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ON CERTAIN CLASSES OF DIRICHLET SERIES WITH REAL COEFFICIENTS ABSOLUTELY CONVERGENT IN A HALF-PLANE

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For $h>0$, $\alpha\in[0,h)$ and $\mu\in\mathbb{R}$ denote by $SD_h(\mu,\alpha)$ a class of absolutely convergent in the half-plane $\Pi_0=\{s:\ \operatorname{Re}s<0\}$ Dirichlet series $F(s)=e^{sh}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}$ such that

$$\operatorname{Re}\left\{\frac{(\mu-1)F'(s)-\mu F''(s)/h}{(\mu-1)F(s)-\mu F'(s)/h}\right\}>\alpha \text{ for all } s\in\Pi_0,$$

and let $\Sigma D_h(\mu,\alpha)$ be a class of absolutely convergent in half-plane Π_0 Dirichlet series $F(s)=e^{-sh}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}$ such that

$$\operatorname{Re}\left\{\frac{(\mu-1)F'(s)+\mu F''(s)/h}{(\mu-1)F(s)+\mu F'(s)/h}\right\}<-\alpha \text{ for all } s\in\Pi_0.$$

Then $SD_h(0,\alpha)$ consists of pseudostarlike functions of order α and $SD_h(1,\alpha)$ consists of pseudoconvex functions of order α.

For functions from the classes $SD_h(\mu,\alpha)$ and $\Sigma D_h(\mu,\alpha)$, estimates for the coefficients and growth estimates are obtained. In particular, it is proved the following statements: 1) In order that function $F(s)=e^{sh}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}$ belongs to $SD_h(\mu,\alpha)$, it is sufficient, and in the case when $f_k(\mu\lambda_k/h-\mu+1)\le 0$ for all $k\ge 1$, it is necessary that

$$\sum_{k=1}^{\infty}\left|f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\right|(\lambda_k-\alpha)\le h-\alpha,$$

where $h>0,\alpha\in[0,h)$ (Theorem 1).
2) In order that function $F(s)=e^{-sh}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}$ belongs to $\Sigma D_h(\mu,\alpha)$, it is sufficient, and in the case when $f_k(\mu\lambda_k/h+\mu-1)\le 0$ for all $k\ge 1$, it is necessary that

$$\sum_{k=1}^{\infty}\left|f_k\left(\frac{\mu\lambda_k}{h}+\mu-1\right)\right|(\lambda_k+\alpha)\le h-\alpha,$$

where $h>0,\alpha\in[0,h)$ (Theorem 2). Neighborhoods of such functions are investigated. Ordinary Hadamard compositions and Hadamard compositions of the genus m were also studied.

1. Introduction and definitions. Let $h>0$, (λ_k) be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1>h$) and SD_h be the class of Dirichlet series

$$F(s)=e^{hs}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}\quad (s=\sigma+it) \tag{1}$$

absolutely convergent in the half-plane $\Pi_0=\{s\colon \operatorname{Re}\ s<0\}$ and either $(\forall k\ge 1)\colon f_k\le 0$ or $(\forall k\ge 1)\colon f_k\ge 0$.

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It is known [1], [2, p.135] that each function $F \in SD_h$ is non-univalent in Π_0, but there exist functions $F \in SD_h$ conformal in Π_0, that is conformal at every point $z_0 \in \Pi_0$; if $\sum_{k=2}^{\infty} \lambda_k |f_k| \le h$ then function F is conformal in Π_0. A conformal function F in Π_0 is said to be *pseudostarlike of order* $\alpha \in [0, h)$ if

$$\operatorname{Re}\{F'(s)/F(s)\} > \alpha \quad (s \in \Pi_0)$$

and is said to be *pseudoconvex of order* $\alpha \in [0, h)$ if

$$\operatorname{Re}\{F''(s)/F'(s)\} > \alpha \quad (s \in \Pi_0).$$

It is known [3] that if $\sum_{k=1}^{\infty} (\lambda_k - \alpha)|f_k| \le h - \alpha$ then F is pseudostarlike of the order α, and if $\sum_{k=1}^{\infty} \lambda_k(\lambda_k - \alpha)|f_k| \le h(h - \alpha)$ then F is pseudoconvex of the order α. To combine the concepts of pseudostarlikeness and pseudoconvexity, we introduce the class $SD_h(\mu, \alpha)$ of Dirichlet series from the class SD_h such that

$$\operatorname{Re}\left\{\frac{(\mu - 1)F'(s) - \mu F''(s)/h}{(\mu - 1)F(s) - \mu F'(s)/h}\right\} > \alpha \quad (s \in \Pi_0). \tag{2}$$

Remark, the class $SD_h(0, \alpha)$ coincides with the class of pseudostarlike functions of order α and $SD_h(1, \alpha)$ coincides with the class of pseudoconvex functions of order α.

In addition to the class SD_h we consider the class ΣD_h of Dirichlet series

$$F(s) = e^{-hs} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \quad (s = \sigma + it) \tag{3}$$

absolutely convergent in Π_0 and either $(\forall k \ge 1)\colon f_k \le 0$ or $(\forall k \ge 1)\colon f_k \ge 0$.

By $\Sigma D_h(\mu, \alpha)$ we denote a class of functions $F \in \Sigma D_h$ such that

$$\operatorname{Re}\left\{\frac{(\mu - 1)F'(s) + \mu F''(s)/h}{(\mu - 1)F(s) + \mu F'(s)/h}\right\} < -\alpha \quad (s \in \Pi_0). \tag{4}$$

Hence for $\mu = 0$ it follows that $\operatorname{Re}\{F'(s)/F(s)\} < -\alpha$ $(s \in \Pi_0)$, i.e. F is [3] Σ-pseudostarlike function of order α. If $\mu = 1$ then (4) implies $\operatorname{Re}\{F''(s)/F'(s)\} < -\alpha$ $(s \in \Pi_0)$, i.e. F is [3] Σ-pseudoconvex function of order α.

It is known [3] that if $\sum_{k=1}^{\infty} (\lambda_k + \alpha)|f_k| \le h - \alpha$ then F is Σ-pseudostarlike of order α, and if $\sum_{k=1}^{\infty} \lambda_k(\lambda_k + \alpha)|f_k| \le h(h - \alpha)$ then F is Σ-pseudoconvex of order α.

By the way, let us draw reader's attention to the paper [13], in which pseudostarlike and pseudoconvexe on the direction of multiple Dirichlet series are considered.

In the present article, we will study the properties of functions from the classes $SD_h(\mu, \alpha)$ and $\Sigma D_h(\mu, \alpha)$.

2. Coefficient inequalities in the class $SD_h(\mu, \alpha)$**.** The following theorem is true.

Theorem 1. *Let $h > 0$ and $\alpha \in [0, h)$. In order that function* (1) *belongs to $SD_h(\mu, \alpha)$, it is sufficient, and in the case when $f_k(\mu\lambda_k/h - \mu + 1) \le 0$ for all $k \ge 1$, it is necessary that*

$$\sum_{k=1}^{\infty} \left| f_k\Big(\frac{\mu\lambda_k}{h} - \mu + 1\Big)\right| (\lambda_k - \alpha) \le h - \alpha. \tag{5}$$

Proof. Since the inequality $|w - h| < |w - (2\alpha - h)|$ holds if and only if $\operatorname{Re} w > \alpha$ for $\alpha < h$, condition (2) holds if and only if

$$\left|\frac{(\mu - 1)F'(s) - \mu F''(s)/h}{(\mu - 1)F(s) - \mu F'(s)/h} - h\right| < \left|\frac{(\mu - 1)F'(s) - \mu F''(s)/h}{(\mu - 1)F(s) - \mu F'(s)/h} - (2\alpha - h)\right| \quad (s \in \Pi_0). \tag{6}$$

In view of (1)

$$(\mu-1)F'(s)-\frac{\mu}{h}F''(s)-h\left((\mu-1)F(s)-\frac{\mu}{h}F'(s)\right)=$$
$$=-\sum_{k=1}^{\infty}(\lambda_k-h)f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\exp\{s\lambda_k\},$$
$$(\mu-1)F'(s)-\frac{\mu}{h}F''(s)-(2\alpha-h)\left((\mu-1)F(s)-\frac{\mu}{h}F'(s)\right)=$$
$$=2(\alpha-h)e^{hs}-\sum_{k=1}^{\infty}(\lambda_k-2\alpha+h)f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\exp\{s\lambda_k\}\quad (s\in\Pi_0).$$

Therefore, (6) holds if and only if

$$\Big|-\sum_{k=1}^{\infty}(\lambda_k-h)f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\exp\{s\lambda_k\}\Big|-$$
$$-\Big|2(\alpha-h)e^{hs}-\sum_{k=1}^{\infty}(\lambda_k-2\alpha+h)f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\exp\{s\lambda_k\}\Big|<0\quad (s\in\Pi_0), \tag{7}$$

i.e.

$$|e^{hs}|\Big|\sum_{k=1}^{\infty}(\lambda_k-h)f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\exp\{s(\lambda_k-h)\}\Big|-$$
$$-|e^{hs}|\Big|2(\alpha-h)-\sum_{k=1}^{\infty}(\lambda_k-2\alpha+h)f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\exp\{s(\lambda_k-h)\}\Big|<0.$$

Since $-|a+b|\le -a+|b|$ for $a>0$, the last inequality holds if

$$\Big|\sum_{k=1}^{\infty}(\lambda_k-h)f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\exp\{s(\lambda_k-h)\}\Big|+$$
$$+\Big|\sum_{k=1}^{\infty}(\lambda_k-2\alpha+h)f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\exp\{s(\lambda_k-h)\}\Big|-2(h-\alpha)<0$$

i.e. if for all $\sigma<0$

$$\sum_{k=1}^{\infty}(\lambda_k-h)\Big|f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\Big|\exp\{\sigma(\lambda_k-h)\}+$$
$$+\sum_{k=1}^{\infty}(\lambda_k-2\alpha+h)\left|f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\right|\exp\{\sigma(\lambda_k-h)\}<2(h-\alpha)$$

Therefore, (7) holds if

$$\sum_{k=1}^{\infty}(2\lambda_k-2\alpha)\big|f_k\big(\frac{\mu\lambda_k}{h}-\mu+1\big)\big|\le 2(h-\alpha),$$

that is, if (5) holds. The sufficiency of condition (5) is proved.

Let now $f_k(\mu\lambda_k/h-\mu+1)\le 0$ for all $k\ge 1$ and (2) holds. Then (7) holds, i.e.

$$\frac{\Big|\sum\limits_{k=1}^{\infty}(\lambda_k-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|\exp\{s\lambda_k\}\Big|}{\Big|2(h-\alpha)e^{hs}-\sum\limits_{k=1}^{\infty}(\lambda_k-2\alpha+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|\exp\{s\lambda_k\}\Big|}<1,$$

whence

$$\operatorname{Re}\left\{\frac{\sum\limits_{k=1}^{\infty}(\lambda_k-h)\left|f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\right|\exp\{s\lambda_k\}}{2(h-\alpha)e^{hs}-\sum\limits_{k=1}^{\infty}(\lambda_k-2\alpha+h)\left|f_k\left(\frac{\mu\lambda_k}{h}-\mu+1\right)\right|\exp\{s\lambda_k\}}\right\}<1$$

for all $s\in\Pi_0$. In particular, we get for all $\sigma<0$

$$\frac{\sum\limits_{k=1}^{\infty}(\lambda_k-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|\exp\{\sigma\lambda_k\}}{2(h-\alpha)e^{h\sigma}-\sum\limits_{k=1}^{\infty}(\lambda_k-2\alpha+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|\exp\{\sigma\lambda_k\}}<1.$$

Letting $\sigma\uparrow 0$ we arrive at the inequality

$$\frac{\sum\limits_{k=1}^{\infty}(\lambda_k-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|}{2(h-\alpha)-\sum\limits_{k=1}^{\infty}(\lambda_k-2\alpha+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\Big|}\le 1,$$

which is equivalent to the inequality (5). □

Remark 1. If $\mu\ge 0$ then $\mu\lambda_k/h-\mu+1\ge 0$, and if $\mu\le -h/(\lambda_1-h)$ then $\mu\lambda_k/h-\mu+1\le 0$ for all $k\ge 1$. Therefore, the condition $f_k(\mu\lambda_k/h-\mu+1)\le 0$ for all $k\ge 1$ holds if either $f_k\le 0$ for all $k\ge 1$ and $\mu\ge 0$ or $f_k\ge 0$ for all $k\ge 1$ and $\mu\le -h/(\lambda_1-h)$.

3. Coefficient inequalities in the class $\Sigma D_h(\mu,\alpha)$. The following analog of Theorem 1 is true.

Theorem 2. *Let $h>0$ and $\alpha\in[0,h)$. In order that function (3) belongs to $\Sigma D_h(\mu,\alpha)$, it is sufficient, and in the case when $f_k(\mu\lambda_k/h+\mu-1)\le 0$ for all $k\ge 1$, it is necessary that*

$$\sum_{k=1}^{\infty}\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|(\lambda_k+\alpha)\le h-\alpha. \tag{8}$$

Proof. Since the inequality $|w+h|<|w+2\alpha-h|$ holds if and only if $\operatorname{Re} w<-\alpha$ for $\alpha<h$, condition (4) holds if and only if

$$\left|\frac{(\mu-1)F'(s)+\mu F''(s)/h}{(\mu-1)F(s)+\mu F'(s)/h}+h\right|<\left|\frac{(\mu-1)F'(s)+\mu F''(s)/h}{(\mu-1)F(s)+\mu F'(s)/h}+2\alpha-h\right|\quad (s\in\Pi_0). \tag{9}$$

In view of (3)

$$(\mu-1)F'(s)+\frac{\mu}{h}F''(s)+h\left((\mu-1)F(s)+\frac{\mu}{h}F'(s)\right)=$$
$$=\sum_{k=1}^{\infty}(\lambda_k+h)f_k\left(\frac{\mu\lambda_k}{h}+\mu-1\right)\exp\{s\lambda_k\},$$
$$(\mu-1)F'(s)+\frac{\mu}{h}F''(s)+(2\alpha-h)\left((\mu-1)F(s)+\frac{\mu}{h}F'(s)\right)=$$
$$=2(h-\alpha)e^{-hs}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\left(\frac{\mu\lambda_k}{h}+\mu-1\right)\exp\{s\lambda_k\}\quad (s\in\Pi_0).$$

Therefore, (9) holds if and only if

$$\Big|\sum_{k=1}^{\infty}(\lambda_k+h)f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\exp\{s\lambda_k\}\Big|-$$

$$-\Big|2(h-\alpha)e^{-hs}-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\exp\{s\lambda_k\}\Big|<0\quad(s\in\Pi_0),$$

As in the proof of Theorem 1, it is possible to show that () holds if

$$\sum_{k=1}^{\infty}(\lambda_k+\alpha)\Big|f_k\left(\frac{\mu\lambda_k}{h}+\mu-1\right)\Big|\exp\{\sigma(\lambda_k+h)\}<h-\alpha\quad(\sigma<0).$$

The last inequality follows from (8). The sufficiency of (8) is proved.

On the other hand, if $f_k(\mu\lambda_k/h+\mu-1)\le 0$ for all $k\ge 1$ then from () as in the proof of Theorem 1 it follows that

$$\operatorname{Re}\left\{\frac{\sum\limits_{k=1}^{\infty}(\lambda_k+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|\exp\{s\lambda_k\}}{2(h-\alpha)e^{-hs}-\sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|\exp\{s\lambda_k\}}\right\}<1$$

for all $s\in\Pi_0$. In particular, we get for all $\sigma<0$

$$\frac{\sum\limits_{k=1}^{\infty}(\lambda_k+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|\exp\{\sigma\lambda_k\}}{2(h-\alpha)e^{-h\sigma}-\sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|\exp\{\sigma\lambda_k\}}<1.$$

Letting $\sigma\uparrow 0$ we arrive at the inequality

$$\sum_{k=1}^{\infty}(\lambda_k+h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|\le 2(h-\alpha)-\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)\Big|f_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\Big|,$$

which is equivalent to inequality (8). □

Remark 2. If $\mu\le 0$ then $\mu\lambda_k/h+\mu-1\le 0$, and if $\mu\ge h/(\lambda_1+h)$ then $\mu\lambda_k/h+\mu-1\ge 0$ for all $k\ge 1$. Therefore, the condition $f_k(\mu\lambda_k/h+\mu-1)\le 0$ for all $k\ge 1$ holds if either $f_k\ge 0$ for all $k\ge 1$ and $\mu\le 0$ or $f_k\le 0$ for all $k\ge 1$ and $\mu\ge h/(\lambda_1+h)$.

Remark 3. By $M_p(\mu,\alpha)$ denote the class of functions f of the form $f(z)=\frac{1}{z}+\sum\limits_{k=p}^{\infty}f_kz^k$, $f_k\in\mathbb{R}$, $p\in\mathbb{N}$, such that

$$\operatorname{Re}\left\{\frac{(1-2\mu)zf'(z)-\mu z^2f''(z)}{(\mu-1)f(z)+\mu zf'(z)}\right\}>\alpha\in[0,1).\tag{10}$$

In [4] it is proved that if $f_k\le 0$ and $\mu\ge 1/(p+1)$ then $f\in M_p(\mu,\alpha)$ if and only if

$$\sum_{k=p}^{\infty}(k+\alpha)(k\mu+\mu-1)|f_k|\le 1-\alpha.$$

Let us make the substitution $z=e^s$. Then $f(e^s)=F(s)$, where F is represented by series (3) with $h=1$ and $\lambda_1=p$, and condition (10) is equivalent to condition (4). Therefore, Theorem 2 implies the following statement that refines the result cited above from [4].

Corollary 1. *Let $\alpha\in[0,1)$. In order that the function $f(z)=\frac{1}{z}+\sum\limits_{k=p}^{\infty}f_kz^k$, $f_k\in\mathbb{R}$, $p\in\mathbb{N}$, belongs to the class $M_p(\mu,\alpha)$ it is sufficient and in the case when either $f_k\le 0$ $(\forall k\ge 1)$ and $\mu\ge 1/(p+1)$ or $f_k\ge 0$ $(\forall k\ge 1)$ and $\mu\le 0$ it is necessary that*

$$\sum_{k=p}^{\infty}|f_k(k\mu+\mu-1)|(k+\alpha)\le 1-\alpha.$$

4. Estimates of Dirichlet series. Let us start with functions from the class $SD_h(\mu,\alpha)$. If $f_k\le 0$ and $\mu\ge 0$ then $\mu>-h/(\lambda_1-h)$, $\mu\lambda_k/h-\mu+1\ge\mu\lambda_1/h-\mu+1>0$ and, thus, by Theorem 1 in view of Remark 1 we get

$$\Big(\frac{\mu\lambda_1}{h}-\mu+1\Big)\frac{\lambda_1-\alpha}{h-\alpha}\sum_{k=1}^{\infty}|f_k|\le\sum_{k=1}^{\infty}|f_k|\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)\frac{\lambda_k-\alpha}{h-\alpha}\le 1,$$

i.e.

$$\sum_{k=1}^{\infty}|f_k|\le\frac{h(h-\alpha)}{(\mu(\lambda_1-h)+h)(\lambda_1-\alpha)}<1.$$

Therefore, for $\sigma<0$

$$\Big|\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\}\Big|\le\sum_{k=1}^{\infty}|f_k|\exp\{\sigma\lambda_k\}\le\sum_{k=1}^{\infty}|f_k|\le\frac{h(h-\alpha)}{(\mu(\lambda_1-h)+h)(\lambda_1-\alpha)}$$

and, thus,

$$e^{h\sigma}-\frac{h(h-\alpha)}{(\mu(\lambda_1-h)+h)(\lambda_1-\alpha)}\le|F(s)|\le e^{h\sigma}+\frac{h(h-\alpha)}{(\mu(\lambda_1-h)+h)(\lambda_1-\alpha)}\quad(s\in\Pi_0).\quad(11)$$

If $f_k\ge 0$ and $\mu<-h/(\lambda_1-h)$ then $\mu<0$, $|\mu\lambda_k/h-\mu+1|=|-\mu(\lambda_k-h)-h|/h\ge$ $\ge(|\mu|(\lambda_k-h)-h)/h\ge(|\mu|(\lambda_1-h)-h)/h\ge 1$ provided $\mu\le-2h/(\lambda_1-h)$ and, thus,

$$\frac{\lambda_1-\alpha}{h-\alpha}\sum_{k=1}^{\infty}|f_k|\le\sum_{k=1}^{\infty}|f_k|\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|\frac{\lambda_k-\alpha}{h-\alpha}\le 1,$$

whence $\sum_{k=1}^{\infty}|f_k|\le(h-\alpha)/(\lambda_1-\alpha)<1$ and as above

$$e^{h\sigma}-\frac{h-\alpha}{\lambda_1-\alpha}\le|F(s)|\le e^{h\sigma}+\frac{h-\alpha}{\lambda_1-\alpha}\quad(s\in\Pi_0).\qquad(12)$$

Thus, the following statement is correct.

Proposition 1. *Let $\alpha\in[0,h)$ and $F\in SD_h(\mu,\alpha)$. If $f_k\le 0$ for all $k\ge 1$ and $\mu\ge 0$ then (11) hods. If $f_k\ge 0$ for all $k\ge 1$ and $\mu\le-2h/(\lambda_1-h)$ then (12) holds.*

Let's move on to the functions from the class $\Sigma D_h(\mu,\alpha)$. If $f_k\le 0$ and $\mu\ge h/(\lambda_1+h)$ then $\mu\lambda_k/h+\mu-1\ge\mu\lambda_1/h+\mu-1\ge 0$ and, thus, by Theorem 2 in view of Remark 2 we get

$$\Big(\frac{\mu\lambda_1}{h}+\mu-1\Big)\frac{\lambda_1+\alpha}{h-\alpha}\sum_{k=1}^{\infty}|f_k|\le\sum_{k=1}^{\infty}|f_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\frac{\lambda_k+\alpha}{h-\alpha}\le 1,$$

whence $\sum_{k=1}^{\infty}|f_k|\le h(h-\alpha)/\big((\mu(\lambda_1+h)-h)(\lambda_1+\alpha)\big)<1$ provided $\mu>2h/(\lambda_1+h)$, and as above

$$e^{h|\sigma|}-\frac{h(h-\alpha)}{(\mu(\lambda_1+h)-h)(\lambda_1+\alpha)}\le|F(s)|\le e^{h|\sigma|}+\frac{h(h-\alpha)}{(\mu(\lambda_1+h)-h)(\lambda_1+\alpha)}\quad(s\in\Pi_0).\quad(13)$$

If $f_k\ge 0$ and $\mu<0$ then $|\mu\lambda_k/h+\mu-1|=|\mu|(\lambda_k/h+1)+1\ge|\mu|(\lambda_1/h+1)+1$ and, thus,

$$\Big(|\mu|\Big(\frac{\lambda_1}{h}+1\Big)+1\Big)\frac{\lambda_1-\alpha}{h-\alpha}\sum_{k=1}^{\infty}|f_k|\le\sum_{k=1}^{\infty}|f_k|\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|\frac{\lambda_k-\alpha}{h-\alpha}\le 1,$$

whence $\sum_{k=1}^{\infty}|f_k|\le h(h-\alpha)/\big((|\mu|(\lambda_1+h)+h)(\lambda_1+\alpha)\big)<1$ and as above

$$e^{h|\sigma|}-\frac{h(h-\alpha)}{(|\mu|(\lambda_1+h)+h)(\lambda_1+\alpha)}\le|F(s)|\le e^{h|\sigma|}+\frac{h(h-\alpha)}{(|\mu|(\lambda_1+h)+h)(\lambda_1+\alpha)}\quad(s\in\Pi_0).\quad(14)$$

Thus, the following statement is correct.

Proposition 2. *Let $\alpha \in [0, h)$ and $F \in \Sigma D_h(\mu, \alpha)$. If $f_k \le 0$ for all $k \ge 1$ and $\mu > 2h/(\lambda_1+h)$ then (13) hods. If $f_k \ge 0$ for all $k \ge 1$ and $\mu < 0$ then (14) holds.*

5. Neighborhoods of the functions from $SD_h(\mu, \alpha)$ and $\Sigma D_h(\mu, \alpha)$. Let A denote the class of analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ functions $f(z) = z + \sum_{k=2}^{\infty} f_k z^k$. For $f \in A$, following A.W. Goodman [5] and S. Ruscheweyh [6], a set

$$N_\delta(f) = \Big\{g(z) = z + \sum_{k=2}^{\infty} g_k z^k \in A\colon \ \sum_{k=2}^{\infty} k|g_k - f_k| \le \delta\Big\}.$$

is called its neighborhood. For certain classes of Dirichlet series absolutely convergent in the half-plane Π_0, the concept of neighborhoods was introduced and studied in [7] and [8]. When studying neighborhoods of functions of the form (1) or (3), we will distinguish the cases of negative and positive coefficients. In other words, let us say that $F \in SD_h^+$ if $F \in SD_h$ and $f_k \ge 0$ for all $k \ge 1$ and $F \in SD_h^-$ if $F \in SD_h$ and $f_k \le 0$ for all $k \ge 1$. Define classes $SD_h^+(\mu, \alpha)$, $SD_h^-(\mu, \alpha)$, ΣD_h^+, ΣD_h^-, $\Sigma D_h^+(\mu, \alpha)$ and $\Sigma D_h^-(\mu, \alpha)$ in the same way.

For $\delta > 0$ we define the neighborhood of $F \in SD_h^+$ as follows

$$O_{\delta,S}^+(F) = \Big\{G(s) = e^{hs} + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in SD_h^+ : \ \sum_{k=1}^{\infty} \lambda_k^2 |g_k - f_k| \le \delta\Big\} \qquad (15)$$

and similarly we define $O_{\delta,S}^-(F)$ for $F \in SD_h^-$.

For $F \in \Sigma D_h^+$ let

$$O_{\delta,\Sigma}^+(F) = \Big\{G(s) = e^{-hs} + \sum_{k=1}^{\infty} g_k \exp\{s\lambda_k\} \in \Sigma D_h^+ : \ \sum_{k=1}^{\infty} \lambda_k^2 |g_k - f_k| \le \delta\Big\} \qquad (16)$$

and similarly we define $O_{\delta,\Sigma}^-(F)$ for $F \in \Sigma D_h^-$.

Theorem 3. *Let $h > 0$ and $0 \le \beta < \alpha < h$. If $\mu > 0$, $F \in SD_h^-(\mu, \alpha)$ and $G \in O_{\delta,S}^-(F)$ with*

$$\delta = \delta_1 := \frac{h(h-\beta)}{\mu+1}\left(1 - \frac{(\lambda_1-\beta)(h-\alpha)}{(\lambda_1-\alpha)(h-\beta)}\right)$$

then $G \in SD_h^-(\mu, \beta)$. If $\mu \le -h/(\lambda_1 - h)$, $F \in SD_h^+(\mu, \alpha)$ and $G \in O_{\delta,S}^+(F)$ with

$$\delta = \delta_2 := \frac{h(h-\beta)}{|\mu|}\left(1 - \frac{(\lambda_1-\beta)(h-\alpha)}{(\lambda_1-\alpha)(h-\beta)}\right)$$

then $G \in SD_h^+(\mu, \beta)$.

On the contrary, if $\mu > 0$, $F \in SD_h^-(\mu, \alpha)$ and $G \in SD_h^-(\mu, \beta)$ then $G \in O_{\delta,S}^-(F)$ with $\delta = \delta_3 := \frac{2h(h-\beta)\lambda_1^2}{\mu(\lambda_1-\alpha)(\lambda_1-h)}$. If $\mu \le -h/(\lambda_1 - h)$, $F \in SD_h^+(\mu, \alpha)$ and $G \in SD_h^+(\mu, \beta)$ then $G \in O_{\delta,S}^+(F)$ with $\delta = \delta_4 := \frac{2h(h-\beta)\lambda_1^2}{(|\mu|(\lambda_1-h)-h|)(\lambda_1-\alpha)}$.

Proof. If $\mu > 0$, $F \in SD_h^-(\mu, \alpha)$ and $G \in O_{\delta,S}^-(F)$ with $\delta = \delta_1$ then by Theorem 1

$$\sum_{k=1}^{\infty} |g_k| \Big(\frac{\mu\lambda_k}{h} - \mu + 1\Big)(\lambda_k - \beta) \le$$

$$\le \sum_{k=1}^{\infty} |f_k| \Big(\frac{\mu\lambda_k}{h} - \mu + 1\Big)(\lambda_k - \alpha)\frac{\lambda_k - \beta}{\lambda_k - \alpha} + \sum_{k=1}^{\infty} |g_k - f_k| \Big(\frac{\mu\lambda_k}{h} - \mu + 1\Big)(\lambda_k - \beta) \le$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\Big(\frac{\mu}{h}+\frac{1}{\lambda_k}-\frac{\mu}{\lambda_k}\Big)\frac{\lambda_k-\alpha}{\lambda_k}\leq$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\frac{\mu+1}{h}\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\delta_1\frac{\mu+1}{h}=h-\beta,$$

i.e. $G\in SD_h^-(\mu,\beta)$.

If $\mu\leq -h/(\lambda_1-h)$, $F\in SD_h^+(\mu,\alpha)$ and $G\in O_{\delta,S}^+(F)$ with $\delta=\delta_2$ then as above

$$\sum_{k=1}^{\infty}g_k\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\beta)\leq$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}\sum_{k=1}^{\infty}f_k\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\alpha)+\sum_{k=1}^{\infty}|g_k-f_k|\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\beta)\leq$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\sum_{k=1}^{\infty}|g_k-f_k|\Big|\frac{-\mu\lambda_k}{h}+\mu-1\Big|(\lambda_k-\beta)\leq$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\frac{|\mu|(\lambda_k-h)-h}{h\lambda_k}\frac{\lambda_k-\beta}{\lambda_k}\leq$$

$$\leq \frac{\lambda_1-\beta}{\lambda_1-\alpha}(h-\alpha)+\frac{|\mu|}{h}\delta_2=h-\beta,$$

i.e. $G\in SD_h^+(\mu,\beta)$.

On the contrary, if $\mu>0$, $F\in SD_h^-(\mu,\alpha)$ and $G\in SD_h^-(\mu,\beta)$ then

$$\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|=\sum_{k=1}^{\infty}|g_k-f_k|\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)(\lambda_k-\alpha)\frac{h\lambda_k^2}{(\lambda_k-\alpha)(\mu(\lambda_k-h)+h)}\leq$$

$$\leq \sum_{k=1}^{\infty}|g_k-f_k|\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)(\lambda_k-\alpha)\frac{h\lambda_k^2}{\mu(\lambda_k-\alpha)(\lambda_k-h)}\leq$$

$$\leq \frac{h\lambda_1^2}{\mu(\lambda_1-\alpha)(\lambda_1-h)}\Big(\sum_{k=1}^{\infty}|f_k|\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)(\lambda_k-\alpha)+\sum_{k=1}^{\infty}|g_k|\Big(\frac{\mu\lambda_k}{h}-\mu+1\Big)(\lambda_k-\beta)\Big)\leq$$

$$\leq \frac{h\lambda_1^2}{\mu(\lambda_1-\alpha)(\lambda_1-h)}(2h-\alpha-\beta)\leq \frac{2h(h-\beta)\lambda_1^2}{\mu(\lambda_1-\alpha)(\lambda_1-h)}=\delta_3,$$

i.e. $G\in O_{\delta,S}^-(F)$ with $\delta=\delta_3$.

If $\mu\leq -h/(\lambda_1-h)$, $F\in SD_h^+(\mu,\alpha)$ and $G\in SD_h^+(\mu,\beta)$ then as above

$$\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\leq\sum_{k=1}^{\infty}|g_k-f_k|\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\alpha)\frac{h\lambda_k^2}{(|\mu|(\lambda_k-h)-h|)(\lambda_k-\alpha)}\leq$$

$$\leq \frac{h\lambda_1^2}{(|\mu|(\lambda_1-h)-h|)(\lambda_1-\alpha)}\Big(\sum_{k=1}^{\infty}f_k\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\alpha)+\sum_{k=1}^{\infty}g_k\Big|\frac{\mu\lambda_k}{h}-\mu+1\Big|(\lambda_k-\beta)\Big)\leq$$

$$\leq \frac{h\lambda_1^2}{(|\mu|(\lambda_1-h)-h|)(\lambda_1-\alpha)}\Big(2h-\alpha-\beta)\Big)\leq\delta_4,$$

i.e. $G\in O_{\delta,S}^+(F)$ with $\delta=\delta_4$. □

For the class $\Sigma D_h(\mu,\alpha)$, the following theorem is true.

Theorem 4. *Let $h>0$ and $0\le\beta<\alpha<h$. If $\mu\ge h/(\lambda_1+h)$, $F\in\Sigma D^-_h(\mu,\alpha)$ and $G\in O^-_{\delta,\Sigma}(F)$ with $\delta=\delta_5:=\frac{h(\alpha-\beta)\lambda_1^2}{\mu(\lambda_1+h)(\lambda_1+\beta)}$ then $G\in\Sigma D^-_h(\mu,\beta)$. If $\mu\le 0$, $F\in\Sigma D^+_h(\mu,\alpha)$ and $G\in O^+_{\delta,\Sigma}(F)$ with $\delta=\delta_6:=\frac{h(\alpha-\beta)\lambda_1^2}{(|\mu|+1)(\lambda_1+h)(\lambda_1+\beta)}$ then $G\in\Sigma D^+_h(\mu,\beta)$.*

On the contrary, if $\mu\ge 1$, $F\in\Sigma D^-_h(\mu,\alpha)$ and $G\in\Sigma D^-_h(\mu,\beta)$ then $G\in O^-_{\delta,\Sigma}(F)$ with

$$\delta=\delta_7:=\frac{h(h-\alpha)}{\mu}\left(1+\frac{(h-\beta)(\lambda_1+\alpha)}{(h-\alpha)(\lambda_1+\beta)}\right).$$

If $\mu<0$, $F\in\Sigma D^+_h(\mu,\alpha)$ and $G\in\Sigma D^+_h(\mu,\beta)$ then $G\in O^+_{\delta,\Sigma}(F)$ with

$$\delta=\delta_8:=\frac{h(h-\alpha)}{|\mu|}\left(1+\frac{(h-\beta)(\lambda_1+\alpha)}{(h-\alpha)(\lambda_1+\beta)}\right).$$

Proof. If $\mu\ge h/(\lambda_1+h)$, $F\in\Sigma D^-_h(\mu,\alpha)$ and $G\in O^-_{\delta,\Sigma}(F)$ with $\delta=\delta_5$ then by Theorem 2

$$\sum_{k=1}^{\infty}\left|g_k\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)\right|(\lambda_k+\beta)\le$$
$$\le\sum_{k=1}^{\infty}|f_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)(\lambda_k+\alpha)+\sum_{k=1}^{\infty}|g_k-f_k|\frac{\mu(\lambda_k+h)-h}{h}(\lambda_k+\beta)\le$$
$$\le h-\alpha+\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\frac{\mu(\lambda_k+h)-h}{h\lambda_k}\frac{\lambda_k+\beta}{\lambda_k}\le$$
$$\le h-\alpha+\frac{\mu(\lambda_1+h)(\lambda_1+\beta)}{\lambda_1^2}\delta_5=h-\beta,$$

i.e. $G\in\Sigma D^-_h(\mu,\beta)$.

If $\mu\le 0$, $F\in\Sigma D^+_h(\mu,\alpha)$ and $G\in O^+_{\delta,\Sigma}(F)$ with $\delta=\delta_6$ then as above

$$\sum_{k=1}^{\infty}g_k\left|\frac{\mu\lambda_k}{h}+\mu-1\right|(\lambda_k+\beta)\le$$
$$\le h-\alpha+\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|\frac{|\mu|(\lambda_k+h)+h}{h\lambda_k}\frac{\lambda_k+\beta}{\lambda_k}\le$$
$$\le h-\alpha+\frac{|\mu|(\lambda_1+h)+h}{h\lambda_1}\frac{\lambda_1+\beta}{\lambda_1}\delta_6<h-\alpha+\frac{(|\mu|+1)(\lambda_1+h)}{h\lambda_1}\frac{\lambda_1+\beta}{\lambda_1}\delta_6=h-\beta,$$

i.e. $G\in\Sigma D^+_h(\mu,\beta)$.

On the contrary, if $\mu\ge 1\ge h/(\lambda_1+h)$, $F\in\Sigma D^-_h(\mu,\alpha)$ and $G\in\Sigma D^-_h(\mu,\beta)$ then

$$\sum_{k=1}^{\infty}\lambda_k^2|g_k-f_k|=\sum_{k=1}^{\infty}|g_k-f_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)(\lambda_k+\alpha)\frac{h\lambda_k^2}{(\lambda_k+\alpha)(\mu(\lambda_k+h)-h)}\le$$
$$\le\sum_{k=1}^{\infty}|g_k-f_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)(\lambda_k+\alpha)\frac{h\lambda_k}{\mu\lambda_k+(\mu-1)h}\le$$
$$\le\frac{h}{\mu}\Big(\sum_{k=1}^{\infty}|f_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)(\lambda_k+\alpha)+\sum_{k=1}^{\infty}|g_k|\Big(\frac{\mu\lambda_k}{h}+\mu-1\Big)(\lambda_k+\beta)\frac{\lambda_k+\alpha}{\lambda_k+\beta}\Big)\le$$

$$\leq \frac{h}{\mu}\Big(h-\alpha+(h-\beta)\frac{\lambda_1+\alpha}{\lambda_1+\beta}\Big)=\delta_7,$$

i.e. $G\in O^-_{\delta,\Sigma}(F)$ with $\delta=\delta_7$.

Finally, if $\mu<0$, $F\in\Sigma D^+_h(\mu,\alpha)$ and $G\in\Sigma D^+_h(\mu,\beta)$ then as above

$$\sum_{k=1}^\infty \lambda_k^2|g_k-f_k|=\sum_{k=1}^\infty |g_k-f_k|\Big|\frac{\mu\lambda_k}{h}+\mu-1\Big|(\lambda_k+\alpha)\frac{h\lambda_k^2}{(\lambda_k+\alpha)|\mu(\lambda_k+h)-h|}\leq$$

$$\leq \sum_{k=1}^\infty |g_k-f_k|\Big|\frac{\mu\lambda_k}{h}+\mu-1\Big|(\lambda_k+\alpha)\frac{h\lambda_k}{|-\mu(\lambda_k+h)+h|}\leq$$

$$\leq \frac{h}{|\mu|}\Big(\sum_{k=1}^\infty f_k\Big|\frac{\mu\lambda_k}{h}+\mu-1\Big|(\lambda_k+\alpha)+\sum_{k=1}^\infty g_k\Big|\frac{\mu\lambda_k}{h}+\mu-1\Big|(\lambda_k+\beta)\frac{\lambda_k+\alpha}{\lambda_k+\beta}\Big)\leq$$

$$\leq \frac{h}{|\mu|}\Big(h-\alpha+(h-\beta)\frac{\lambda_1+\alpha}{\lambda_1+\beta}\Big)=\delta_8,$$

i.e. $G\in O^+_{\delta,\Sigma}(F)$ with $\delta=\delta_8$. □

6. Hadamard compositions. For power series $f_j(z)=\sum_{k=0}^\infty f_{k,j}z^k$ $(j=1,2)$ the series $(f_1*f_2)(z)=\sum_{k=0}^\infty f_{k,1}f_{k,2}z^k$ is called the *Hadamard composition* (product) [9]. Properties of this composition obtained by J. Hadamard find applications [10, 11] in the theory of the analytic continuation of functions represented by power series. Many authors have used Hadamard compositions to study various properties of entire and analytic in the unit disc of functions. For absolutely convergent in the half-plane Π_0 Dirichlet series Hadamard compositions were used in [3, 8].

Suppose that $F_j\in SD^-_h$ $(j=1,2)$, i.e.

$$F_j(s)=e^{sh}-\sum_{k=1}^\infty f_{k,j}\exp\{s\lambda_k\},\quad f_{k,j}>0. \tag{17}$$

Then the Hadamard composition of F_1 and F_2 is defined by

$$(F_1*F_2)(s)=e^{sh}-\sum_{k=1}^\infty f_{k,1}f_{k,2}\exp\{s\lambda_k\}. \tag{18}$$

If $F_j\in SD^+_h$ $(j=1,2)$, i.e.

$$F_j(s)=e^{sh}+\sum_{k=1}^\infty f_{k,j}\exp\{s\lambda_k\},\quad f_{k,j}>0, \tag{19}$$

then the Hadamard composition of F_1 and F_2 is defined by

$$(F_1*F_2)(s)=e^{sh}+\sum_{k=1}^\infty f_{k,1}f_{k,2}\exp\{s\lambda_k\}. \tag{20}$$

The definitions of the Hadamard composition in the classes ΣD^-_h and ΣD^+_h are similar.

Theorem 5. *If $F_j \in SD_h^-(\mu_j, \alpha_j)\,(j = 1,2)$, where $\mu_j \geq 0$ and $0 \leq \alpha_j < h$ then $(F_1 * F_2) \in SD_h^-(\mu, \alpha)$, where $\mu = \min\{\mu_1, \mu_2\}$ and*

$$\alpha = h\frac{\mu(\lambda_1 - h)(\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2) - \lambda_1(h - \alpha_1)(h - \alpha_2)}{\mu(\lambda_1 - h)(\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2) - h(h - \alpha_1)(h - \alpha_2)}. \tag{21}$$

If $F_j \in SD_h^+(\mu_j, \alpha_j)\,(j = 1,2)$, where $\mu \leq -h/(\lambda_1 - h)$ and $0 \leq \alpha_j < h$ then $(F_1 * F_2) \in SD_h^+(\mu, \alpha)$, where $\mu = \max\{\mu_1, \mu_2\}$ and

$$\alpha = h\frac{(|\mu|(\lambda_1 - h) - h))(\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2) - \lambda_1(h - \alpha_1)(h - \alpha_2)}{(|\mu|(\lambda_1 - h) - h)(\lambda_1 - \alpha_1))(\lambda_1 - \alpha_2) - h(h - \alpha_1)(h - \alpha_2)}.$$

Proof. If $F_j \in SD_h^-(\mu_j, \alpha_j)$ and $\mu = \min\{\mu_1, \mu_2\}$ then by Theorem 1

$$\sum_{k=1}^{\infty} \frac{(\mu(\lambda_k - h) + h))(\lambda_k - \alpha_j)}{h(h - \alpha_j)} f_{k,j} \leq \sum_{k=1}^{\infty} \frac{(\mu_j(\lambda_k - h) + h))(\lambda_k - \alpha_j)}{h(h - \alpha_j)} f_{k,j} \leq 1 \tag{22}$$

and by Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{\infty} \sqrt{\frac{(\mu(\lambda_k - h) + h)(\lambda_k - \alpha_1)}{h(h - \alpha_1)} f_{k,1} \frac{(\mu(\lambda_k - h) + h)(\lambda_k - \alpha_2)}{h(h - \alpha_2)} f_{k,2}} \leq$$

$$\leq \sqrt{\sum_{k=1}^{\infty} \frac{(\mu(\lambda_k - h) + h)(\lambda_k - \alpha_1)}{h(h - \alpha_1)} f_{k,1} \sum_{k=1}^{\infty} \frac{(\mu(\lambda_k - h) + h)(\lambda_k - \alpha_2)}{h(h - \alpha_2)} f_{k,2}} \leq 1,$$

i.e.

$$\sum_{k=1}^{\infty} \frac{(\mu(\lambda_k - h) + h)}{h} \sqrt{\frac{(\lambda_k - \alpha_1)(\lambda_k - \alpha_2)}{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1, \tag{23}$$

whence

$$\frac{(\mu(\lambda_k - h) + h)}{h} \sqrt{\frac{(\lambda_k - \alpha_1)(\lambda_k - \alpha_2)}{(h - \alpha_1)(h - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}} \leq 1,$$

i.e.

$$\sqrt{f_{k,1} f_{k,2}} \leq \frac{h}{(\mu(\lambda_k - h) + h)} \sqrt{\frac{(h - \alpha_1)(h - \alpha_2)}{(\lambda_k - \alpha_1)(\lambda_k - \alpha_2)}}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\mu(\lambda_k - h) + h}{h} \frac{\lambda_k - \alpha}{h - \alpha} f_{k,1} f_{k,2} = \sum_{k=1}^{\infty} \frac{\mu(\lambda_k - h) + h}{h} \frac{\lambda_k - \alpha}{h - \alpha} \sqrt{f_{k,1} f_{k,2}} \sqrt{f_{k,1} f_{k,2}} \leq$$

$$\leq \sum_{k=1}^{\infty} \frac{\lambda_k - \alpha}{h - \alpha} \sqrt{\frac{(h - \alpha_1)(h - \alpha_2)}{(\lambda_k - \alpha_1)(\lambda_k - \alpha_2)}} \sqrt{f_{k,1} f_{k,2}}. \tag{24}$$

From (21) it follows that

$$\frac{\lambda_1 - \alpha}{h - \alpha} = \frac{\mu(\lambda_1 - h)}{h} \frac{(\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2)}{(h - \alpha_1)(h - \alpha_2)},$$

and, since $\frac{\mu(\lambda_k-h)+h}{\lambda_k-\alpha} \geq \frac{\mu(\lambda_k-h)}{\lambda_k-\alpha} \geq \frac{\mu(\lambda_1-h)}{\lambda_1-\alpha}$, we have

$$\frac{\lambda_k-\alpha}{h-\alpha} \leq \frac{(\mu(\lambda_k-h)+h)}{h}\frac{(\lambda_k-\alpha_1)(\lambda_k-\alpha_2)}{(h-\alpha_1)(h-\alpha_2)},$$

i.e.

$$\frac{\lambda_k-\alpha}{h-\alpha}\sqrt{\frac{(h-\alpha_1)(h-\alpha_2)}{(\lambda_k-\alpha_1)(\lambda_k-\alpha_2)}} \leq \frac{(\mu(\lambda_k-h)+h)}{h}\sqrt{\frac{(\lambda_k-\alpha_1)(\lambda_k-\alpha_2)}{(h-\alpha_1)(h-\alpha_2)}},$$

and in view of (24) and (23) we get

$$\sum_{k=1}^{\infty}\frac{\mu(\lambda_k-h)+h}{h}\frac{\lambda_k-\alpha}{h-\alpha}f_{k,1}f_{k,2} \leq \sum_{k=1}^{\infty}\frac{(\mu(\lambda_k-h)+h)}{h}\sqrt{\frac{(\lambda_k-\alpha_1)(\lambda_k-\alpha_2)}{(h-\alpha_1)(h-\alpha_2)}}\sqrt{f_{k,1}f_{k,2}} \leq 1.$$

Thus, by Theorem 1 $(F_1 * F_2) \in SD_h^-(\mu,\alpha)$. The first part of Theorem 5 is proved.

Let us prove the second part. Since $\mu \leq -h/(\lambda_1-h)$, we have

$$|\mu(\lambda_k-h)+h| = |-\mu(\lambda_k-h)-h| = ||\mu|(\lambda_k-h)-h| = |\mu|(\lambda_k-h)-h.$$

Therefore, if $F_j \in SD_h^+(\mu_j,\alpha_j)$ and $\mu = \max\{\mu_1,\mu_2\}$ then $|\mu| = \min\{|\mu_1|,|\mu_2|\}$ and by Theorem 1

$$\sum_{k=1}^{\infty}\frac{(|\mu|(\lambda_k-h)-h)(\lambda_k-\alpha_j)}{h(h-\alpha_j)}f_{k,j} \leq \sum_{k=1}^{\infty}\frac{(|\mu_j|(\lambda_k-h)-h)(\lambda_k-\alpha_j)}{h(h-\alpha_j)}f_{k,j} \leq 1.$$

This inequality differs from inequality (22) only in that now instead of $\mu(\lambda_k-h)+h$ there is $|\mu|(\lambda_k-h)-h$. We note that now $\frac{|\mu|(\lambda_k-h)-h}{\lambda_k-\alpha} \geq \frac{|\mu|(\lambda_1-h)-h}{\lambda_1-\alpha}$. Therefore, repeating the proof of the first part, we arrive to the validity of the second part. The proof of Theorem 5 is complete. □

The following theorem is an analog of Theorem 5 for the classes ΣD_h^- and ΣD_h^+.

Theorem 6. *If $F_j \in \Sigma D_h^-(\mu_j,\alpha_j)\,(j=1,2)$, where $\mu_j \geq 1$ and $0 \leq \alpha_j < h$ then $(F_1 * F_2) \in \Sigma D_h^-(\mu,\alpha)$, where $\mu = \min\{\mu_1,\mu_2\}$ and*

$$\alpha = h\frac{\lambda_1(\lambda_1+\alpha_1)(\lambda_1+\alpha_2)-\lambda_1(h-\alpha_1)(h-\alpha_2)}{\lambda_1(\lambda_1+\alpha_1)(\lambda_1+\alpha_2)+h(h-\alpha_1)(h-\alpha_2)}. \tag{25}$$

If $F_j \in \Sigma D_h^+(\mu_j,\alpha_j)\,(j=1,2)$, where $\mu_j \leq -1 < 0$ and $0 \leq \alpha_j < h$ then $(F_1 * F_2) \in \Sigma D_h^+(\mu,\alpha)$, where $\mu = \max\{\mu_1,\mu_2\}$ and

$$\alpha = h\left(1-\frac{(h-\alpha_1)(h-\alpha_2)}{|\mu|(\lambda_1+\alpha_1)(\lambda_1+\alpha_2)}\right).$$

Proof. If $F_j \in \Sigma D_h^-(\mu_j,\alpha_j)$ and $\mu = \min\{\mu_1,\mu_2\}$ then by Theorem 2 and Remark 2

$$\sum_{k=1}^{\infty}\frac{(\mu(\lambda_k+h)-h)(\lambda_k+\alpha_j)}{h(h-\alpha_j)}f_{k,j} \leq \sum_{k=1}^{\infty}\frac{(\mu_j(\lambda_k+h)-h)(\lambda_k+\alpha_j)}{h(h-\alpha_j)}f_{k,j} \leq 1. \tag{26}$$

and by Cauchy-Schwarz inequality we obtain as above

$$\sum_{k=1}^{\infty}\frac{\mu(\lambda_k+h)-h)}{h}\sqrt{\frac{(\lambda_k+\alpha_1)(\lambda_k+\alpha_2)}{(h-\alpha_1)(h-\alpha_2)}}\sqrt{f_{k,1}f_{k,2}}\le 1, \tag{27}$$

whence

$$\sqrt{f_{k,1}f_{k,2}}\le\frac{h}{\mu(\lambda_k+h)-h}\sqrt{\frac{(h-\alpha_1)(h-\alpha_2)}{(\lambda_k+\alpha_1)(\lambda_k+\alpha_2)}}.$$

Therefore, we get as above

$$\sum_{k=1}^{\infty}\frac{\mu(\lambda_k+h)-h}{h}\frac{\lambda_k+\alpha}{h-\alpha}f_{k,1}f_{k,2}\le\sum_{k=1}^{\infty}\frac{\lambda_k+\alpha}{h-\alpha}\sqrt{\frac{(h-\alpha_1)(h-\alpha_2)}{(\lambda_k+\alpha_1)(\lambda_k+\alpha_2)}}\sqrt{f_{k,1}f_{k,2}}. \tag{28}$$

From (25) it follows that

$$\frac{\lambda_1+\alpha}{h-\alpha}=\frac{\lambda_1}{h}\frac{(\lambda_1+\alpha_1)(\lambda_1+\alpha_2)}{(h-\alpha_1)(h-\alpha_2)},$$

and, since $\dfrac{\mu(\lambda_k+h)-h}{\lambda_k+\alpha}\ge\dfrac{\lambda_k}{\lambda_k+\alpha}\ge\dfrac{\lambda_1}{\lambda_1+\alpha}$, we have

$$\frac{\lambda_k+\alpha}{h-\alpha}\le\frac{\mu(\lambda_k+h)-h}{h}\frac{(\lambda_k+\alpha_1)(\lambda_k+\alpha_2)}{(h-\alpha_1)(h-\alpha_2)}.$$

Therefore, in view of (28) and (27) we get

$$\sum_{k=1}^{\infty}\frac{\mu(\lambda_k+h)-h}{h}\frac{\lambda_k+\alpha}{h-\alpha}f_{k,1}f_{k,2}\le\sum_{k=1}^{\infty}\frac{\mu(\lambda_k+h)-h)}{h}\sqrt{\frac{(\lambda_k+\alpha_1)(\lambda_k+\alpha_2)}{(h-\alpha_1)(h-\alpha_2)}}\sqrt{f_{k,1}f_{k,2}}\le 1$$

and, thus, by Theorem 2 $(F_1*F_2)\in\Sigma D_h^-(\mu,\alpha)$. The first part of Theorem 6 is proved.

Let us prove its second part. Since $\mu\le 0$, we have

$$|\mu(\lambda_k+h)-h|=|-\mu(\lambda_k+h)+h|=|\mu|(\lambda_k+h)+h.$$

Therefore, if $F_j\in\Sigma D_h^+(\mu_j,\alpha_j)$ and $\mu=\max\{\mu_1,\mu_2\}$ then $|\mu|=\min\{|\mu_1|,|\mu_2|\}$ and by Theorem 2

$$\sum_{k=1}^{\infty}\frac{(|\mu|(\lambda_k+h)+h)(\lambda_k+\alpha_j)}{h(h-\alpha_j)}f_{k,j}\le\sum_{k=1}^{\infty}\frac{(|\mu_j|(\lambda_k+h)+h)(\lambda_k+\alpha_j)}{h(h-\alpha_j)}f_{k,j}\le 1.$$

This inequality differs from inequality (22) only in that now instead of $\mu(\lambda_k+h)-h$ there is $|\mu|(\lambda_k+h)+h$. We remark that now $|\mu|(\lambda_k+h)+h\ge|\mu|\lambda_k+\alpha$. Therefore, repeating the proof of the first part, we arrive to the validity of the second part. □

7. Hadamard compositions of the genus m**.** Let $m\in\mathbb{N}$ and $P(x_1,...,x_p)$ be a homogeneous polynomial of degree m, that is $P(tx_1,...,tx_p)=t^mP(x_1,...,x_p)$ for all t from the field above that a polynomial is defined.

Dirichlet series (1) is called [12] a Hadamard composition of genus m of Dirichlet series $F_j(s)=e^{sh}+\sum_{k=1}^{\infty}f_{k,j}\exp\{s\lambda_k\}$ $(1\le j\le p)$ if $f_k=P(f_{k,1},...,f_{k,p})$ for all $k\ge 1$, where

$$P(x_1,...,x_p)=\sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}x_1^{k_1}\cdot\ldots\cdot x_p^{k_p}.$$

is a homogeneous polynomial of degree $m\ge 1$. We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m=2$.

Suppose that all $c_{k_1\dots k_p}>0$ and $F_j\in SD_h^-$ for all $1\le j\le p$, that is (17) holds for all j. If the function $F\in SD_h^-$ is Hadamard composition of genus $m\ge 1$ of the functions F_j then

$$f_k=\sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}f_{k,1}^{k_1}\cdot\ldots\cdot f_{k,p}^{k_p}. \tag{29}$$

In the class SD_h^- the following theorem is true.

Theorem 7. *Let all $c_{k_1\dots k_p}>0$, $\sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}=C<\infty$ and $0\le\alpha_j<h$. If $\mu>0$, $\sum_{k=1}^{\infty}(\lambda_k-h)^{-2}\le h^{-2}\mu/C$ and $F_j\in SD_h^-(\mu,\alpha_j)$ for all $1\le j\le p$ then Hadamard composition F of genus $m\ge 2$ of the functions F_j belongs to $SD_h^-(\mu,\alpha)$, where $\alpha=\min\{\alpha_j:\ 1\le j\le p\}$.*

If $\mu\le -2h/(\lambda_1-h)$, $\sum\limits_{k=1}^{\infty}\big((\lambda_k-h)((|\mu|(\lambda_k-h)-h)\big)^{-1}\le h^{-2}/C$ and $F_j\in SD_h^+(\mu,\alpha_j)$ for all $1\le j\le p$ then Hadamard composition F of genus $m\ge 2$ of the functions F_j belongs to $SD_h^+(\mu,\alpha)$, where $\alpha=\min\{\alpha_j:\ 1\le j\le p\}$.

Proof. If all $F_j\in SD_h^-(\mu,\alpha_j)$ and $\mu>0$ then by Theorem 1

$$\sum_{k=1}^{\infty}\frac{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha_j)}{h(h-\alpha_j)}f_{k,j}\le 1,$$

whence $f_{k,j}\le \frac{h(h-\alpha_j)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha_j)}\le \frac{h(h-\alpha)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha)}<1$. Therefore, (29) implies

$$f_k\le \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}\left(\frac{h(h-\alpha)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha)}\right)^m = C\left(\frac{h(h-\alpha)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha)}\right)^m$$

and, thus, for $m\ge 2$

$$\sum_{k=1}^{\infty}\frac{(\mu(\lambda_k+h)-h)(\lambda_k-\alpha)}{h(h-\alpha)}f_k\le C\sum_{k=1}^{\infty}\left(\frac{h(h-\alpha)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha)}\right)^{m-1}\le$$

$$\le C\sum_{k=1}^{\infty}\frac{h(h-\alpha)}{(\mu(\lambda_k-h)+h)(\lambda_k-\alpha)}\le C\sum_{k=1}^{\infty}\frac{h^2}{\mu(\lambda_k-h)(\lambda_k-h)}\le 1,$$

i.e. $F\in SD_h^-(\mu,\alpha)$. The first part of Theorem 7 is proved.

If all $F_j\in SD_h^+(\mu,\alpha_j)$ and $\mu\le -2h/(\lambda_1-h)$ then $\mu\le -h/(\lambda_1-h)$ and as above in view of Theorem 1 we have

$$\sum_{k=1}^{\infty}\frac{(|\mu|(\lambda_k-h)-h)(\lambda_k-\alpha_j)}{h(h-\alpha_j)}f_{k,j}\le 1,$$

whence as above for $m\ge 2$ we get

$$\sum_{k=1}^{\infty}\frac{(|\mu|(\lambda_k-h)-h)(\lambda_k-\alpha)}{h(h-\alpha)}f_k\le C\sum_{k=1}^{\infty}\frac{h^2}{(|\mu|(\lambda_k-h)-h)(\lambda_k-h)}\le 1,$$

i.e. $F\in SD_h^+(\mu,\alpha)$. □

In the class ΣD_h^- the following theorem is true.

Theorem 8. *Let all* $c_{k_1\ldots k_p} > 0$, $\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} = C < \infty$ *and* $0 \le \alpha_j < h$. *If* $\mu \ge 1$, $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \le \frac{\mu}{Ch^2}$ *and* $F_j \in \Sigma D_h^-(\mu, \alpha_j)$ *for all* $1 \le j \le p$ *then Hadamard composition* F *of genus* $m \ge 2$ *of the functions* F_j *belongs to* $\Sigma D_h^-(\mu, \alpha)$, *where* $\alpha = \min\{\alpha_j : 1 \le j \le p\}$.

If $\mu < 0$, $\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k^2} \le \frac{|\mu|}{Ch^2}$ *and* $F_j \in \Sigma D_h^+(\mu, \alpha_j)$ *for all* $1 \le j \le p$ *then Hadamard composition* F *of genus* $m \ge 2$ *of the functions* F_j *belongs to* $\Sigma D_h^+(\mu, \alpha)$, *where* $\alpha = \min\{\alpha_j : 1 \le j \le p\}$.

Proof. If all $F_j \in \Sigma D_h^-(\mu, \alpha_j)$ and $\mu \ge 1$ then $\mu \ge h/(\lambda_1 + h)$ and by Theorem 2

$$\sum_{k=1}^{\infty} \frac{(\mu(\lambda_k + h) - h)(\lambda_k + \alpha_j)}{h(h - \alpha_j)} f_{k,j} \le 1,$$

whence $f_{k,j} \le \frac{h(h-\alpha)}{(\mu(\lambda_k+h)-h)(\lambda_k+\alpha)} < 1$ and (29) implies $f_k \le C(\frac{h(h-\alpha)}{(\mu(\lambda_k+h)-h)(\lambda_k+\alpha)})^m$. Thus, for $m \ge 2$ we have

$$\sum_{k=1}^{\infty} \frac{(\mu(\lambda_k + h) - h)(\lambda_k + \alpha)}{h(h - \alpha)} f_k \le C \sum_{k=1}^{\infty} \left(\frac{h(h - \alpha)}{(\mu(\lambda_k + h) - h)(\lambda_k + \alpha)} \right)^{m-1} \le$$

$$\le C \sum_{k=1}^{\infty} \frac{h(h - \alpha)}{(\mu(\lambda_k + h) - h)(\lambda_k + \alpha)} \le C \sum_{k=1}^{\infty} \frac{h^2}{\mu \lambda_k^2} \le 1,$$

i.e. $F \in \Sigma D_h^-(\mu, \alpha)$. The first part of Theorem 8 is proved.

If all $F_j \in \Sigma D_h^+(\mu, \alpha_j)$ and $\mu < 0$ then as above in view of Theorem 2 we have

$$\sum_{k=1}^{\infty} \frac{(|\mu|(\lambda_k + h) + h)(\lambda_k + \alpha_j)}{h(h - \alpha_j)} f_{k,j} \le 1,$$

and as above for $m \ge 2$ we get

$$\sum_{k=1}^{\infty} \frac{(|\mu|(\lambda_k + h) + h)(\lambda_k + \alpha)}{h(h - \alpha)} f_k \le C \sum_{k=1}^{\infty} \frac{h(h - \alpha)}{(|\mu|(\lambda_k + h) + h)(\lambda_k + \alpha)} \le C \sum_{k=1}^{\infty} \frac{h^2}{|\mu| \lambda_k^2} \le 1,$$

i.e. $F \in SD_h^+(\mu, \alpha)$. □

Finally, we consider Hadamard compositions of genus 1, i.e. case when $f_k = \sum_{j=1}^{p} c_j f_{k,j}$, where $c_j > 0$ and $\sum_{j=1}^{p} c_j = 1$. It is clear that if $f_{k,j}(\mu\lambda_k/h - \mu + 1) \le 0$ for all $k \ge 1$ and all $1 \le j \le p$ then $f_k(\mu\lambda_k/h - \mu + 1) \le 0$ for all $k \ge 1$. Therefore, by Theorem 1 the following statement is true.

Proposition 3. *If* $F_j \in SD_h(\mu, \alpha_j)$ *for all* j *and* $f_{k,j}(\mu\lambda_k/h - \mu + 1) \le 0$ *for all* $k \ge 1$ *and all* j *then Hadamard composition* F *of genus* $m = 1$ *of the functions* F_j *belongs to* $SD_h(\mu, \alpha)$, *where* $\alpha = \min\{\alpha_j : 1 \le j \le p\}$.

Proof. If $F_j \in SD_h(\mu, \alpha_j)$ for all j then using Theorem 1 we have

$$\sum_{k=1}^{\infty} \left| f_k \left(\frac{\mu \lambda_k}{h} - \mu + 1 \right) \right| \frac{\lambda_k - \alpha}{h - \alpha} \le \sum_{k=1}^{\infty} \left| \sum_{j=1}^{p} c_j f_{k,j} \left(\frac{\mu \lambda_k}{h} - \mu + 1 \right) \right| \frac{\lambda_k - \alpha}{h - \alpha} \le$$

$$\leq \sum_{j=1}^{p} c_j \sum_{k=1}^{\infty} \left| f_{k,j}\left(\frac{\mu\lambda_k}{h} - \mu + 1\right)\right| \frac{\lambda_k - \alpha_j}{h - \alpha_j} \leq \sum_{j=1}^{p} c_j = 1,$$

i.e. $F \in SD_h(\mu, \alpha)$. □

Similarly, if $f_{k,j}(\mu\lambda_k/h+\mu-1) \leq 0$ for all $k \geq 1$ and all $1 \leq j \leq p$ then $f_k(\mu\lambda_k/h+\mu-1) \leq$ ≤ 0 for all $k \geq 1$. Therefore, using Theorem 2, we can similarly prove the following assertion.

Proposition 4. *If $F_j \in \Sigma D_h(\mu, \alpha_j)$ for all j and $f_{k,j}(\mu\lambda_k/h+\mu-1) \leq 0$ for all $k \geq 1$ and all j then Hadamard composition F of genus $m = 1$ of the functions F_j belongs to $\Sigma D_h(\mu, \alpha)$, where $\alpha = \min\{\alpha_j : 1 \leq j \leq p\}$.*

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