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ALMOST PERIODIC DISTRIBUTIONS AND CRYSTALLINE MEASURES

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We study temperate distributions and measures with discrete support in Euclidean space and their Fourier transforms with special attention to almost periodic distributions. In particular, we prove that if distances between points of the support of a measure do not quickly approach 0 at infinity, then this measure is a Fourier quasicrystal (Theorem 1).

We also introduce a new class of almost periodicity of distributions, close to the previous one, and study its properties. Actually, we introduce the concept of s-almost periodicity of temperate distributions. We establish the conditions for a measure μ to be s-almost periodic (Theorem 2), a connection between s-almost periodicity and usual almost periodicity of distributions (Theorem 3). We also prove that the Fourier transform of an almost periodic distribution with locally finite support is a measure (Theorem 4), and prove a necessary and sufficient condition on a locally finite set E for each measure with support on E to have s-almost periodic Fourier transform (Theorem 5).

1. Introduction. The Fourier quasicrystal concept was inspired by the experimental discovery of nonperiodic atomic structures with diffraction patterns consisting of spots, made in the middle of 80's. A number of papers has appeared, in which the properties of Fourier quasicrystals are studied. Conditions for support of Fourier quasicrystals to be a finite union of discrete lattices were found, and nontrivial examples of Fourier quasicrystals were constructed ([2, 4–6, 9–11, 13–16, 18–20, 22])

These studies have been extended to a more general setting of temperate distributions with discrete support and spectrum ([3, 7, 8, 17, 21]). Note that these studies used, most often implicitly, the properties of almost periodic measures and distributions. The goal of the paper proposed here is to make these connections explicit.

The structure of this article is as follows. In Section 2 we give the necessary notation and definitions. In Section 3 we prove some properties of distributions with locally finite support and spectrum. These results are very close to the results of [3]. Here we also prove that if distances between points of the support of a measure do not quickly approach 0 at infinity, then this measure is a Fourier quasicrystal (Theorem 1). In Section 4 we introduce the concept of s-almost periodicity of temperate distributions. Here we show conditions for a measure μ to be s-almost periodic (Theorem 2), a connection between s-almost periodicity and usual almost periodicity of distributions (Theorem 3). We also prove that the Fourier transform of an almost periodic distribution with locally finite support is a measure (Theorem 4), and prove a necessary and sufficient condition on a locally finite set E for each measure

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with support on E to have s-almost periodic Fourier transform (Theorem 5). In Section 5 we give proofs of theorems from Section 4.

2. Notation and definitions. Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\varphi\in C^\infty(\mathbb{R}^d)$ with the finite norms

$$N_{n,m}(\varphi)=\sup_{\mathbb{R}^d}\{\max\{1,|x|^n\}\max_{\|k\|\le m}|D^k\varphi(x)|\},\quad n,m=0,1,2,\dots,$$

where

$$k=(k_1,\dots,k_d)\in(\mathbb{N}\cup\{0\})^d,\ \ \|k\|=k_1+\dots+k_d,\ \ D^k=\partial_{x_1}^{k_1}\dots\partial_{x_d}^{k_d}.$$

These norms generate the topology on $S(\mathbb{R}^d)$. Elements of the space $S^*(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called temperate distributions. For each temperate distribution f there are $C<\infty$ and n, $m\in\mathbb{N}\cup\{0\}$ such that for all $\varphi\in S(\mathbb{R}^d)$

$$|f(\varphi)|\le CN_{n,m}(\varphi). \tag{1}$$

Moreover, this estimate is sufficient for the distribution f to belong to $S^*(\mathbb{R}^d)$ (see [25, Ch.3]).

The Fourier transform of a temperate distribution f is defined by the equality

$$\hat f(\varphi)=f(\hat\varphi)\quad\text{for all}\quad \varphi\in S(\mathbb{R}^d),$$

where

$$\hat\varphi(y)=\int_{\mathbb{R}^d}\varphi(x)\exp\{-2\pi i\langle x,y\rangle\}dx$$

is the Fourier transform of the function φ. By $\check\varphi$ we denote the inverse Fourier transform of φ. The Fourier transform is the bijection of $S(\mathbb{R}^d)$ on itself and the bijection of $S^*(\mathbb{R}^d)$ on itself. The support of $\hat f$ is called *spectrum* of f.

We will say that a set $A\subset\mathbb{R}^d$ is *locally finite* if the intersection of A with any ball is finite, A is *relatively dense* if there is $R<\infty$ such that A intersects with each ball of radius R, and A is *uniformly discrete*, if A is locally finite and has a strictly positive separating constant

$$\eta(A):=\inf\{|x-x'|:\ x,\ x'\in A,\ x\ne x'\}.$$

Also, we will say that A is *polynomially discrete*, or shortly *p-discrete*, if there are positive numbers c,h such that

$$|x-x'|\ge c\min\{1,\ |x|^{-h},|x'|^{-h}\}\qquad\forall x,x'\in A,\quad x\ne x'. \tag{2}$$

A set A is of *bounded density* if it is locally finite and

$$\sup_{x\in\mathbb{R}^d}\#A\cap B(x,1)<\infty.$$

As usual, $\#E$ is a number of elements of the finite set E, and $B(x,r)$ is the ball with center at the point x and radius r.

An element $f\in S^*(\mathbb{R}^d)$ is called *a crystalline measure* if f and $\hat f$ are complex-valued measures on $\mathbb{R}^d$ with locally finite supports.

Denote by $|\mu|(A)$ the variation of the complex-valued measure μ on A. If both measures $|\mu|$ and $|\hat\mu|$ have locally finite supports and belong to $S^*(\mathbb{R}^d)$, we say that μ is a *Fourier quasicrystal*. A measure $\mu=\sum_{\lambda\in\Lambda}a_\lambda\delta_\lambda$ with $a_\lambda\in\mathbb{C}$ and countable Λ is called *purely point*, here δ_y is the unit mass at the point $y \in \mathbb{R}^d$. If this is the case, we will replace a_λ with $\mu(\lambda)$ and write $\operatorname{supp}\mu = \{\lambda : \mu(\lambda) \neq 0\}$.

3. Temperate distributions with locally finite support. By [24], every distribution f on $\mathbb{R}^d$ with locally finite support Λ has the form

$$f = \sum_{\lambda\in\Lambda}\left(\sum_{\|k\|\le K_\lambda} p_k(\lambda)D^k\right)\delta_\lambda \quad \text{with} \quad p_k(\lambda)\in\mathbb{C}, \quad K_\lambda<\infty.$$

Note that $\operatorname{ord} f = \sup_\lambda K_\lambda(x) \le \infty$.

Proposition 1. *Suppose $f \in S^*(\mathbb{R}^d)$ has a locally finite support Λ. Then*
i) ord $f < \infty$, hence,

$$f = \sum_{\lambda\in\Lambda}\sum_{\|k\|\le K} p_k(\lambda)D^k\delta_\lambda, \quad k\in(\mathbb{N}\cup\{0\})^d, \quad K = \mathit{ord}\, f; \tag{3}$$

in particular, if f has a locally finite spectrum Γ, then ord $\hat f < \infty$ and

$$\hat f = \sum_{\gamma\in\Gamma}\sum_{\|j\|\le J} q_j(\gamma)D^j\delta_\gamma, \quad j\in(\mathbb{N}\cup\{0\})^d, \quad J = \mathit{ord}\, \hat f. \tag{4}$$

ii) If Λ is p-discrete, then there exist C, $T<\infty$ such that for all k

$$|p_k(\lambda)| \le C\max\{1, |\lambda|^T\} \quad \textit{for all } \lambda\in\Lambda. \tag{5}$$

Moreover, there exists $T_1<\infty$ such that

$$\sum_{\lambda\in\Lambda, |\lambda|<R}\ \sum_{\|k\|\le K} |p_k(\lambda)| = O(R^{T_1}) \quad \text{as} \quad R\to\infty. \tag{6}$$

Proof of Proposition 1. i) Let $\lambda\in\Lambda$ and $\varepsilon\in(0,1)$ be such that

$$\inf\{|\lambda-\lambda'| : \lambda'\in\Lambda,\ \lambda'\neq\lambda\} > \varepsilon.$$

Let φ be a non-negative function on $\mathbb{R}$ such that

$$\varphi(|x|)\in C^\infty(\mathbb{R}^d), \quad \varphi(|x|)=0 \text{ for } |x|>1/2, \quad \varphi(|x|)=1 \text{ for } |x|\le 1/3. \tag{7}$$

Then set

$$\varphi_{\lambda,k,\varepsilon}(x) = \frac{(x-\lambda)^k}{k!}\varphi\left(\frac{|x-\lambda|}{\varepsilon}\right)\in S(\mathbb{R}^d),$$

where, as usual, $k! = k_1!\cdots k_d!$. It is easily shown that

$$f(\varphi_{\lambda,k,\varepsilon}) = (-1)^{\|k\|}p_k(\lambda).$$

Let f satisfy (1) with some m, n. We get

$$|f(\varphi_{\lambda,k,\varepsilon})| \le C\sup_{|x-\lambda|<\varepsilon}\max\{1,|x|^n\}\sum_{\|\alpha+\beta\|\le m} c(\alpha,\beta)\left|D^\alpha\varphi\left(\frac{|x-\lambda|}{\varepsilon}\right)D^\beta\left(\frac{(x-\lambda)^k}{k!}\right)\right|,$$

where $\alpha,\beta\in(\mathbb{N}\cup\{0\})^d$ and $c(\alpha,\beta)<\infty$. Note that

$$\left|D^\alpha\varphi\left(\frac{|x-\lambda|}{\varepsilon}\right)\right|\le\varepsilon^{-\|\alpha\|}c(\alpha)\quad\text{for}\quad|\lambda-x|<\varepsilon/3,$$

and this derivative vanishes for $|\lambda-x|\ge\varepsilon/2$. Also,

$$D^\beta(x-\lambda)^k=\begin{cases}0 & \text{if } k_j<\beta_j \text{ for at least one } j;\\ c(k,\beta)(x-\lambda)^{k-\beta} & \text{if } k_j\ge\beta_j\ \forall j.\end{cases}.$$

Since

$$\max\{1,|x|^n\}\le2^n\max\{1,|\lambda|^n\}$$

for $x\in\mathrm{supp}\varphi_{\lambda,k,\varepsilon}$, we get

$$|p_k(\lambda)|\le\sum_{\|\alpha+\beta\|\le m,\,\beta_j\le k_j\,\forall j}c(k,\alpha,\beta)\max\{1,|\lambda|^n\}\varepsilon^{\|k\|-\|\alpha+\beta\|}.$$

For $\|k\|>m$ we take $\varepsilon\to0$ and obtain $p_k(\lambda)=0$.

Since $\hat f\in S^*(\mathbb{R}^d)$, we obtain (4).

ii) Let Λ be p-discrete and (5) be not satisfy. Then there is k, $\|k\|\le K$, and a sequence $\lambda_s\to\infty$ such that $|\lambda_{s+1}|>1+|\lambda_s|$ for all s and

$$\frac{\log|p_k(\lambda_s)|}{\log|\lambda_s|}\to\infty,\quad s\to\infty. \tag{8}$$

Put $\beta_s=c|\lambda_s|^{-h}$ with c from (2) and

$$\psi_{s,k}(x)=\frac{(x-\lambda_s)^k}{k!}\varphi\left(\frac{|x-\lambda_s|}{\beta_s}\right),\qquad\Psi_k(x)=\sum_{s=1}^\infty\frac{\psi_{s,k}(x)}{p_k(\lambda_s)}.$$

We can assume that λ_1 is so large that $\mathrm{supp}\,\psi_{s,k}\cap\mathrm{supp}\,\psi_{s',k}=\varnothing,\ s\ne s'$. Then by (8),

$$1/p_k(\lambda_s)=o(1/|\lambda_s|^T),\quad|\lambda_s|\to\infty,\quad\text{for every } T<\infty.$$

Since

$$D^j(\psi_{s,k}(x))=O(|\lambda_s|^{h\|j\|}),\ j\in(\mathbb{N}\cup\{0\})^d,$$

we see that

$$D^j(\Psi_k(x))=o(1/|x|^{T-h\|j\|}),\quad x\to\infty,$$

and $\Psi_k\in S(\mathbb{R}^d)$.

Since Λ is p-discrete, we get $\lambda\notin B(\lambda_s,c|\lambda_s|^{-h})$ for all $\lambda\in\Lambda\setminus\{\lambda_s\}$. Therefore, $f(\Psi_k)$ is equal to

$$\sum_{\lambda\in\Lambda}\sum_{\|l\|\le K}\sum_s(-1)^{\|l\|}p_l(\lambda)p_k(\lambda_s)^{-1}D^l(\psi_{s,k})(\lambda)=\sum_s\sum_{\|l\|\le K}(-1)^{\|l\|}p_l(\lambda_s)p_k(\lambda_s)^{-1}D^l(\psi_{s,k})(\lambda_s).$$

Since $D^l(\psi_{s,k})(\lambda_s)=0$ for $l\ne k$ and $D^k(\psi_{s,k})(\lambda_s)=1$, we obtain the contradiction. □

Estimate (6) follows immediately from (5) and the following simple lemma:

Lemma 1 (cf. [8], a part of the proof of Theorem 6). *If S is p-discrete set, then $\#S \cap B(0,R) = O(R^{T'})$ as $R \to \infty$ with $T' < \infty$.*

Remark. V. Palamodov [21] proved Proposition 1 for temperate distributions with uniformly discrete support.

Proposition 2. *Let $\mu \in S^*(\mathbb{R}^d)$ be a measure. Then $|\mu|$ belongs to $S^*(\mathbb{R}^d)$ if and only if there is $T < \infty$ such that $|\mu|(B(0,R)) = O(R^T)$ as $R \to \infty$.*

Indeed, any non-negative measure ν on $\mathbb{R}^d$ satisfying the condition $\nu(B(0,R)) = O(R^T)$ as $R \to \infty$ belongs to $S^*(\mathbb{R}^d)$ (cf. [24]). The converse statement see, for example, [6, Lemma 1].

It follows from Propositions 1 and 2.

Theorem 1 (cf. [8]). *If $\mu \in S^*(\mathbb{R}^d)$ is a measure with p-discrete support, then $|\mu| \in S^*(\mathbb{R}^d)$. In particular, every crystalline measure with p-discrete support and p-discrete spectrum is a Fourier quasicrystal.*

M. Kolountzakis, J. Lagarias proved in [10] that the Fourier transform of every measure μ on the line $\mathbb{R}$ with locally finite support of bounded density, bounded masses $\mu(x)$, and locally finite spectrum is also a measure

$$\hat{\mu} = \textstyle\sum_{\gamma\in\Gamma} q_\gamma \delta_\gamma$$

with uniformly bounded q_γ. The following proposition generalizes this result for distributions from $S^*(\mathbb{R}^d)$.

Proposition 3. *Suppose $f \in S^*(\mathbb{R}^d)$ has form (3) with some K and countable Λ, and $\hat{f}$ has form (4) with the locally finite support Γ. If*

$$\rho_f(r) := \sum_{|\lambda|<r} \sum_{\|k\|\le K} |p_k(\lambda)| = O(r^{d+H}), \quad r \to \infty, \quad H \ge 0,$$

then $\operatorname{ord} \hat{f} \le H$; if $\rho_f(r) = o(r^{d+H})$ as $r \to \infty$, then $\operatorname{ord} \hat{f} < H$.

Furthermore, in the case of integer H and $\|j\| = H$ we get

$$|q_j(\gamma)| \le C' \max\{1, |\gamma|^K\};$$

for the case of uniformly discrete Γ this estimate with the same K takes place for all j.

Corollary 1. *If $f \in S^*(\mathbb{R}^d)$ has form (3) with countable Λ, locally finite spectrum Γ, and $\rho_f(r) = O(r^d)$ as $r \to \infty$, then $\hat{f}$ is a measure, and*

$$\hat{f} = \sum_{\gamma\in\Gamma} q(\gamma)\delta_\gamma, \ \ |q(\gamma)| \le C' \max\{1, |\gamma|^K\}.$$

Proof of Proposition 3. Let $\gamma \in \Gamma$ and pick $\varepsilon \in (0,1)$ such that

$$\inf\{|\gamma - \gamma'| : \gamma' \in \Gamma, \gamma' \ne \gamma\} > \varepsilon.$$

Let φ be the same as in the proof of Proposition 1. Put

$$\varphi_{\gamma,l,\varepsilon}(y) = \frac{(y-\gamma)^l}{l!}\varphi(|y-\gamma|/\varepsilon) \in S(\mathbb{R}^d).$$

We have
$$(-1)^{\|l\|}q_l(\gamma) = \sum_{\|j\|\le J} q_j(\gamma)D^j\delta_\gamma(\varphi_{\gamma,l,\varepsilon}(y)) = (\hat{f},\varphi_{\gamma,l,\varepsilon}) = (f,\hat{\varphi}_{\gamma,l,\varepsilon}).$$
Note that
$$\hat{\varphi}_{\gamma,l,\varepsilon}(x) = e^{-2\pi i\langle x,\gamma\rangle}(l!)^{-1}(-2\pi i)^{-\|l\|}D^l(\widehat{\varphi(\cdot/\varepsilon)}) = c(l)e^{-2\pi i\langle x,\gamma\rangle}\varepsilon^{d+\|l\|}(D^l\hat{\varphi})(\varepsilon x).$$
Therefore,
$$D^k(\hat{\varphi}_{\gamma,l,\varepsilon})(x) = \varepsilon^{d+\|l\|}\sum_{\alpha+\beta=k} c(\alpha,\beta)D^\alpha\left[e^{-2\pi i\langle x,\gamma\rangle}\right]D^\beta[(D^l\hat{\varphi})(\varepsilon x)] =$$
$$= \sum_{\alpha+\beta=k} c(\alpha,\beta)(-2\pi i)^{\|\alpha\|}\gamma^\alpha e^{-2\pi i\langle x,\gamma\rangle}\varepsilon^{d+\|l\|+\|\beta\|}(D^{\beta+l}\hat{\varphi})(\varepsilon x).$$
Since $\hat{\varphi}(\varepsilon x)\in S(\mathbb{R}^d)$, we get for every $x\in\mathbb{R}^d$ and $n\in\mathbb{N}\cup\{0\}$
$$|D^{\beta+l}(\hat{\varphi})(\varepsilon x)| \le N_{n,\|\beta+l\|}(\hat{\varphi})(\max\{1,|\varepsilon x|^n\})^{-1}.$$
Therefore for every k, $\|k\|\le K$,
$$|D^k(\hat{\varphi}_{\gamma,l,\varepsilon})(x)| \le C(K,n)\varepsilon^{d+\|l\|}\max\{1,|\gamma|^K\}(\max\{1,|\varepsilon x|^n\})^{-1},$$
where $C(k,n)$ depends on φ. Now we may estimate $(f,\hat{\varphi}_{\gamma,l,\varepsilon})$ as
$$\left|\sum_k\sum_\lambda p_k(\lambda)D^k(\hat{\varphi}_{\gamma,l,\varepsilon})(\lambda)\right| \le C(K,n)\varepsilon^{d+\|l\|}\max\{1,|\gamma|^K\}\int_0^\infty \frac{\rho_f(dt)}{\max\{1;\,(\varepsilon t)^n\}}. \tag{9}$$
If $\rho_f(r)=O(r^{d+H})$ for $r\to\infty$, take t_0 such that
$$|\rho(t)|<C_0t^{d+H} \text{ for } t>t_0.$$
If $\rho_f(r)=o(r^{d+H})$, fix any $\eta>0$ and take $t_0=t_0(\eta)$ such that
$$|\rho(t)|<\eta t^{d+H} \text{ for } t>t_0.$$
Then pick $n>d+H$ and $\varepsilon<1/t_0$. Integrating by parts and using the estimate for $\rho_f(t)$, we obtain
$$\int_0^\infty \max\{1,(\varepsilon t)^n\}^{-1}\rho_f(dt) = \rho_f(1/\varepsilon)+\int_{1/\varepsilon}^\infty(\varepsilon t)^{-n}\rho_f(dt) \le \frac{nC_0}{\varepsilon^n}\int_{1/\varepsilon}^\infty t^{d+H-n-1}dt.$$
Therefore, the left-hand side of (9) not more than $\varepsilon^{\|l\|-H}C_0C'\max\{1,|\gamma|^K\}$, and
$$|q_l(\gamma)| \le C'C_0\max\{1,|\gamma|^K\}\varepsilon^{\|l\|-H}.$$

If $\|l\|>H$, we take $\varepsilon\to0$ and get $q_l(\gamma)=0$, hence, $J=\operatorname{ord}\hat{f}\le H$.

If H is integer, we get $|q_l(\gamma)|\le C'C_0\max\{1,|\gamma|^K\}$ for $\|l\|=H$.

If $\rho_f(r)=o(r^{d+H})$, we replace C_0 by η and note that η is arbitrary small for ε small enough. Hence, $q_l(\gamma)=0$ for $\|l\|=H$.

Finally, if Γ is uniformly discrete, we take $\varepsilon=\varepsilon_0<\eta(\Gamma)/2$ for all $\gamma\in\Gamma$ and obtain the bound
$$|q_l(\gamma)| \le \varepsilon_0^{-H}C'C_0\max\{1,|\gamma|^K\} \quad \forall l,\|l\|\le J.$$

□

4. Almost periodic distributions and their properties. Recall that a continuous function g on $\mathbb{R}^d$ is almost periodic if for any $\varepsilon > 0$ the set of ε-almost periods of g

$$\left\{\tau \in \mathbb{R}^d : \sup_{x\in\mathbb{R}^d} |g(x+\tau) - g(x)| < \varepsilon\right\}$$

is a relatively dense set in $\mathbb{R}^d$ (cf.,for example, [1]).

Almost periodic functions are uniformly bounded on $\mathbb{R}^d$. The class of almost periodic functions is closed with respect to taking absolute values, and finite linear combinations; the limit of a uniformly in $\mathbb{R}^d$ convergent sequence of almost periodic functions is also almost periodic.

A typical example of an almost periodic function is an absolutely convergent exponential sum

$$\sum c_n \exp\{2\pi i\langle x, \omega_n\rangle\}$$

with $\omega_n \in \mathbb{R}^d$, $c_n \in \mathbb{C}$.

A measure μ on $\mathbb{R}^d$ is called almost periodic if the function

$$(\psi \star \mu)(t) = \int_{\mathbb{R}^d} \psi(t-x)d\mu(x)$$

is almost periodic in $t \in \mathbb{R}^d$ for each continuous function ψ on $\mathbb{R}^d$ with compact support. A distribution $f \in S^*(\mathbb{R}^d)$ is almost periodic if the function $(\psi \star f)(t) = f(\psi(t-\cdot))$ is almost periodic in $t \in \mathbb{R}^d$ for each $\psi \in C^\infty$ with compact support (see [12, 18–20, 23]). Clearly, every almost periodic distribution has a relatively dense support. But there are measures that are almost periodic temperate distributions, but are not almost periodic as measures (see [18]).

Definition. A distribution $f \in S^*(\mathbb{R}^d)$ is *s-almost periodic*, if the function

$$(\psi \star f)(t) = f(\psi(t-\cdot))$$

is almost periodic in $t \in \mathbb{R}^d$ for each $\psi \in S(\mathbb{R}^d)$.

The following theorem plays a very important role in our investigations.

Theorem 2. *If f is a temperate distribution and its Fourier transform $\hat f$ is a purely point measure such that*

$$|\hat f|(B(0,r)) = O(r^T)$$

for $r \to \infty$ with some $T < \infty$, then f is s-almost periodic distribution.

Proof of Theorem 2. It is very easy. Let $\hat f = \sum_{\gamma\in\Gamma} b(\gamma)\delta_\gamma$, $M(r) = |\hat f|(B(0,r))$. For any $\psi \in S(\mathbb{R}^d)$ we get

$$f(\psi(t-\cdot)) = (\hat f(y), \hat\psi(y)e^{2\pi i\langle t,y\rangle}) = \sum_{\gamma\in\Gamma} b(\gamma)\hat\psi(\gamma)e^{2\pi i\langle t,\gamma\rangle}. \qquad (10)$$

Since $|\hat\psi(y)| \le N_{T+1,0}(\hat\psi)|y|^{-T-1}$ for $|y| > 1$ and

$$\sum_{\gamma\in\Gamma} |b(\gamma)||\hat\psi(\gamma)| \le C_0 + C_1\int_1^\infty r^{-T-1}M(dr) < \infty,$$

we see that the series in (10) absolutely converges, and the function $(f \star \psi)(t)$ is almost periodic. □

From Proposition 2 it follows the following corollary.

Corollary 2. *If $f \in S^*(\mathbb{R}^d)$, $\hat{f}$ is purely point measure, and $|\hat{f}| \in S^*(\mathbb{R}^d)$, then f is s-almost periodic distribution. In particular, every Fourier quasicrystal is s-almost periodic distribution.*

Using Proposition 1, we also get

Corollary 3. *Let $f \in S^*(\mathbb{R}^d)$ have p-discrete spectrum, and the Fourier transform $\hat{f}$ is a measure. Then f is s-almost periodic distribution.*

Corollary 4. *Let $f \in S^*(\mathbb{R}^d)$ of form (3) have locally finite spectrum Γ with polynomial growth of numbers $\#(\Gamma \cap B(0,r))$. If*

$$\sum_{\|k\| \le K} \sum_{|\lambda|<r} |p_k(\lambda)| = O(r^d) \quad \text{for} \quad r \to \infty,$$

then f is s-almost periodic distribution.

Indeed, by Corollary 1, $\hat{f}$ is a measure with polynomially growth coefficients.

Evidently, every s-almost periodic distribution is an almost periodic distribution too. Conversely, the following assertion is valid:

Theorem 3. *Every almost periodic (in sense of distributions) non-negative measure $\mu \in S^*(\mathbb{R}^d)$ is s-almost periodic distribution. The same implication is valid if μ is a complex-valued measure on $\mathbb{R}^d$ such that*

$$\sup_{x \in \mathbb{R}^d} |\mu|(B(x,1)) < \infty. \tag{11}$$

It is easy to check that every almost periodic in the sense of distributions measure μ under condition (11) is almost periodic in sense of measures.

Proofs of Theorem 3 and the following ones are given in the next Section 5.

Theorem 4. *If $f \in S^*(\mathbb{R}^d)$ is an almost periodic distribution with locally finite spectrum Γ, then $\hat{f}$ is a measure.*

Show that p-discreteness of support of a measure is closely connected with s-almost periodicity of its Fourier transform:

Theorem 5. *In order for each measure $\mu \in S^*(\mathbb{R}^d)$ with support in a fixed locally finite set $A \subset \mathbb{R}^d$ to have s-almost periodic Fourier transform $\hat{\mu}$, it is necessary and sufficient that A be p-discrete.*

Moreover, if $\hat{\mu} \star \psi(t)$ is bounded for all $\psi \in S(\mathbb{R}^d)$ and $\mu \in S^*(\mathbb{R}^d)$ with supp $\mu \subset A$, then A is p-discrete.

5. Proofs of Theorems 3–5.

Proof of Theorem 3. Let φ be C^∞ non-negative function with compact support such that $\varphi(x) \equiv 1$ for $x \in B(0,1)$. Since $\varphi \star \mu(t)$ is an almost periodic function, we see that it is uniformly bounded. If $\mu \geq 0$, we get

$$\mu(B(x,1)) < C$$

for all $x \in \mathbb{R}^d$. Set $\mu^t(x) := \mu(t-x)$ with $t \in \mathbb{R}^d$. For every complex-valued measure μ subject to (11) we get

$$M(r) := |\mu^t|(B(0,r)) < Cr^d$$

for all $r > 1$, where the constant C is the same for all t. Take $\psi \in S(\mathbb{R}^d)$. Then

$$|\psi(x)| \leq C_1 |x|^{-d-1}$$

for $|x| > 1$. For any $\varepsilon > 0$ there is $R < \infty$ that does not depend on t and such that

$$\left| \int_{|x|>R} \psi(x)\mu^t(dx) \right| \leq C_1 \int_R^\infty r^{-d-1} M(dr) \leq C_1(d+1) \int_R^\infty M(r) r^{-d-2} dr < \varepsilon/3.$$

Therefore for all $t \in \mathbb{R}^d$

$$\left| \int_{|t-x|>R} \psi(t-x)\mu(dx) \right| = \left| \int_{|x'|>R} \psi(x')\mu^t(dx') \right| < \varepsilon/3. \tag{12}$$

Let $\xi(x)$ be C^∞-function on $\mathbb{R}^d$ such that

$$0 \leq \xi \leq 1, \quad \xi(x) \equiv 1 \quad \text{for} \quad |x| < R, \quad \xi(x) \equiv 0 \quad \text{for} \quad |x| > R+1.$$

The function $(\xi\psi) \star \mu(t)$ is almost periodic, hence there are a relatively dense set $E \subset \mathbb{R}^d$ such that for any $\tau \in E$ and all $t \in \mathbb{R}^d$

$$|(\xi\psi) \star \mu(t+\tau) - (\xi\psi) \star \mu(t)| < \varepsilon/3.$$

Applying (12) to $(1-\xi(t+\tau))\psi(t+\tau)$ and $(1-\xi(t))\psi(t)$, we obtain

$$|\psi\star\mu(t+\tau) - \psi\star\mu(t)| \leq |(\xi\psi)\star\mu(t+\tau) - (\xi\psi)\star\mu(t)| + |(1-\xi)\psi\star\mu(t+\tau)| + |(1-\xi)\psi\star\mu(t)| < \varepsilon.$$

Hence, E is the set of ε-almost periods for the function $\psi \star \mu$. □

Proof of Theorem 4. Let f be an almost periodic temperate distribution with a locally finite spectrum Γ. By Proposition 1, $\hat{f}$ has form (4). Suppose that $J \neq 0$ and $q_{j'}(\gamma') \neq 0$ for some $\gamma' \in \Gamma$ and $j' = (j'_1, \dots, j'_d)$, $\|j'\| = J$. Without loss of generality suppose that $j'_1 \neq 0$. Set

$$e_1 = (1,0,\dots,0), \quad e_2 = (0,1,\dots,0), \dots, \; e_d = (0,\dots,0,1), \tag{13}$$

and $j'' = j' - e_1$. Pick $\varepsilon < \min\{|\gamma' - \gamma| : \gamma \in \Gamma\}$, and set

$$\varphi_{\gamma',j'',\varepsilon}(y) = \frac{(y-\gamma')^{j''}}{j''!} \varphi\left(\frac{|y-\gamma'|}{\varepsilon}\right),$$

where φ is defined in (7). We have

$$\hat{f}(e^{2\pi i\langle y,t\rangle}\varphi_{\gamma',j'',\varepsilon}(y)) = \sum_{\gamma\in\Gamma} \sum_{\|j\|\leq J} (-1)^{\|j\|} q_j(\gamma) D^j(e^{2\pi i\langle y,t\rangle}\varphi_{\gamma',j'',\varepsilon}(y))(\gamma) \tag{14}$$

Since

$$D^j(\varphi_{\gamma',j'',\varepsilon}(y))(\gamma) = \begin{cases} 0, & \text{if } \gamma \neq \gamma' \text{ or } j \neq j''; \\ 1, & \text{if } \gamma = \gamma' \text{ and } j = j'', \end{cases}$$

we see that expression (14) is equal to

$$(-1)^J q_{j'}(\gamma')2\pi i t_1 e^{2\pi i\langle\gamma',t\rangle} + (-1)^J \sum_{s=2}^{d} q_{j''+e_s}(\gamma')2\pi i t_s e^{2\pi i\langle\gamma',t\rangle} + (-1)^{J-1} q_{j''}(\gamma')e^{2\pi i\langle\gamma',t\rangle}.$$

The first summand is unbounded in $t_1 \in \mathbb{R}$, hence the function

$$f(\hat{\varphi}_{\gamma',j'',\varepsilon}(x-t)) = \hat{f}(e^{2\pi i\langle y,t\rangle}\varphi_{\gamma',j'',\varepsilon}(-y))$$

is unbounded and not almost periodic. We obtain the contradiction, therefore, $J = 0$ and $\hat{f}$ is a measure. □

To prove Theorems 5 we need the following proposition:

Proposition 4. *Let $\lambda_n, \tau_n \in \mathbb{R}^d$ be two sequences such that $\tau_n \to 0$, $|\lambda_n| > |\lambda_{n-1}| + 1$ for all n, and*

$$\frac{\log|\tau_n|}{\log|\lambda_n|} \to -\infty \quad \text{as} \quad n \to \infty. \tag{15}$$

Let μ be any measure from $S^*(\mathbb{R}^d)$ such that its restriction for each ball $B(\lambda_n, 1/(2|\lambda_n|))$ equals $|\tau_n|^{-2/3}(\delta_{\lambda_n+\tau_n} - \delta_{\lambda_n})$. Then there is $\psi \in S(\mathbb{R}^d)$ such that $\hat{\mu} \star \hat{\psi}(t)$ is unbounded. In particular, $\hat{\mu}$ is not s-almost periodic distribution.

Proof of Proposition 4. By thinning out the sequence τ_n, we can assume that for all n

$$\sum_{p<n} |\tau_p|^{-1/3} < (1/3)|\tau_n|^{-1/3}, \tag{16}$$

and

$$\sum_{p>n} |\tau_p|^{2/3} < (2/(3\pi))|\tau_n|^{2/3}. \tag{17}$$

Set

$$\psi(x) = \sum_n |\tau_n|^{1/3}\varphi(|\lambda_n||x - \lambda_n|),$$

where φ is defined in (7). By (15), $|\tau_n| = o(1/|\lambda_n|^T)$ as $n \to \infty$ for every $T < \infty$. Therefore, for all $k \in (\mathbb{N} \cup \{0\})^d$, $N \in \mathbb{N}$ we get $D^k\psi(x) = o(|\lambda_n|^{-N})$ for $x \in B(\lambda_n, 1/(2|\lambda_n|))$. Hence, $(D^k\psi)(x)(1 + |x|^N)$ is bounded on $\mathbb{R}^d$ for all N and k, i.e., $\psi \in S(\mathbb{R}^d)$. By (7), $\psi(x) = 0$ for $x \notin \cup_n B(\lambda_n, 1/(2\lambda_n))$. Hence, for every $t \in \mathbb{R}^d$

$$\hat{\mu}(\hat{\psi}(t-y)) = \mu(\psi(x)e^{-2\pi i\langle x,t\rangle}) = \sum_{n=1}^{\infty} |\tau_n|^{-1/3}[\varphi(|\tau_n||\lambda_n|)e^{-2\pi i\langle(\lambda_n+\tau_n),t\rangle} - \varphi(0)e^{-2\pi i\langle\lambda_n,t\rangle}].$$

$|\tau_n| < 1/(3|\lambda_n|)$ for large n, therefore, $\varphi(|\tau_n||\lambda_n|) = \varphi(0) = 1$. Besides, for $t = \tau_n/(2|\tau_n|^2)$

$$|e^{-2\pi i\langle(\lambda_n+\tau_n),t\rangle} - e^{-2\pi i\langle\lambda_n,t\rangle}| = |e^{-2\pi i\langle\tau_n,t\rangle} - 1| = 2.$$

Therefore,

$$|\hat{\mu}(\hat{\psi}(t-y))| \geq 2|\tau_n|^{-1/3} - 2\sum_{p<n}|\tau_p|^{-1/3} - \sum_{p>n}|\tau_p|^{-1/3}|e^{-2\pi i\langle\tau_p,t\rangle}-1|. \tag{18}$$

Taking into account (16), (17), and the estmates

$$|e^{-2\pi i\langle\tau_p,t\rangle}-1| \leq 2\pi|\tau_p||t| = \pi|\tau_p||\tau_n|^{-1},$$

we obtain that (18) is more than $2|\tau_n|^{-1/3}/3$. Hence the convolution $(\hat{\mu}\star\hat{\psi})(t)$ is unbounded on the sequence $\tau_n/(2|\tau_n|^2)$, and the distribution $\hat{\mu}$ is not s-almost periodic. □

Proof of Theorem 5. Suppose that A is a not p-discrete set. Then there are two sequences $\lambda_n, \lambda'_n \in A$ such that λ_n and $\tau_n := \lambda'_n - \lambda_n$ satisfy (15) and $|\lambda_n| > 1 + |\lambda_{n-1}|$. Check that the measure

$$\mu = \sum_n |\tau_n|^{-2/3}[\delta_{\lambda'_n} - \delta_{\lambda_n}]$$

belongs to $S^*(\mathbb{R}^d)$.

For any $\phi \in S(\mathbb{R}^d)$

$$|(\mu,\phi)| \leq \sum_n |\tau_n|^{-2/3}|\phi(\lambda'_n)-\phi(\lambda_n)| \leq \sum_n |\tau_n|^{1/3}N_{0,1}(\phi),$$

where $N_{0,1}(\phi)$ is defined in (1). By (15), $\tau_n = O(n^{-T})$ for any $T < \infty$, therefore the sum converges, μ satisfies (1), and μ is a temperate distribution. Applying Proposition 4, we obtain that $\hat{\mu}$ is not s-almost periodic.

Proposition 4 actually implies that the convolution $\hat{\mu}\star\hat{\psi}$ with some $\psi \in S(\mathbb{R}^d)$ is unbounded, which proves the last part of the theorem.

Sufficiency follows from Corollary 3. □

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REFERENCES

1. C. Corduneanu, Almost periodic functions, Second English ed. Chelsea, New-York, 1989 (Distributed by AMS and Oxford University Press).
2. S.Yu. Favorov, *Fourier quasicrystals and Lagarias' conjecture*, Proc. Amer. Math. Soc., **144** (2016), 3527–3536.
3. S.Yu. Favorov, *Tempered distributions with discrete support and spectrum*, Bulletin of the Hellenic Mathematical Society, **62** (2018), 66–79.
4. S.Yu. Favorov, *Large Fourier quasicrystals and Wiener's theorem*, Journal of Fourier Analysis and Applications, **25** (2019), №2, 377–392.
5. S.Yu. Favorov, *Local Wiener's theorem and coherent sets of frequencies*, Analysis Math., **46** (2020), №4, 737–746. DOI: 10.1007/s10476-020-0042-x

6. S.Yu. Favorov, *Uniqueness theorems for Fourier quasicrystals and temperate distributions with discrete support*, Proc. Amer. Math. Soc., **149** (2021), 4431–4440.
7. S.Yu. Favorov, *Temperate distributions with locally finite support and spectrum on Euclidean spaces*, Israel Journal of Mathematics TBD, (2022), 1–24.
8. S.Yu. Favorov, *Fourier quasicrystals and distributions on Euclidean spaces with spectrum of bounded density*, Analysis Mathematica **49** (2023), №3, 747-764. DOI 10.7471007/s10476-023-0228-0.
9. M.N. Kolountzakis, *On the structure of multiple translations tilings by polygonal regions*, Preprint, 1999, 16 p.
10. M.N. Kolountzakis, J.C. Lagarias, *Structure of tilings of the line by a function*, Duke Math. Journal, **82** (1996), 653–678.
11. J.C. Lagarias, *Geometric models for quasicrystals I.Delone set of finite type*, Discr. and Comp. Geometry, **21** (1999)
12. J.G. de Lamadrid, L.N. Argabright, Almost periodic measures, Memoirs of the AMS, №428, Providence RI, 1990, 218 p.
13. N. Lev, A. Olevskii, *Measures with uniformly discrete support and spectrum*, C.R. Acad. Sci., ser.1., **351** (2013), 599–603.
14. N. Lev, A. Olevskii, *Quasicrystals and Poisson's summation formula*, Invent. Math., **200** (2015), 585–606.
15. N. Lev, A. Olevskii, *Quasicrystals with discrete support and spectrum*, Rev. Mat. Iberoam., **32** (2016), №4, 1341–1252.
16. N. Lev, A. Olevskii, *Fourier quasicrystals and discreteness of the diffraction spectrum*, Advances in Mathematics, **315** (2017), 1–26.
17. N. Lev, G. Reti, *Crystalline temperate distribution with uniformly discrete support and spectrum*, Journal of Functional Analysis **281** (2021), №4, 109072.
18. Y. Meyer, *Quasicrystals, almost periodic patterns, mean–periodic functions, and irregular sampling*, African Diaspora Journal of Mathematics, **13** (2012), №1, 1–45.
19. Y. Meyer, *Measures with locally finite support and spectrum*, Proc. Natl. Acad. Sci. USA, **113** (2016), №12, 31523158, DOI 10.1073/pnas.1600685113
20. Y. Meyer, *Global and local estimates on trigonometric sums*, Trans. R. Norw. Soc. Sci. Lett., (2018), 1–25.
21. V.P. Palamodov, *A geometric characterization of a class of Poisson type distributions*, Journal of Fourier Analysis and Applications, **23** (2017), №5, 1227–1237.
22. Quasicrystals and discrete geometry. J. Patera ed., Fields Institute Monographs, AMS, Providence RI, 289 p.
23. L.I. Ronkin, *Almost periodic distributions and divisors in tube domains*, Zap. Nauchn. Sem. POMI, **247** (1997), 210–236.
24. W. Rudin, Functional analysis, McGraw-Hill Book Company, New York, St. Louis, Sun Francisco, 1973, 443 p.
25. V.S. Vladimirov, Equations of mathematical physics, Marcel Dekker, Inc., New-York, 1971, 418 p.

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