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E. Kochubinska, A. Oliynyk

MONOGENIC FREE INVERSE SEMIGROUPS AND PARTIAL AUTOMORPHISMS OF REGULAR ROOTED TREES

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For a one-to-one partial mapping on an infinite set, we present a criterion in terms of its cycle-chain decomposition that the inverse subsemigroup generated by this mapping is monogenic free inverse.

We also give a sufficient condition for a regular rooted tree partial automorphism to extend to a partial automorphism of another regular rooted tree so that the inverse semigroup generated by this extended partial automorphism is monogenic free inverse. The extension procedure we develop is then applied to n-ary adding machines.

1. Introduction. Inverse semigroups of partial automorphisms of (infinite) regular rooted trees naturally generalize the important and intensively investigated notion of automorphism groups of regular rooted trees [4]. One can define them by using partial invertible automata over finite alphabets [14, 12, 2]. Alternatively, they can be regarded as inverse limits of inverse semigroups of partial automorphisms of finite regular rooted trees. The latter, in turn, can be constructed as partial wreath powers of the inverse symmetric semigroup IS_n [5, 6, 7].

A natural problem arising in this context is to find faithful representations of given inverse semigroups in terms of regular rooted tree partial automorphisms. One of the possible direction here is to consider free inverse semigroups, in particular monogenic free ones (c.f. [10, 11]).

A particularly interesting question is when the inverse semigroup generated by a one-to-one partial mapping f on an infinite set Ω is monogenic free inverse. We find a useful necessary and sufficient condition for this in terms of the cycle-chain decomposition of f. Then we apply this criterion to partial automorphisms of regular rooted trees, which yields a sufficient condition that a level transitive automorphism can be extended to a partial automorphism generating monogenic free inverse semigroup.

The paper is organized as follows. In Section 2 we recall required definitions on inverse semigroups including the cycle-chain decomposition of one-to-one mappings. Then we prove a criterion when the inverse semigroup generated by a one-to-one mapping is monogenic free inverse. In Section 3 we recall required definitions on automorphisms and partial automorphism of regular rooted trees; for details, the reader can refer to [4, 12, 2]. Then we show how a level transitive regular rooted tree automorphism satisfying certain conditions can be

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extended to a partial automorphism of another regular rooted tree so that the inverse semigroup generated by this extended partial automorphism is monogenic free inverse. We apply this extension technique to n-ary adding machines. In Section 4, we formulate a few open questions arisen during our research.

2. Monogenic free inverse semigroups. Let Ω be a non-empty set. For a partial mapping f on Ω, we denote by $\operatorname{dom} f$ and $\operatorname{ran} f$ the domain and the range of f correspondingly. We write $f(\omega) = \varnothing$ if $\omega \in \Omega \setminus \operatorname{dom} f$. We will use the right actions of mappings, i.e. for $\omega \in \Omega$ and partial mappings f, g on Ω we denote the element $g(f(\omega))$ by $(fg)(\omega)$.

A partial mapping $f\colon \Omega \to \Omega$ is called a partial one-to-one mapping if $f(\omega_1) \neq f(\omega_2)$ for distinct $\omega_1, \omega_2 \in \operatorname{dom} f$. For a partial one-to-one mapping f, the partial inverse f^{-1} is well-defined. We use the notation f^{-n} for the nth iteration of f^{-1}, $n \geq 1$. The inverse semigroup of all partial one-to-one mappings on Ω is denoted by $IS(\Omega)$.

A partial one-to-one mapping $f \in IS(\Omega)$ is called:

- a *finite cycle* of length $k \geq 1$ if $\operatorname{dom} f = \{\omega_1, \ldots, \omega_k\}$ and
$$f(\omega_1) = \omega_2, \ldots, f(\omega_{k-1}) = \omega_k, f(\omega_k) = \omega_1;$$
- an *infinite cycle* if $\operatorname{dom} f = \{\omega_i \colon i \in \mathbb{Z}\}$ and
$$f(\omega_i) = \omega_{i+1}, \quad i \in \mathbb{Z};$$
- a *finite chain* of length $k \geq 0$ if $\operatorname{dom} f = \{\omega_1, \ldots, \omega_k\}$ and for some ω_{k+1}
$$f(\omega_1) = \omega_2, \ldots, f(\omega_k) = \omega_{k+1}, f(\omega_{k+1}) = \varnothing;$$
- an *infinite left chain* if $\operatorname{dom} f = \{\omega_i \colon i \in \mathbb{N}\}$ and
$$f(\omega_1) = \varnothing, \quad f(\omega_i) = \omega_{i-1}, \quad i \geq 2;$$
- an *infinite right chain*, if $\operatorname{dom} f = \{\omega_i \colon i \in \mathbb{N}\}$ and
$$f(\omega_i) = \omega_{i+1}, \quad i \geq 1.$$

The cycle–chain decomposition of $f \in IS(\Omega)$ ([9, 3, 8]) is the unique partition of the set Ω as a disjoint union of non-empty subsets such that the restriction of f on each of them is either a finite or an infinite cycle, or a finite or an infinite chain.

The following auxiliary statement gives a useful method to prove that the inverse semigroup generated by a given partial one-to-one mapping is free.

Theorem 1. *Let Ω be an infinite set, $f \in IS(\Omega)$. The following statements are equivalent:*

(i) the inverse subsemigroup generated by f is monogenic free inverse;

(iii) for arbitrary $n \geq 1$, there exist $\omega_1, \omega_2 \in \Omega$ such that $f^n(\omega_1) \in \operatorname{dom} f, f^{-n}(\omega_2) \in \operatorname{ran} f$, but $f^{n+1}(\omega_1) \notin \operatorname{dom} f, f^{-(n+1)}(\omega_2) \notin \operatorname{ran} f$;

(iii) the cycle–chain decomposition of f contains either both infinite left and right chains or for arbitrary $n \geq 1$ a finite chain of length greater than n.

Proof. I. Assume that (ii) holds. We will show that the monogenic inverse subsemigroup S generated by f is free. The proof is inspired by [2, Example 23].

Let

$$I = \{(m,n,k) \in \mathbb{Z}^3 \colon m \geq 0, n \geq 0, m+n > 0, -m \leq k \leq n\}.$$

Following [13, IX.1.9], each element $s \in S$ can be written in the form $s = f^m f^{-(m+n)} f^{n-k}$ for some triple $(m,n,k) \in I$, and it is sufficient to show uniqueness of such representation.

Let us consider two different triples $(m_1,n_1,k_1),(m_2,n_2,k_2) \in I$ and show that the products $s_1 = f^{m_1} f^{-(m_1+n_1)} f^{n_1-k_1}$ and $s_2 = f^{m_2} f^{-(m_2+n_2)} f^{n_2-k_2}$ are different. We have three cases.

Case 1: $k_1 \neq k_2$. According to (ii), there exists

$$\omega \in \operatorname{dom} f^{\max(m_1,m_2)} \cap \operatorname{ran} f^{\max(n_1,n_2)}$$

such that $\omega \notin \operatorname{dom} f^{\max(m_1,m_2)+1}$. Then, $\omega \in \operatorname{dom} s_1 \cap \operatorname{dom} s_2$, and we have

$$s_1(\omega) = f^{m_1} f^{-(m_1+n_1)} f^{n_1-k_1}(\omega) = f^{-n_1} f^{n_1-k_1}(\omega) = f^{-k_1}(\omega),$$
$$s_2(\omega) = f^{m_2} f^{-(m_2+n_2)} f^{n_2-k_2}(\omega) = f^{-n_2} f^{n_2-k_2}(\omega) = f^{-k_2}(\omega).$$

Since f is one-to-one and ω does not belong to a cycle, $f^{-k_1}(\omega) \neq f^{-k_2}(\omega)$. Hence, $s_1 \neq s_2$.

Case 2: $n_1 \neq n_2$. Without loss of generality, we may assume that $n_1 < n_2$. Due to (ii), there exists

$$\omega \in \operatorname{dom} f^{\max(m_1,m_2)} \cap \operatorname{ran} f^{n_1}$$

such that $\omega \notin \operatorname{ran} f^{n_2}$. Then, on the one hand, we have

$$s_1(\omega) = f^{m_1} f^{-(m_1+n_1)} f^{n_1-k_1}(\omega) = f^{-n_1} f^{n_1-k_1}(\omega) = f^{-k_1}(\omega)$$

and $\omega \in \operatorname{dom} s_1$. On the other hand,

$$s_2(\omega) = f^{m_2} f^{-(m_2+n_2)} f^{n_2-k_2}(\omega) = f^{-n_2} f^{n_2-k_2}(\omega).$$

Since $\omega \notin \operatorname{dom} f^{-n_2}$, we have $s_2(\omega) = \varnothing$ and $\omega \notin \operatorname{dom} s_2$. Hence, $\operatorname{dom} s_1 \neq \operatorname{dom} s_2$, consequently $s_1 \neq s_2$.

Case 3: $m_1 \neq m_2$. Without loss of generality, we may assume that $m_1 < m_2$. Using the assumption from the second statement we can find $\omega \in \operatorname{dom} f^{m_1} \cap \operatorname{ran} f^{n_1}$ such that $\omega \notin \operatorname{dom} f^{m_2}$. Then, we have

$$s_1(\omega) = f^{m_1} f^{-(m_1+n_1)} f^{n_1-k_1}(\omega) = f^{-n_1} f^{n_1-k_1}(\omega) = f^{-k_1}(\omega),$$

and $\omega \in \operatorname{dom} s_1$. At the same time, since

$$s_2(\omega) = f^{m_2} f^{-(m_2+n_2)} f^{n_2-k_2}(\omega).$$

and $\omega \notin \operatorname{dom} f^{m_2}$, we have $s_2(\omega) = \varnothing$ and $\omega \notin \operatorname{dom} s_2$. Hence, $\operatorname{dom} s_1 \neq \operatorname{dom} s_2$ and $s_1 \neq s_2$.

II. Assume ad absurdum that (iii) does not hold but the inverse subsemigroup generated by f is free. Then there exists $n \geq 1$ such that in the cycle–chain decomposition of f the lengths of finite chains are bounded by n. Then the cycle–chain decomposition of f^n contains only trivial finite chains. Moreover, besides cycles it can contain only infinite left chains or only infinite right chains. Without loss of generality we assume that it contains no infinite left chain. Then the inverse subsemigroup generated by f^n is either a bicyclic semigroup or a cyclic group. This contradicts the assumption that the inverse subsemigroup generated by f is free inverse.

III. The equivalence of the second and the third statements is straightforward. □

As an immediate corollary we generalize the example from [13, IX.1.11]. For arbitrary disjoint infinite subsets $A = \{a_i : i \geq 1\}$ and $B = \{b_i : i \geq 1\}$ of $\mathbb{Z}$ define $f_{A,B} : \mathbb{Z} \to \mathbb{Z}$ by

$$f_{A,B}(x) = \begin{cases} a_{i+1}, & \text{if } x = a_i, i \geq 1; \\ b_{i-1}, & \text{if } x = b_i, i \geq 2; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then $f_{A,B}$ is a partial one-to-one mapping. Its cycle-chain decomposition consists of a left infinite chain, a right infinite chain and trivial finite chains. Applying Theorem 1, we obtain that the inverse subsemigroup of $IS(\mathbb{Z})$, generated by $f_{A,B}$, is monogenic free inverse.

3. Partial automorphisms of regular rooted trees. Let X be a finite set, $n = |\mathsf{X}| \geq 2$. The set

$$\mathsf{X}^* = \bigcup_{m=0}^{\infty} \mathsf{X}^m$$

of all finite words over X including the empty word Λ is a free monoid under concatenation. The length of a word $w \in \mathsf{X}^*$ will be denoted by $|w|$. The set X^* is the set of vertices of an n-regular rooted tree $\mathcal{T}_n$. The empty word is the root of $\mathcal{T}_n$, and two words $u, v \in \mathsf{X}^*$ are connected by an edge if and only if $u = vx$ or $v = ux$ for some $x \in \mathsf{X}$.

A partial automorphism of the tree $\mathcal{T}_n$ is a partial one-to-one correspondence on X^* which is defined on a subtree containing root and which preserves the structure of the rooted tree, i.e. the root is a fixed point and images of connected vertices are connected whenever they are well defined. All partial automorphisms of $\mathcal{T}_n$ form an inverse semigroup $\operatorname{PAut}\mathcal{T}_n$ under superposition. All total automorphisms form the group of automorphisms of $\mathcal{T}_n$ denoted by $\operatorname{Aut}\mathcal{T}_n$. Since each partial automorphism f preserves lengths of words, the cycle-chain decomposition of f consists of finite cycles and chains. It consists of cycles if and only if f is total.

One of the ways to define partial automorphisms of $\mathcal{T}_n$ is given by partial invertible automata over X. A partial invertible automaton over X is a triple $\mathcal{A} = (Q, \lambda, \mu)$, where Q is the set of states, $\lambda : Q \times \mathsf{X} \to Q$ is the partial transition function and $\mu : Q \times \mathsf{X} \to \mathsf{X}$ is the partial output function, such that the following two conditions hold:

1. the domains of partial transition and partial output functions are equal;
2. for arbitrary $q \in Q$, $\mu(q, \cdot)$ defines a partial one-to-one mapping on X.

A partial invertible automaton $\mathcal{A}$ is called finite if the set Q of its states is finite. Partial functions λ and μ admit recursive extensions to the set $Q \times \mathsf{X}^*$, defined by the following rules: for $q \in Q$, $\lambda(q, \Lambda) = q$, $\mu(q, \Lambda) = \Lambda$; for $x \in \mathsf{X}$, $w \in \mathsf{X}^*$,

$$(q, xw) \in \operatorname{dom}\lambda \iff (q, xw) \in \operatorname{dom}\mu \iff (q, x) \in \operatorname{dom}\mu, (\lambda(q, x), w) \in \operatorname{dom}\mu,$$

and

$$\lambda(q, xw) = \lambda(\lambda(q, x), w), \quad \mu(q, xw) = \mu(q, x)\mu(\lambda(q, x), w),$$

whenever $(q, xw) \in \operatorname{dom}\mu$. Given this extension, for every state $q \in Q$, $\mu(q, \cdot)$ defines a partial mapping on X^*:

$$f_{\mathcal{A},q}(w) = \mu(q, w), \quad w \in \mathsf{X}^*.$$

It can be directly verified that for each $f \in \mathrm{PAut}\,\mathcal{T}_n$ there exists a partial invertible automaton $\mathcal{A}$ over X and its state q such that $f = f_{\mathcal{A},q}$. In this case we say that $\mathcal{A}$ defines f at its state q. If there exists a finite invertible automaton defining f at some of its states, then f is called finite state partial automorphism. All finite state partial automorphisms form a countable inverse subsemigroup $\mathrm{FPAut}\,\mathcal{T}_n$ of $\mathrm{PAut}\,\mathcal{T}_n$. All finite state total automorphisms form a countable subgroup $\mathrm{FAut}\,\mathcal{T}_n$ of $\mathrm{Aut}\,\mathcal{T}_n$.

As the main application of Theorem 1 we now construct monogenic free inverse semigroups generated by partial automorphisms of regular rooted trees.

We start with some auxiliary definitions. Let $f \in \mathrm{PAut}\,\mathcal{T}_n$. Consider an automaton $\mathcal{A} = (Q, \lambda, \mu)$ over X such that $f = f_{\mathcal{A},q}$ for some state $q \in Q$. The state $r \in Q$ is called essential with respect to q if there exists a word $w \in \mathsf{X}^*$ such that $\lambda(q, w) = r$. Denote by $\mathrm{Ess}(\mathcal{A}, q)$ the set of all essential states with respect to q. Then the set of partial automorphisms

$$\{f_{\mathcal{A},r} \colon r \in \mathrm{Ess}(\mathcal{A}, q)\}$$

does not depend on the automaton $\mathcal{A}$ and its state q. This set is denoted by $Q(f)$ and its elements are called sections of f. For each section $g \in Q(f)$ define the frequency sequence $t_g(m)$, $m \geq 0$, of g as follows

$$t_g(m) = |\{w \in \mathsf{X}^m \colon f_{\mathcal{A},\lambda(q,w)} = g\}|, \quad m \geq 0.$$

Let $\mathcal{A} = (Q, \lambda, \mu)$ be a partial invertible automaton over X. Assume that Y is a finite alphabet such that $\mathsf{X} \subset \mathsf{Y}$. A partial invertible automaton $\mathcal{A}_1 = (Q, \lambda_1, \mu_1)$ over Y is called an extension of $\mathcal{A}$ if restrictions of λ_1 and μ_1 to $Q \times \mathsf{X}$ coincide with λ and μ correspondingly.

A sequence a_m, $m \geq 0$, of non-negative integers is called non-finitary if the set of indices m, such that $a_m > 0$, is infinite. The sequence a_m, $m \geq 0$, is called polynomially bounded if there exists a polynomial $p(x)$ with integer coefficients such that for some $M > 0$ the inequalities $a_m \leq p(m)$, $m \geq M$, hold.

Theorem 2. *Let $\mathcal{A} = (Q, \lambda, \mu)$ be an invertible automaton over X, $|\mathsf{X}| = n \geq 2$, $f \in \mathrm{Aut}\,\mathcal{T}_n$ be a level transitive automorphism defined by $\mathcal{A}$ at its state q. Assume that there exists a section $g \in Q(f)$ such that the frequency sequence $t_g(m)$, $m \geq 0$, is non-finitary and polynomially bounded. Then there exists an extension $\mathcal{A}_1$ of $\mathcal{A}$ over an alphabet Y, $|\mathsf{Y}| = n + 2$, such that the inverse semigroup generated by the partial automorphism, defined by $\mathcal{A}_1$ at q, is monogenic free inverse.*

Proof. Since $g \in Q(f)$, the automaton $\mathcal{A}$ defines g at some of its states. We may assume that such a state is unique. Denote by r the state of $\mathcal{A}$ such that $f_{\mathcal{A},r} = g$.

Let y_0, y_1 be different symbols not contained in X. Then define $\mathsf{Y} = \mathsf{X} \cup \{y_0, y_1\}$ and the automaton $\mathcal{A}_1 = (Q, \lambda_1, \mu_1)$ over Y such that for arbitrary $y \in \mathsf{Y}, a \in Q$:

$$\lambda_1(a, y) = \begin{cases} \lambda(a, y), & \text{if } y \in \mathsf{X}; \\ \varnothing, & \text{if } a = r, y = y_1; \\ a, & \text{otherwise} \end{cases}, \quad \mu_1(a, y) = \begin{cases} \mu(a, y), & \text{if } y \in \mathsf{X}; \\ y_1, & \text{if } a = r, y = y_0; \\ \varnothing, & \text{if } a = r, y = y_1; \\ x, & \text{otherwise} \end{cases}.$$

The automaton $\mathcal{A}_1$ is a partial invertible automaton and it is an extension of $\mathcal{A}$. Denote by f_1 the partial automorphism, defined by $\mathcal{A}_1$ at q. It is sufficient to show that f_1 as a one-to-one mapping on Y^* satisfies the third condition of Theorem 1.

Let $l \geq 1$. We will show that f_1 contains a chain of length greater than l. Since the restriction of f_1 on X^* coincides with f and f is level transitive, the cycle-chain decomposition of f_1 for each $m \geq 1$ contains a cycle on the set X^m. The length of this cycle is n^m. Since the frequency sequence $t_g(m)$, $m \geq 0$, is non-finitary, the set M of indices m such that $f_{\mathcal{A},\lambda(q,u)} = g$ for some $u \in \mathsf{X}^m$ is infinite. For each $m \in M$, denote by W_m the subset of all $u \in \mathsf{X}^m$ such that $f_{\mathcal{A},\lambda(q,u)} = g$. Since the frequency sequence $t_g(m)$, $m \geq 0$, is polynomially bounded there exist $m \in M$ and words $u_1, u_2 \in W_m$ such that

$$f^k(u_1) = u_2 \text{ and } k > l.$$

Consider the word $w_1 = u_1 y_0$. The definition of the automaton $\mathcal{A}_1$ immediately implies that

$$g^i(w_1) = f^i(u_1)y_1, \quad 1 \leq i \leq k-1, \quad g^k(w_1) = \varnothing.$$

Since $k > l$ we obtain a chain of required length. The proof is complete. □

As an example let us apply Theorem 2 to the n-ary adding machine (see e.g. [4]). Consider $\mathsf{X} = \{x_0, \ldots, x_{n-1}\}$ and the invertible automaton $\mathcal{A} = (\{q_0, q_1\}, \lambda, \mu)$ over X such that

$$\lambda(q_0, x_i) = q_1, 0 \leq i \leq n-2, \quad \lambda(q_0, x_{n-1}) = q_0, \quad \lambda(q_1, x) = q_1, x \in \mathsf{X},$$

$$\mu(q_0, x_i) = x_{i+1}, 0 \leq i \leq n-2, \quad \mu(q_0, x_{n-1}) = x_0, \quad \mu(q_1, x) = x, x \in \mathsf{X}.$$

Then $\mathcal{A}$ defines at its state q_0 the level transitive automorphism f, called n-ary adding machine. For $w \in \mathsf{X}^*$ the equality $\lambda(q_0, w) = q_0$ holds if and only if $w = x_{n-1} \ldots x_{n-1}$. It means that for the section $f \in Q(f)$, the frequency sequence $t_f(m)$, $m \geq 0$, is constant, i.e. $t_f(m) = 1$, $m \geq 0$. Hence, it is non-finitary and polynomially bounded. Applying the construction from the proof of Theorem 2 to $\mathcal{A}$ we obtain an extension $\mathcal{A}_1$ of $\mathcal{A}$ over an alphabet $\mathsf{X} \cup \{y_0, y_1\}$ such that the inverse semigroup generated by the partial automorphism f_1, defined by $\mathcal{A}_1$ at q_0, is monogenic free inverse.

4. Open questions. In [2, Example 23] the authors construct a partial invertible automaton with two states over an alphabet of cardinality four such that the inverse semigroup generated by partial automorphisms at its states is monogenic free inverse monoid.

Problem 1. *Does there exist a finite partial invertible automaton over an alphabet of cardinality two such that the inverse semigroup generated by partial automorphisms at its states is monogenic free inverse semigroup (monoid)?*

A subsemigroup $S \subseteq \mathrm{PAut}\,\mathcal{T}_n$ is called level transitive if S acts transitively on sets X^m, $m \geq 0$. It is called level semi-transitive [1] if S acts semi-transitively on sets X^m, $m \geq 0$. If S is level transitive then it is level semi-transitive, but not vice versa. If semigroup S is inverse then its level semi-transitivity implies level transitivity. Let $f \in \mathrm{PAut}\,\mathcal{T}_n$ be a partial automorphism such that f is not total. Then the monogenic inverse semigroup generated by f is not level transitive.

Problem 2. *Describe minimal under inclusion level transitive inverse semigroups (level semitransitive semigroups) that contain monogenic free inverse semigroups constructed in Theorem 2.*

Finite automata is a natural source of numerous algorithmic problems. Among others let us mention the following one.

Problem 3. *Does there exist an algorithm that decides whether the inverse semigroup generated by the partial automorphism defined by a given finite partial automaton at its state is monogenic free inverse?*

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Taras Shevchenko National University of Kyiv
Kyiv Ukraine
ekochubinska@knu.ua
aolijnyk@gmail.com

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