generality we may assume that the neighbourhood $V(\alpha)$ has the compact closure $\overline{V(\alpha)}$. Then the family $\{V(\alpha) \colon \alpha \in I\}$ is an open cover of I. Since I is compact, $I \subseteq V(\alpha_1) \cup \ldots \cup V(\alpha_n)$ for finitely many $\alpha_1, \ldots, \alpha_n \in I$. Put $V_1(I) = V(\alpha_1) \cup \ldots \cup V(\alpha_n)$. Then $\overline{V_1(I)} = \overline{V(\alpha_1)} \cup \ldots \cup \overline{V(\alpha_n)}$ is a compact subset of S such that $x \cdot V_1(I) \subseteq U(I)$. Similarly we get that there exists an open neighbourhood $V_2(I)$ of I with the compact closure $\overline{V_2(I)}$ such that $V_2(I) \cdot x \subseteq U(I)$. Put $V(I) = V_1(I) \cap V_2(I)$. Then $V(I)$ is an open neighbourhood of I with the compact closure $\overline{V(I)} = \overline{V_1(I)} \cap \overline{V_2(I)}$ such that $x \cdot V(I) \subseteq U(I)$ and $V(I) \cdot x \subseteq U(I)$. □

A subset A of $\boldsymbol{B}_{[0,\infty)}$ is called *unbounded* if for any positive real number a there exist $(x, y) \in A$ such that $x \geqslant a$ and $y \geqslant a$.

Lemma 2. *For any open neighbourhood $U(I)$ of the ideal I in S_1^I the set $U(I) \cap \boldsymbol{B}_{[0,\infty)}$ is unbounded.*

Proof. Suppose to the contrary that there exists a positive real number m such that $x < m$ or $y < m$ for any $(x, y) \in U(I) \cap \boldsymbol{B}_{[0,\infty)}$. Lemma 1 implies that there exists an open neighbourhood $V(I) \subseteq U(I)$ of I such that $V(I) \cdot (0, 2m) \subseteq U(I)$. Again, by Lemma 1, there exists an open neighbourhood $W(I) \subseteq V(I)$ of I such that $(2m, 0) \cdot W(I) \subseteq V(I)$. Then choose any $(x, y) \in W(I) \setminus I$ and observe that $(a, b) = (2m, 0)(x, y)(0, 2m)$ has the desired property: $\min\{a, b\} \geqslant m$. The obtained contradiction implies the statement of the lemma. □

Proposition 1. *For any open neighbourhood $U(I)$ of the ideal I in S_1^I there exists a compact subset $A_a = [0, a] \times [0, a]$ in $\boldsymbol{B}^1_{[0,\infty)}$ such that $S_1^I \setminus U(I) \subseteq A_a$.*

Proof. Suppose to the contrary that there exists an open neighbourhood $U(I)$ of the ideal I in S_1^I such that $S_1^I \setminus U(I) \nsubseteq A_n$ for any positive integer n. By Lemma 1 without loss of generality we may assume that the closure $\overline{U(I)}$ is a compact subset of S_1^I. By Lemma 2 the set $U(I) \cap \boldsymbol{B}_{[0,\infty)}$ is unbounded in $\boldsymbol{B}_{[0,\infty)}$. Since $\boldsymbol{B}^1_{[0,\infty)} \cap U(I)$ is an open subset in $\boldsymbol{B}^1_{[0,\infty)}$, the assumption of the proposition implies that for any positive integer n there exists an element $(x_n, y_n) \in \overline{U(I)} \setminus U(I)$ such that $(x_n, y_n) \notin A_n$. This implies that the set $\overline{U(I)} \setminus U(I)$ is unbounded in $\boldsymbol{B}_{[0,\infty)}$. But $\overline{U(I)} \setminus U(I)$ is a compact subspace of the metric space $\boldsymbol{B}^1_{[0,\infty)}$, a contradiction. □

Proposition 1 implies the following theorem.

Theorem 1. *Let S_1^I be a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined compact ideal I. Then either I is an open subset of S_1^I or the space S_1^I is compact.*

Example 1 and Proposition 2 show that if the ideal I of the semigroup S_1^I is trivial, i.e., the ideal I is a singleton, then the semigroup S_1^I admits the unique Hausdorff compact shift-continuous topology.

Example 1. Let $S_1^{\mathbf{0}}$ be the semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. We extend the topology of $\boldsymbol{B}^1_{[0,\infty)}$ up to a compact topology τ^1_{Ac} on $S_1^{\mathbf{0}}$ in the following way. We define

$$\mathscr{B}^1_{\mathsf{Ac}}(\mathbf{0}) = \{U_n(\mathbf{0}) = \{0\} \cup \{(x,y)\colon x > n \text{ or } y > n\} : n \in \mathbb{N}\}$$

is the system of open neighbourhoods of zero in τ^1_{Ac}.

Proposition 2. $(S_1^{\mathbf{0}}, \tau^1_{\mathsf{Ac}})$ *is a compact Hausdorff semitopological semigroup with continuous inversion.*

Proof. By [2, 3], $\boldsymbol{B}^1_{[0,\infty)}$ is a topological inverse semigroup, and hence it sufficient to show that the semigroup operation on $(S_1^{\mathbf{0}}, \tau^1_{\mathsf{Ac}})$ is separately continuous at zero.

It is obvious that $\mathbf{0} \cdot U_n(\mathbf{0}) = U_n(\mathbf{0}) \cdot \mathbf{0} = \{\mathbf{0}\} \subseteq U_n(\mathbf{0})$ for any positive integer n.

Next we shall show that $(x,y) \cdot U_{2n}(\mathbf{0}) \subseteq U_n(\mathbf{0})$ for any positive integer $n > \max\{x,y\}+1$. We consider the possible cases.

1. Suppose that $a > 2n$. Then for any $b \in \mathbb{R}$ the equality

$$(x,y)(a,b) = \begin{cases} (x-y+a,b), & \text{if } y < a; \\ (x,b), & \text{if } y = a; \\ (x,y-a+b), & \text{if } y > a, \end{cases} \tag{1}$$

implies that $(x,y)(a,b) = (x-y+a,b)$. By the assumptions $n > \max\{x,y\}+1$ and $a > 2n$, we get that $x-y+a > -n+2n = n$, and hence $(x-y+a,b) \in U_n(\mathbf{0})$.

2. Suppose that $n \leqslant a \leqslant 2n$ and $b > 2n$. By (1) we have that $(x,y)(a,b) = (x-y+a,b)$. The assumption $n > \max\{x,y\}+1$ implies that $x-y+a > -n+n = 0$. Since $b > 2n$ we get that $(x-y+a,b) \in U_n(\mathbf{0})$.

3. Suppose that $0 \leqslant a < n$ and $b > 2n$. By (1) we have that

$$(x,y)(a,b) = (x-y+a,b) \in U_n(\mathbf{0})$$

in the case when $y < a$, and if $y \geqslant a$ then $y-a+b > 2n$, and hence $(x, y-a+b) \in U_n(\mathbf{0})$.

Similar arguments and the equality

$$(a,b)(x,y) = \begin{cases} (a-b+x,y), & \text{if } b < x; \\ (a,y), & \text{if } b = x; \\ (a,b-x+y), & \text{if } b > x, \end{cases}$$

imply that for any positive integer $n > \max\{x,y\}+1$ the inclusion $U_{2n}(\mathbf{0}) \cdot (x,y) \subseteq U_n(\mathbf{0})$ holds. The above inclusions imply that the semigroup operation on $(S_1^{\mathbf{0}}, \tau^1_{\mathsf{Ac}})$ is separate continuous.

Since $(U_n(\mathbf{0}))^{-1} = U_n(\mathbf{0})$ for any $n \in \mathbb{N}$ the inversion on $(S_1^{\mathbf{0}}, \tau^1_{\mathsf{Ac}})$ is continuous.

It is obvious that τ^1_{Ac} is a compact Hausdorff topology on $S_1^{\mathbf{0}}$. Moreover $(S_1^{\mathbf{0}}, \tau^1_{\mathsf{Ac}})$ is the one-point Alexandroff compactification of the locally compact space $\boldsymbol{B}^1_{[0,\infty)}$ such that the singleton set $\{\mathbf{0}\}$ which consists of the zero of $S_1^{\mathbf{0}}$ is its remainder. □

Theorem 1 and Proposition 2 imply the following theorem.

Theorem 2. *Let $S_1^{\mathbf{0}}$ be a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. Then either $\mathbf{0}$ is an isolated point of $S_1^{\mathbf{0}}$ or the topology of $S_1^{\mathbf{0}}$ coincides with τ^1_{Ac}.*

Since the bicyclic monoid does not embeds into any Hausdorff compact topological semigroup [5] and the semigroup contains many isomorphic copies of the bicyclic semigroup, Theorems 1 and 2 imply the following corollaries.

Corollary 1. *Let S_1^I be a Hausdorff locally compact topological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined compact ideal I. Then I is an open subset of S_1^I.*

Corollary 2. *Let $S_1^{\mathbf{0}}$ be a Hausdorff locally compact topological semigroup $\boldsymbol{B}^1_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. Then $\mathbf{0}$ is an isolated point of $S_1^{\mathbf{0}}$.*

3. A locally compact semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined compact ideal. Later in this section by S_2^I we denote a Hausdorff locally compact semitopological semigroup which is the semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined non-open compact ideal I.

The proof of Lemma 3 is similar to Lemma 2.

Lemma 3. *For any open neighbourhood $U(I)$ of the ideal I in S_1^I the set $U(I)\cap\boldsymbol{B}_{[0,\infty)}$ is unbounded.*

Lemma 4. *Let $U(I)$ be any open neighbourhood of the ideal I in S_2^I with the compact closure $\overline{U(I)}$. Then there exist finite subsets B and C of non-negative real numbers such that*

$$S_2^I\setminus U(I)\subseteq\bigsqcup_{\alpha\in B}L_\alpha^+\sqcup\bigsqcup_{\alpha\in C}L_\alpha^-.$$

Proof. Since $\overline{U(I)}\setminus U(I)$ is compact subset in S_2^I, $\overline{U(I)}\setminus U(I)$ is compact subset in $\boldsymbol{B}^2_{[0,\infty)}$. The equality $\boldsymbol{B}^2_{[0,\infty)}=\bigoplus_{\alpha\in[0,+\infty)}L_\alpha^+\oplus\bigoplus_{\alpha\in(0,+\infty)}L_\alpha^-$ implies the statement of the lemma. □

Lemma 5. *For any non-negative real number α the sets $L_\alpha^+\cup I$ and $L_\alpha^-\cup I$ are compact.*

Proof. First we show that there exists a non-negative real number α_0 such that the sets $L_{\alpha_0}^+\cup I$ and $L_{\alpha_0}^-\cup I$ are compact. We fix an arbitrary open neighbourhood $U(I)$ of the ideal I in S_2^I. By Lemma 4 $L_\alpha^+\cup L_\alpha^-\subseteq U(I)$ for almost all but finitely many $\alpha\in[0,+\infty)$. Without loss of generality we may assume that the closure $\overline{U(I)}$ of $U(I)$ is a compact subset of S_2^I. Fix $\alpha_0\in[0,+\infty)$ such that $L_{\alpha_0}^+\cup L_{\alpha_0}^-\subseteq U(I)$. Since $L_{\alpha_0}^+$ and $L_{\alpha_0}^-$are open subsets of S_2^I, we get that

$$L_\alpha^+\cup I=S_2^I\setminus\left(\bigcup_{\alpha_0\neq\alpha\geqslant0}L_\alpha^+\cup\bigcup_{\alpha>0}L_\alpha^-\right)\qquad\text{and}\qquad L_\alpha^-\cup I=S_2^I\setminus\left(\bigcup_{\alpha>0}L_\alpha^+\cup\bigcup_{\alpha_0\neq\alpha\geqslant0}L_\alpha^-\right)$$

are closed subsets of $\overline{U(I)}$, and hence they are compact.

We observe that

$$(x,x+\alpha_0)\cdot(\alpha_0,\alpha)=(x,x+\alpha)\qquad\text{and}\qquad(\alpha,\alpha_0)\cdot(x+\alpha_0,x)=(x+\alpha,x)$$

in $\boldsymbol{B}_{[0,\infty)}$ for any non-negative real numbers α, α_0 and x. This implies that $\rho_{(\alpha_0,\alpha)}(L_{\alpha_0}^+)=L_\alpha^+$ and $\lambda_{(\alpha,\alpha_0)}(L_{\alpha_0}^-)=L_\alpha^-$, where $\rho_{(\alpha_0,\alpha)}\colon S_2^I\to S_2^I$ and $\lambda_{(\alpha,\alpha_0)}\colon S_2^I\to S_2^I$ are right and left shifts on elements (α_0,α) and (α,α_0), respectively. Since S_2^I is a semitopological semigroup, the sets $\rho_{(\alpha_0,\alpha)}(L_{\alpha_0}^+\cup I)\cup I=L_\alpha^+\cup I$ and $\lambda_{(\alpha,\alpha_0)}(L_{\alpha_0}^-\cup I)\cup I=L_\alpha^-\cup I$ are compact. □

Lemma 6. *Let $U(I)$ be any open neighbourhood of the ideal I in S_2^I with compact closure $\overline{U(I)}$. Then for any non-negative real number α the sets $L_\alpha^+\setminus U(I)$ and $L_\alpha^-\setminus U(I)$ are compact.*

Proof. By Lemma 5 for any non-negative real number α the sets $L^+_\alpha \cup I$ and $L^-_\alpha \cup I$ are compact. Since $L^+_\alpha \setminus U(I)$ and $L^-_\alpha \setminus U(I)$ are closed subsets of $L^+_\alpha \cup I$ and $L^-_\alpha \cup I$, they are compact. □

Lemmas 3, 4, 5, and 6 imply the following theorem.

Theorem 3. *Let S_2^I be a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined compact ideal I. Then either I is an open subset of S_2^I or the space S_2^I is compact.*

Next we need some notions for the further construction. For the natural partial order $\preccurlyeq$ on the semigroup $\boldsymbol{B}_{[0,\infty)}$ and any $(a,b)\in\boldsymbol{B}_{[0,\infty)}$ we denote

$$\begin{aligned} {\uparrow}_{\preccurlyeq}(a,b) &= \left\{(x,y)\in\boldsymbol{B}_{[0,\infty)}\colon (a,b)\preccurlyeq(x,y)\right\};\\ {\downarrow}_{\preccurlyeq}(a,b) &= \left\{(x,y)\in\boldsymbol{B}_{[0,\infty)}\colon (x,y)\preccurlyeq(a,b)\right\};\\ {\downarrow}^{\circ}_{\preccurlyeq}(a,b) &= {\downarrow}_{\preccurlyeq}(a,b)\setminus\{(a,b)\}. \end{aligned}$$

The following statement describes the natural partial order $\preccurlyeq$ on the semigroup $\boldsymbol{B}_{[0,\infty)}$ and it follows from Lemma 1 of [25].

Lemma 7. *Let (a,b) and (c,d) be arbitrary elements of the semigroup $\boldsymbol{B}_{[0,\infty)}$. Then the following statements are equivalent:*

(i) $(a,b)\preccurlyeq(c,d)$*;*

(ii) $a\geqslant c$ *and* $a-b=c-d$*;*

(iii) $b\geqslant d$ *and* $a-b=c-d$*.*

Lemma 7 implies that for any non-negative real number α the set L^+_α coincides with all elements of $\boldsymbol{B}_{[0,\infty)}$ which are comparable with $(0,\alpha)$, and the set L^-_α coincides with all elements of $\boldsymbol{B}_{[0,\infty)}$ which are comparable with $(\alpha,0)$ with the respact to the natural partial order $\preccurlyeq$ on the semigroup $\boldsymbol{B}_{[0,\infty)}$. Hence we have that $L^+_\alpha={\downarrow}_{\preccurlyeq}(0,\alpha)$ and $L^-_\alpha={\downarrow}_{\preccurlyeq}(\alpha,0)$.

Simple calculations and routine verifications show the following proposition.

Proposition 3. *Let α and β be non-negative real numbers. Then the following statements hold:*

(i) $L^+_\alpha\cdot L^+_\beta=L^+_{\alpha+\beta}$*;*

(ii) $L^-_\alpha\cdot L^-_\beta=L^-_{\alpha+\beta}$*;*

(iii) $L^+_\alpha\cdot L^-_\beta=\begin{cases} L^+_{\alpha-\beta}, & \text{if } \alpha\geqslant\beta;\\ L^-_{\beta-\alpha}, & \text{if } \alpha\leqslant\beta;\end{cases}$

(iv) $L^-_\beta\cdot L^+_\alpha={\downarrow}_{\preccurlyeq}(\beta,\alpha)\subseteq\begin{cases} L^+_{\alpha-\beta}, & \text{if } \alpha\geqslant\beta;\\ L^-_{\beta-\alpha}, & \text{if } \alpha\leqslant\beta.\end{cases}$

Lemma 8. *For arbitrary $(a_0,b_0),(a_1,b_1)\in\boldsymbol{B}_{[0,\infty)}$ there exists $(c,d)\in\boldsymbol{B}_{[0,\infty)}$ such that*

$$(a_0,b_0)\cdot(c,d)\preccurlyeq(a_1,b_1)\;\;[(c,d)\cdot(a_0,b_0)\preccurlyeq(a_1,b_1)].$$

Moreover, $(a_0,b_0)\cdot(x,y)\preccurlyeq(a_1,b_1)$ $[(x,y)\cdot(a_0,b_0)\preccurlyeq(a_1,b_1)]$ for any $(x,y)\preccurlyeq(c,d)$ in $\boldsymbol{B}_{[0,\infty)}$.

Proof. We assume that $c \geqslant a_1+a_0+b_0$ and $d = a_0+c-b_0-a_1+b_1$. The semigroup operation of $\boldsymbol{B}_{[0,\infty)}$ implies that

$$(a_0,b_0)\cdot(c,d) = (a_0,b_0)\cdot(c,a_0+c-b_0-a_1+b_1) = (a_0-b_0+c,a_0+c-b_0-a_1+b_1).$$

Then $a_0-b_0+c \geqslant a_1$ and

$$(a_0-b_0+c)-(a_0+c-b_0-a_1+b_1) = a_0-b_0+c-a_0-c+b_0+a_1-b_1 = a_1-b_1,$$

and hence by Lemma 7 we get that $(a_0,b_0)\cdot(c,d) \preccurlyeq (a_1,b_1)$. The last statement of the lemma follows from Proposition 1.4.7 of [32]. The proof of the dual statement is similar. □

Lemma 8 implies the following proposition.

Proposition 4. *If* $(a_0,b_0)\cdot{\downarrow_{\preccurlyeq}}(c_0,d_0) \subseteq {\downarrow_{\preccurlyeq}}(a_1,b_1)$ $[{\downarrow_{\preccurlyeq}}(c_0,d_0)\cdot(a_0,b_0) \subseteq {\downarrow_{\preccurlyeq}}(a_1,b_1)]$ *for some* (a_0,b_0), $(a_1,b_1),(c_0,d_0) \in \boldsymbol{B}_{[0,\infty)}$, *then*

$$(a_0,b_0)\cdot{\downarrow^{\circ}_{\preccurlyeq}}(c_0,d_0) \subseteq {\downarrow^{\circ}_{\preccurlyeq}}(a_1,b_1) \ \left[{\downarrow^{\circ}_{\preccurlyeq}}(c_0,d_0)\cdot(a_0,b_0) \subseteq {\downarrow^{\circ}_{\preccurlyeq}}(a_1,b_1)\right].$$

Example 2. Let $S_2^{\mathbf{0}}$ be the semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. We extend the topology of $\boldsymbol{B}^2_{[0,\infty)}$ up to a compact topology τ^2_{Ac} on the semigroup $S_2^{\mathbf{0}}$ in the following way. For any $(a_1,b_1),\ldots,(a_k,b_k) \in \boldsymbol{B}^1_{[0,\infty)}$ we put

$$U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)] = S_2^{\mathbf{0}} \setminus ({\uparrow_{\preccurlyeq}}(a_1,b_1)\cup\cdots\cup{\uparrow_{\preccurlyeq}}(a_k,b_k))$$

and define

$$\mathscr{B}^2_{\mathsf{Ac}}(\mathbf{0}) = \left\{U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)] \colon (a_1,b_1),\ldots,(a_k,b_k) \in \boldsymbol{B}_{[0,\infty)}, k\in\mathbb{N}\right\}$$

is the system of open neighbourhoods of zero in τ^2_{Ac}.

Proposition 5. $(S_2^{\mathbf{0}},\tau^2_{\mathsf{Ac}})$ *is a compact Hausdorff semitopological semigroup with continuous inversion.*

Proof. It is obvious that τ^2_{Ac} is a compact Hausdorff topology on $S_2^{\mathbf{0}}$. Moreover $(S_2^{\mathbf{0}},\tau^2_{\mathsf{Ac}})$ is the one-point Alexandroff compactification of the locally compact space $\boldsymbol{B}^2_{[0,\infty)}$ such that the singleton set $\{\mathbf{0}\}$ which consists of the zero of $S_2^{\mathbf{0}}$ is its remainder.

By [4], $\boldsymbol{B}^2_{[0,\infty)}$ is a topological inverse semigroup, and hence it sufficient to show that the the semigroup operation on $(S_2^{\mathbf{0}},\tau^2_{\mathsf{Ac}})$ is separately continuous at zero.

Fix an arbitrary $U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)] \in \mathscr{B}^2_{\mathsf{Ac}}(\mathbf{0})$.

It is obvious that

$$\mathbf{0}\cdot U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)] = U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)]\cdot\mathbf{0} = \{\mathbf{0}\} \subseteq U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)].$$

By Lemma 8 for an arbitrary $(a,b)\in\boldsymbol{B}_{[0,\infty)}$ there exist

$$(c_1,d_1),\ldots,(c_k,d_k),(x_1,y_1),\ldots,(x_k,y_k) \in \boldsymbol{B}_{[0,\infty)}$$

such that $(a,b)\cdot(c_i,d_i) \preccurlyeq (a_i,b_i)$ and $(x_i,y_i)\cdot(a,b) \preccurlyeq (a_i,b_i)$ for all $i=1,\ldots,k$. By Proposition 4 we have that $(a,b)\cdot{\downarrow^{\circ}_{\preccurlyeq}}(c_i,d_i) \subseteq {\downarrow^{\circ}_{\preccurlyeq}}(a_i,b_i)$ and ${\downarrow^{\circ}_{\preccurlyeq}}(x_i,y_i)\cdot(a,b) \subseteq {\downarrow^{\circ}_{\preccurlyeq}}(a_i,b_i)$ for all $i=1,\ldots,k$. This and Proposition 3 imply that

$$(a,b)\cdot U_{\mathbf{0}}[(c_1,d_1),\ldots,(c_k,d_k)] \subseteq U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)]$$

and

$$U_{\mathbf{0}}[(x_1,y_1),\ldots,(x_k,y_k)]\cdot(a,b) \subseteq U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)],$$

and hence the semigroup operation on $(S_2^{\mathbf{0}},\tau^2_{\mathsf{Ac}})$ is separately continuous.

Since $(U_{\mathbf{0}}[(a_1,b_1),\ldots,(a_k,b_k)])^{-1} = U_{\mathbf{0}}[(b_1,a_1),\ldots,(b_k,a_k)$ for any $(a_1,b_1),\ldots,(a_k,b_k) \in \boldsymbol{B}_{[0,\infty)}$ the inversion on $(S_2^{\mathbf{0}},\tau^2_{\mathsf{Ac}})$ is continuous. □

Theorem 3 and Proposition 5 imply the following theorem.

Theorem 4. *Let $S_2^{\mathbf{0}}$ be a Hausdorff locally compact semitopological semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. Then either $\mathbf{0}$ is an isolated point of $S_2^{\mathbf{0}}$ or the topology of $S_2^{\mathbf{0}}$ coincides with τ^2_{Ac}.*

Since the bicyclic monoid does not embeds into any Hausdorff compact topological semigroup [5] and the semigroup $\boldsymbol{B}_{[0,\infty)}$ contains many isomorphic copies of the bicyclic semigroup, Theorems 3 and 4 imply the following corollaries.

Corollary 3. *Let S_2^I be a Hausdorff locally compact topological semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined compact ideal I. Then I is an open subset of S_2^I.*

Corollary 4. *Let $S_2^{\mathbf{0}}$ be a Hausdorff locally compact topological semigroup $\boldsymbol{B}^2_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$. Then $\mathbf{0}$ is an isolated point of $S_2^{\mathbf{0}}$.*

4. A locally compact semigroup $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ with an adjoined compact ideal. Later in this section by $S_{\mathfrak{d}}^{\mathbf{0}}$ we denote a Hausdorff locally compact semitopological semigroup which is the semigroup $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ with an adjoined zero $\mathbf{0}$.

Lemma 9. *Let $U(\mathbf{0})$ be an open neighbourhood of zero with the compact closure $\overline{U(\mathbf{0})}$ in $S_{\mathfrak{d}}^{\mathbf{0}}$. Then for any $(a,b)\in\boldsymbol{B}_{[0,\infty)}$ the set ${\uparrow_{\preccurlyeq}}(a,b)\cap U(\mathbf{0})$ is finite.*

Proof. Suppose to the contrary that there exists an open neighbourhood of zero with the compact closure $\overline{U(\mathbf{0})}$ in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that the set ${\uparrow_{\preccurlyeq}}(a,b)\cap U(\mathbf{0})$ is infinite. By Remark 1 we have that

$$
{\uparrow_{\preccurlyeq}}(a,b)=\left\{(x,y)\in\boldsymbol{B}_{[0,\infty)}\colon (a,a)(x,y)=(a,b)\right\},
$$

and hence the Hausdorffness of $S_{\mathfrak{d}}^{\mathbf{0}}$ and separate continuity of the semigroup operation on $S_{\mathfrak{d}}^{\mathbf{0}}$ imply that ${\uparrow_{\preccurlyeq}}(a,b)$ is a closed subset of $S_{\mathfrak{d}}^{\mathbf{0}}$. Hence, ${\uparrow_{\preccurlyeq}}(a,b)\cap U(\mathbf{0})$ is a compact infinite discrete space, a contradiction. The obtained contradiction implies the statement of the lemma. □

We observe that since $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ is a discrete subspace of $S_{\mathfrak{d}}^{\mathbf{0}}$, any open neighbourhood of zero $U(\mathbf{0})$ is closed. Lemma 9 implies the following corollary.

Corollary 5. *For any open compact neighbourhood $U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ and any real number $\alpha\in[0,\infty)$ the set $L^+_\alpha\cap U(\mathbf{0})$ ($L^-_\alpha\cap U(\mathbf{0})$) either contains a maximal elements (with the respect to the natural partial order on $\boldsymbol{B}_{[0,\infty)}$) or is empty.*

Lemma 10. *If $S_{\mathfrak{d}}^{\mathbf{0}}$ admits the structure of a Hausdorff locally compact semitopological semigroup with a nonisolated zero, then there exists no open compact neighbourhood $U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that the sets $L^+_\alpha\cap U(\mathbf{0})$ and $L^+_\alpha\cap U(\mathbf{0})$ are finite for all $\alpha\in[0,\infty)$.*

Proof. Suppose to the contrary that there exists an open compact neighbourhood $U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that the sets $L^+_\alpha\cap U(\mathbf{0})$ and $L^+_\alpha\cap U(\mathbf{0})$ are finite for all $\alpha\in[0,\infty)$. Separate continuity of the semigroup operation in $S_{\mathfrak{d}}^{\mathbf{0}}$ implies that there exists an open compact neighbourhood $V(\mathbf{0})\subseteq U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that $(1,0)\cdot V(\mathbf{0})\cdot(0,1)\subseteq U(\mathbf{0})$. This inclusion implies that $U(\mathbf{0})\setminus V(\mathbf{0})$ is an infinite subsets of isolated points, which contradicts the compactness of $U(\mathbf{0})$. The obtained contradiction implies the statement of the lemma. □

Lemma 11. *If $S_{\mathfrak{d}}^{\mathbf{0}}$ admits the structure of a Hausdorff locally compact semitopological semigroup with a nonisolated zero, then for any open compact neighbourhood $U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ the sets $L_{\alpha}^{+} \cap U(\mathbf{0})$ and $L_{\alpha}^{-} \cap U(\mathbf{0})$ are infinite for all $\alpha \in [0,\infty)$.*

Proof. By Lemma 10 there exists $\alpha_0 \in [0,\infty)$ such that at least one of the sets $L_{\alpha_0}^{+} \cap U(\mathbf{0})$ or $L_{\alpha_0}^{-} \cap U(\mathbf{0})$ is infinite. Without loss of generality we may assume that the set $L_{\alpha_0}^{+} \cap U(\mathbf{0})$ is infinite. Separate continuity of the semigroup operation of $S_{\mathfrak{d}}^{\mathbf{0}}$ implies that there exists an open compact neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that $V(\mathbf{0}) \cdot (\alpha_0, 0) \subseteq U(\mathbf{0})$. Since $\boldsymbol{B}_{[0,\infty)}^{\mathfrak{d}}$ is a discrete subspace of $S_{\mathfrak{d}}^{\mathbf{0}}$ and $U(\mathbf{0})$ is compact, the set $L_0^{+} \cap U(\mathbf{0})$ is infinite. By the similar way we get that for any $\beta_0 \in (0,\infty)$ there exists an open compact neighbourhood $W(\mathbf{0}) \subseteq U(\mathbf{0})$ such that $(\beta_0, 0) \cdot W(\mathbf{0}) \subseteq U(\mathbf{0})$ and $W(\mathbf{0}) \cdot (0, \beta_0) \subseteq U(\mathbf{0})$. Since $W(\mathbf{0})$ and $U(\mathbf{0})$ are compact, $L_0^{+} \cap W(\mathbf{0})$ is an infinite set, and hence the sets $L_{\beta_0}^{+} \cap U(\mathbf{0})$ and $L_{\beta_0}^{-} \cap U(\mathbf{0})$ are infinite. □

Lemma 12. *If $S_{\mathfrak{d}}^{\mathbf{0}}$ admits the structure of a Hausdorff locally compact semitopological semigroup with a nonisolated zero, then there exists an open compact neighbourhood $U(\mathbf{0})$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ such that $L_0^{+} \cap U(\mathbf{0}) = \varnothing$.*

Proof. By Lemma 11 for any compact open neighbourhood $U(0)$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ the set $L_0^{+} \cap U(\mathbf{0})$ is infinite. For any positive integer n_0 by Lemma 9 the set $\uparrow_{\preccurlyeq}(n_0, n_0) \cap U(\mathbf{0})$ is finite. This implies that the set $L_0^{+} \cap U(\mathbf{0})$ is countable. Let $L_0^{+} \cap U(\mathbf{0}) = \{(a_i, a_i) \colon a_i \in \boldsymbol{B}_{[0,\infty)}^{\mathfrak{d}}, i \in \omega\}$. Put $M = \{a_j - a_i \colon i, j \in \omega, i < j\}$. The set M is countable as a countable union of a family of countable sets. Then there exists $\alpha \in (0,\infty) \setminus M$. Then for any open compact neighbourhood $V(\mathbf{0}) \subseteq U(0)$ of zero in $S_{\mathfrak{d}}^{\mathbf{0}}$ the following inclusion $(\alpha, 0) \cdot V(\mathbf{0}) \cdot (0, \alpha) \subseteq U(0)$ does not hold, because $(\alpha, 0) \cdot L_0^{+} \cdot (0, \alpha) \subseteq L_0^{+}$. This contradicts the separate continuity of the semigroup operation of $S_{\mathfrak{d}}^{\mathbf{0}}$. The obtained contradiction implies the statement of the lemma. □

If we assume that $S_{\mathfrak{d}}^{\mathbf{0}}$ admits the structure of a Hausdorff locally compact semitopological semigroup with a nonisolated zero, then we get Lemma 12 and Lemma 11. But the statement of Lemma 12 contradicts to Lemma 11. Hence the following theorem holds.

Theorem 5. *Let $S_{\mathfrak{d}}^{\mathbf{0}}$ be a Hausdorff locally compact semitopological semigroup which is the semigroup $\boldsymbol{B}_{[0,\infty)}^{\mathfrak{d}}$ with an adjoined zero $\mathbf{0}$. Then $\mathbf{0}$ is an isolated point of $S_{\mathfrak{d}}^{\mathbf{0}}$.*

Later we need the following trivial lemma, which follows from separate continuity of the semigroup operation in semitopological semigroups.

Lemma 13. *Let S be a Hausdorff semitopological semigroup and I be a compact ideal in S. Then the Rees-quotient semigroup S/I with the quotient topology is a Hausdorff semitopological semigroup.*

Theorem 6. *Let $S_{\mathfrak{d}}^{I} = \boldsymbol{B}_{[0,\infty)}^{\mathfrak{d}} \sqcup I$ be a Hausdorff locally compact semitopological semigroup which is the semigroup $\boldsymbol{B}_{[0,\infty)}^{\mathfrak{d}}$ with an adjoined compact ideal I. Then I is an open subset of $S_{\mathfrak{d}}^{I}$.*

Proof. Suppose to the contrary that I is not open $S_{\mathfrak{d}}^{I}$. By Lemma 13 the Rees-quotient semigroup $S_{\mathfrak{d}}^{I}/I$ with the quotient topology τ_q is a semitopological semigroup. Let $\pi \colon S_{\mathfrak{d}}^{I} \to S_{\mathfrak{d}}^{I}/I$ be the natural homomorphism which is a quotient map. It is obvious that the Rees-quotient semigroup $S_{\mathfrak{d}}^{I}/I$ is isomorphic to the semigroup $S_{\mathfrak{d}}^{\mathbf{0}}$, and hence without loss of generality we may assume that $\pi(S_{\mathfrak{d}}^{I}) = S_{\mathfrak{d}}^{\mathbf{0}}$ and the image $\pi(I)$ is zero of $S_{\mathfrak{d}}^{\mathbf{0}}$.

By Lemma 3.16 of [24] there exists an open neighbourhood $U(I)$ of the ideal I with the compact closure $\overline{U(I)}$. Since every point of $\boldsymbol{B}^{\mathfrak{d}}_{[0,\infty)}$ is isolated in $S^I_{\mathfrak{d}}$ we have that $U(I) = \overline{U(I)}$ and its image $\pi(U(I))$ is a compact-and-open neighbourhood of zero in $S^{\mathbf{0}}_{\mathfrak{d}}$. Hence $S^{\mathbf{0}}_{\mathfrak{d}}$ is Hausdorff locally compact space. By Theorem 5, $\mathbf{0}$ is an isolated point of $S^{\mathbf{0}}_{\mathfrak{d}}$. Since $\pi\colon S^I_{\mathfrak{d}} \to S^I_{\mathfrak{d}}/I$ is a quotient map, I is an open subset of $S^I_{\mathfrak{d}}$. $\square$

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