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NONLOCAL HYPERBOLIC STOKES SYSTEM WITH VARIABLE EXPONENT OF NONLINEARITY

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In this paper, we study the problem for a nonlinear hyperbolic Stokes system of the second order with an integral term. Sufficient conditions for the uniqueness of the weak solution of this problem are found in a bounded domain. The nonlinear term of the system contains a variable exponent of nonlinearity, which is a function of spatial variables. The problem is studied in ordinary Sobolev spaces and generalized Lebesgue spaces, which is quite natural in this case.

Introduction. In the present paper we consider the problem of the finding a pair of the functions $\{u,\pi\}$ that satisfy the following relations:

$$u_{tt}-\sum_{i,j=1}^{n}\Big(A_{ij}(x,t)u_{x_i}\Big)_{x_j}+G(x,t,u_t)+\int\limits_{\Omega}\mathfrak{M}(x,t,y)u_t(y,t)\,dy+\nabla\pi=f(x,t)\quad\text{in}\quad Q_{0,T},\tag{1}$$

$$\operatorname{div}u=0\quad\text{in}\quad Q_{0,T},\tag{2}$$

$$\int\limits_{\Omega}\pi(x,t)\,dx=0\quad\text{in}\quad(0,T),\tag{3}$$

$$u\Big|_{\partial\Omega\times[0,T]}=0,\tag{4}$$

$$u\Big|_{t=0}=u_0(x)\quad\text{in}\quad\Omega,\tag{5}$$

$$u_t\Big|_{t=0}=u_1(x)\quad\text{in}\quad\Omega,\tag{6}$$

where $u=(u_1,\dots,u_n)\colon Q_{0,T}\to\mathbb{R}^n$, $\pi\colon Q_{0,T}\to\mathbb{R}$, $\nabla\pi=(\frac{\partial\pi}{\partial x_1},\dots,\frac{\partial\pi}{\partial x_n})$, A_{ij} is some matrix,

$$G(x,t,u_t)=(g_1(x,t)|(u_1)_t|^{q(x)-2}(u_1)_t,\ \ \dots,\ \ g_n(x,t)|(u_n)_t|^{q(x)-2}(u_n)_t),$$

f is some vector, $\operatorname{div}u=\frac{\partial u_1}{\partial x_1}+\dots+\frac{\partial u_n}{\partial x_n}$, $\Omega\subset\mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega$, $n\geq 2$, $Q_{0,T}=\Omega\times(0,T)$, and $T>0$ is some number. The function $q=q(x)$ is called the variable exponent of the nonlinearity to system (1).

System of the Navier-Stokes partial differential equations describes motion and heat transfer of the viscous incompressible fluids. Besides, it is used in mathematical modelling of nature phenomena and various technical problems. That is why problems for the parabolic Navier-Stokes equations with the constant exponents of nonlinearity are widely studied in

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scientific literature (see, for example, [1]–[3], and the references given there). In papers [4]–[6] the authors introduced and investigated the hyperbolic perturbed Navier-Stokes system with the constant exponent of nonlinearity. The local-in-time existence result and global existence of smooth solutions for small data are proved.

Weak solutions of the parabolic Stokes equations with constant and variable exponents of the nonlinearities are studied in [7]–[10]. Some classes of nonlocal nonlinear problems for the equations with variable exponents of the nonlinearity are investigated in [11], [12]. The initial-boundary value problem for linear hyperbolic Stokes equations is considered in [13]. Some local energy decay estimate of solutions for such problem is obtained. Hyperbolic equations of third order with variable exponent of nonlinearity is considered in [14].

We found sufficient conditions of uniqueness of the weak solution for problem (1)–(6). As we know the weak solutions of the hyperbolic Stokes system with variable exponent of the nonlinearity and nonlocal term are not studied yet. The article is organized as follows. In Chapter 2 we derive the model of hyperbolic Stokes system on the basis of the classical system of the Navier-Stokes equations and some laws of physics. The auxiliary statements are given in Chapter 3. Finally, in Chapter 4 we formulate and prove the main statements.

1. Physical background and motivation. The classical Navier-Stokes system for the space $\mathbb{R}^n$ ($n = 3$), provided that the fluid density $\rho \equiv 1$ and in the absence of the external forces, has the following form

$$u_t - \mu\triangle u + (u,\nabla)u + \nabla\pi = 0, \quad \mathbb{R}^n \times (0,\infty), \tag{7}$$

$$\operatorname{div} u = 0, \quad \mathbb{R}^n \times (0,\infty), \tag{8}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))\colon \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^n$ is the fluid velocity vector, $\mu > 0$ is the viscosity of the medium, $\pi = \pi(x,t)\colon \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ is the fluid pressure, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator, $(u,\nabla) = u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}$.

Condition (8) is the condition of fluid incompressibility. Equation (7) arises from the general equation of motion of a continuous medium

$$u_t + (u,\nabla)u = \operatorname{div} N, \tag{9}$$

where $N = (N_{ij})$ is the stress tensor

$$(\operatorname{div} N)_i := \sum_{j=1}^{3} \frac{\partial N_{ij}}{\partial x_j}, \quad i = 1,2,3.$$

For simplicity, we drop in (9) the nonlinear term $(u,\nabla)u$ and obtain

$$u_t = \operatorname{div} N. \tag{10}$$

Let us transform (10). First, we know that

$$N_{ij} = R_{ij} + S_{ij}, \quad i,j = 1,2,3, \tag{11}$$

where $R_{ij} = -\pi\,\delta_{ij}$, δ_{ij} is the Kronecker symbol that is 1 for $i = j$ and 0 for $i \neq j$.

According to the Fourier law for tensor $S = (S_{ij})$, we have the following:

$$S_{ij} := \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,3, \tag{12}$$

where μ is the parameter that characterizes the viscosity of the medium.

Then we calculate the divergence. It is clear that

$$(\operatorname{div} R)_1 = \sum_{j=1}^{3} \frac{\partial R_{1j}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(-\pi\,\delta_{1j}\Big) = -\frac{\partial \pi}{\partial x_1}.$$

Let us similarly transform the second and third components of a vector $\operatorname{div} R$ and have that

$$\operatorname{div} R = -\nabla\pi. \tag{13}$$

In addition,

$$(\operatorname{div} S)_1 = \sum_{j=1}^{3} \frac{\partial S_{1j}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\mu\Big(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\Big)\Big) =$$

$$= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\mu\Big(\frac{\partial u_1}{\partial x_j}\Big)\Big) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\Big(\mu\Big(\frac{\partial u_j}{\partial x_1}\Big)\Big) = \mu\,\Delta u_1 + \mu\,\frac{\partial(\operatorname{div} u)}{\partial x_1}.$$

Using (8), we obtain the following: $(\operatorname{div} S)_1 = \mu\,\Delta v_1$. Let us similarly transform the second and third components of the vector $\operatorname{div} R$. We have that $\operatorname{div} S = \mu\,\Delta v$.

Finally, we obtain the (parabolic) Stokes system:

$$u_t - \mu\triangle u + \nabla\pi = 0, \quad \mathbb{R}^n \times (0,\infty), \tag{14}$$

$$\operatorname{div} u = 0, \quad \mathbb{R}^n \times (0,\infty). \tag{15}$$

Now let us move to the hyperbolic equation. First, let us replace Fourier's law (12) by Cattaneo's law (see [5, p. 195–196] and [21]), which under certain conditions better models the real course of physical processes:

$$\tau\,\frac{\partial S_{ij}}{\partial t} + S_{ij} = \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,3, \tag{16}$$

where $\tau > 0$. Then we differentiate (10) with respect to t and multiply by τ:

$$\tau u_{tt} = \tau \operatorname{div} R_t + \tau \operatorname{div} S_t. \tag{17}$$

From (16) we get that

$$\tau \operatorname{div} S_t = \operatorname{div} \tau S_t = \mu\triangle u - \operatorname{div} S. \tag{18}$$

Using (10), (11) and (13), we obtain from here that $u_t = -\nabla\pi + \operatorname{div} S$. Therefore (18) will take the form

$$\tau \operatorname{div} S_t = \mu\triangle u - u_t - \nabla\pi. \tag{19}$$

Substituting (19) into (17), we obtain the equation

$$\tau u_{tt} - \mu\triangle u + u_t = -\nabla\pi - \tau\nabla\pi_t. \tag{20}$$

Let $\widetilde{\pi} = \pi + \tau\pi_t$. Then from (20) and (15) we obtain such hyperbolic modification of the system of system (14): $\tau u_{tt} - \mu\triangle u + u_t + \nabla\widetilde{\pi} = 0$ in $\mathbb{R}^n \times (0,+\infty)$. Let us divide this equation by $\tau > 0$. We get

$$u_{tt} - a\triangle u + g u_t + \nabla\pi^* = 0, \tag{21}$$

where $a = \frac{\mu}{\tau} > 0$, $g = \frac{1}{\tau} > 0$, $\pi^* = \frac{1}{\tau}(\pi + \tau\pi) = \pi_t + \frac{1}{\tau}\pi$. Clearly, considered in the present paper system (1)–(2) is a generalization of system (21), (15).

3. Auxiliary statements. Suppose that $\mathcal{O} = \Omega$ or $\mathcal{O} = Q_{0,T}$, $\mathcal{M}(\mathcal{O})$ is a set of all measurable functions $v\colon \mathcal{O} \to \mathbb{R}$,

$$\mathcal{B}_+(\mathcal{O}) := \{\mathfrak{q} \in L^\infty(\mathcal{O}) \ \mid \ \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y) > 0\}.$$

For every $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ by definition, put

$$\mathfrak{q}_0 := \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y), \quad \mathfrak{q}^0 := \operatorname*{ess\,sup}_{y\in\mathcal{O}} \mathfrak{q}(y), \quad \rho_{\mathfrak{q}}(v;\mathcal{O}) := \int\limits_{\mathcal{O}} |v(y)|^{\mathfrak{q}(y)}\,dy, \tag{22}$$

$v \in \mathcal{M}(\mathcal{O})$. Assume that $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0 > 1$. The set

$$L^{\mathfrak{q}(y)}(\mathcal{O}) := \{v \in \mathcal{M}(\mathcal{O}) \ \mid \ \rho_{\mathfrak{q}}(v;\mathcal{O}) < +\infty\}$$

with the Luxemburg norm $||v; L^{\mathfrak{q}(y)}(\mathcal{O})|| := \inf\{\lambda > 0 \ \mid \ \rho_{\mathfrak{q}}(v/\lambda;\mathcal{O}) \le 1\}$ is called a Lebesgue space with variable exponent. Properties of the Lebesgue and Sobolev spaces with variable exponent of nonlinearity were widely studied in [15]–[20]. In particular, it is well known that if $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0 > 1$, then $L^{\mathfrak{q}(y)}(\mathcal{O})$ is the Banach space which is reflexive and separable.

The scalar product in $\mathbb{R}^n$ we denote by $(\cdot,\cdot)_{\mathbb{R}^n}$,

$$(u,v)_\Omega := \int\limits_\Omega (u(x), v(x))_{\mathbb{R}^n}\,dx, \quad u = (u_1, ..., u_n), \quad v = (v_1, ..., v_n)\colon \Omega \to \mathbb{R}^n. \tag{23}$$

Let $C_{\rm div} := \{u \in [C_0^\infty(\Omega)]^n \mid \operatorname{div} u = 0\}$, H is a closure of $C_{\rm div}$ in $[L^2(\Omega)]^n$, Z_1 is a closure of $C_{\rm div}$ in $[H^1(\Omega)]^n$. By definition, put

$$V := Z_1 \cap [L^{q(x)}(\Omega)]^n, \ \ U(Q_{0,T}) := L^2(0,T;Z_1) \cap [L^{q(x)}(Q_{0,T})]^n,$$
$$||v;V|| = ||v;Z_1|| + ||v;[L^{q(x)}(\Omega)]^n||,$$
$$||w;U(Q_{0,T})|| = ||w;L^2(0,T;Z_1)|| + ||w;[L^{q(x)}(Q_{0,T})]^n||.$$

We shall need the following assumptions:

(A): $A_{ij}, A_{ij\,t}$, are n order quadratic matrix with the elements from $L^\infty(Q_{0,T})$, $A_{ij} = A_{j\,i}$, $i,j = \overline{1,n}$, for all $\xi^1, \dots, \xi^n \in \mathbb{R}^n$ and almost all $(x,t) \in Q_{0,T}$ the next estimates are true

$$a_{00}\sum_{i=1}^n |\xi^i|^2 \le \sum_{i,j=1}^n (A_{ij}(x,t)\xi^i, \xi^j)_{\mathbb{R}^n} \le a^{00}\sum_{i=1}^n |\xi^i|^2,$$
$$\Big|\sum_{i,j=1}^n (A_{ij\,t}(x,t)\xi^i, \xi^j)_{\mathbb{R}^n}\Big| \le a^1 \sum_{i=1}^n |\xi^i|^2,$$

where $0 < a_{00} \le a^{00} < +\infty$, $0 < a^1 < +\infty$;

(G): $g_l \in L^\infty(Q_{0,T})$, $l = \overline{1,n}$,
$0 < g_0 \le g_l(x,t) \le g^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$ $(l = \overline{1,n})$;

(Q): $q \in \mathcal{B}_+(\Omega)$ and $q_0 > 1$ (see notation (22));

(E): $\mathfrak{M}$ is a quadratic matrix of order n with elements from space $L^\infty(Q_{0,T} \times \Omega)$;

(F): $f \in L^2(0,T;H)$;

(U): $u_0 \in V$, $u_1 \in H \cap [L^{q(x)}(\Omega)]^n$.

We define the operator $A(t)\colon V \to V^*$, $\mathcal{A}\colon U(Q_{0,T}) \to [U(Q_{0,T})]^*$, $E(t)\colon [L^2(\Omega)]^n \to [L^2(\Omega)]^n$, and $\mathtt{E}\colon [L^2(Q_{0,T})]^n \to [L^2(Q_{0,T})]^n$ by the rules

$$\langle A(t)z, w\rangle_V := \int\limits_\Omega \sum_{i,j=1}^n (A_{ij}(x,t) z_{x_i}(x), w_{x_j}(x))_{\mathbb{R}^n}\,dx, \ \ z,w \in V, \ \ t \in (0,T), \tag{24}$$

$$\langle \mathcal{A}u, v\rangle_{U(Q_{0,T})} := \int\limits_0^T \langle A(t)u(t), v(t)\rangle_V \, dt, \quad u, v \in U(Q_{0,T}), \tag{25}$$

$$(E(t)z)(x) := \int\limits_\Omega \mathfrak{M}(x,t,y)z(y)\, dy, \quad x \in \Omega, \quad z \in [L^2(\Omega)]^n, \tag{26}$$

$$(\mathtt{E}u)(x,t) := \big(E(t)u(t)\big)(x) = \int\limits_\Omega \mathfrak{M}(x,t,y)u(y,t)\, dy, \ (x,t) \in Q_{0,T}, \ u \in [L^2(Q_{0,T})]^n. \tag{27}$$

Remark 1 (see [10], Lemma 2)**.** If condition **(E)** holds, then defined in (26), (27) the operators $E(t)\colon [L^2(\Omega)]^n \to [L^2(\Omega)]^n$ and $\mathtt{E}\colon [L^2(Q_{0,T})]^n \to [L^2(Q_{0,T})]^n$ are linear bounded and continuous. Moreover, for all $v \in [L^2(\Omega)]^n$, $z \in [L^2(Q_{0,T})]^n$, $t \in (0,T)$, and $\tau \in (0,T]$ the following estimates are true:

$$\|\, |E(t)v|; L^2(\Omega)\| \le E^0 \|v; [L^2(\Omega)]^n\|, \tag{28}$$

$$\|\, |\mathtt{E}z|; L^2(Q_{0,\tau})\| \le E^{00} \|z; [L^2(Q_{0,\tau})]^n\|, \tag{29}$$

where constants $E^0, E^{00} > 0$ are independent of z, v, τ.

Remark 2 (see [10], Proposition 6)**.** It's easy to see that if $v \in [L^2(Q_{0,T})]^n$, then

$$\|\, |v|; L^2(Q_{0,\tau})\|^2 \le n \|v; [L^2(Q_{0,\tau})]^n\|^2, \ \tau \in [0,T]. \tag{30}$$

A pair $\{u,\, \pi\}$ is called a *weak solution* of problem (1)–(6), if $u \in L^\infty(0,T;V) \cap C([0,T];H)$, $u_t \in L^2(0,T;Z_1) \cap [L^{q(x)}(Q_{0,T})]^n \cap C([0,T];H)$, $u_{tt} \in L^2(0,T;H)$, $\pi \in L^{q^0/(q^0-1)}(Q_{0,T})$, u satisfies initial conditions (5)–(6), for every $v \in V$ and a.e. $t \in (0,T)$ we have

$$(u_{tt}(t), v)_\Omega + \langle A(t)u(t), v\rangle_V + (G(x,u_t), v)_\Omega + (E(t)u_t, v)_\Omega = (f(t), v)_\Omega, \tag{31}$$

π satisfies equality (3) in space $D^*(0,T)$ and satisfies

$$u_{tt} + \mathcal{A}u + G(x,t,u_t) + \mathtt{E}u_t + \nabla \pi = f \quad \text{in space} \quad [D^*(Q_{0,T})]^n. \tag{32}$$

4. Main results. Let us prove the following theorem.

Theorem 1 (uniqueness)**.** *Suppose that conditions* **(A)**–**(U)** *hold. Then problem (1)–(6) can not have more then one weak solution* $\{u,\, \pi\}$.

Proof. Let's assume the opposite. Let $\{u,\, \pi\}$, $\{\widetilde{u},\, \widetilde{\pi}\}$ be weak solutions to problem (1)–(6), $u \ne \widetilde{u}$, $\pi \ne \widetilde{\pi}$ on positive measure subset of $Q_{0,T}$. Take $\omega := u - \widetilde{u}$, Then, for each $\tau \in (0,T]$ it is easy to obtain the equality

$$\int\limits_0^\tau \Big[(\omega_{tt}(t), \omega_t(t))_\Omega + \langle A(t)\omega(t), \omega_t(t)\rangle_V +$$
$$+ \big(G(x,t,u_t(t)) - G(x,t,\widetilde{u}_t(t)), \omega_t(t)\big)_\Omega + (E(t)\omega_t(t), \omega_t(t))_\Omega\Big]\, dt = 0. \tag{33}$$

We get

$$\int\limits_0^\tau (\omega_{tt}(t), \omega_t(t))_\Omega \, dt = \frac{1}{2} \int\limits_{\Omega_t} |\omega_t|^2 \, dx \Big|_{t=0}^{t=\tau} = \frac{1}{2} \int\limits_{\Omega_\tau} |\omega_t|^2 \, dx;$$

$$\int\limits_0^\tau \langle A(t)\omega(t), \omega_t(t)\rangle_V \, dt =$$

$$= \frac{1}{2}\int\limits_\Omega \sum_{i,j=1}^n (A_{ij}\omega_{x_i}, \omega_{x_j})_{\mathbb{R}^n} \, dx\Big|_{t=0}^{t=\tau} - \frac{1}{2}\int\limits_{Q_{0,\tau}} \sum_{i,j=1}^n (A_{ijt}\omega_{x_i}, \omega_{x_j})_{\mathbb{R}^n} \, dxdt \geq$$

$$\geq \frac{1}{2}\int\limits_{\Omega_\tau} a_{00} \sum_{i=1}^n |\omega_{x_i}|^2 \, dx - \frac{1}{2}\int\limits_{Q_{0,\tau}} a^1 \sum_{i=1}^n |\omega_{x_i}|^2 \, dxdt.$$

From **(G)** it follows the estimate $\big(G(x,t,u_t(t)) - G(x,t,\widetilde{u}_t(t)), \omega_t(t)\big)_\Omega \geq 0$. According to the Cauchy-Bunyakovsky-Schwarz inequality, using (29) and (30), we obtain

$$\int\limits_{Q_{0,\tau}} (\mathrm{E}\omega_t, \omega_t)_{\mathbb{R}^n} \, dxdt \leq \| \, |\mathrm{E}\omega_t|; L^2(Q_{0,\tau})\| \cdot \| \, |\omega_t|; L^2(Q_{0,\tau})\| \leq$$

$$\leq n\|\mathrm{E}\omega_t; [L^2(Q_{0,\tau})]^n\| \cdot \|\omega_t; [L^2(Q_{0,\tau})]^n\| \leq nE^{00}\|\omega_t; [L^2(Q_{0,\tau})]^n\| \cdot \|\omega_t; [L^2(Q_{0,\tau})]^n\| =$$

$$= nE^{00}\|\omega_t; [L^2(Q_{0,\tau})]^n\|^2 = nE^{00}\Big(\sum_{l=1}^n \|\omega_{l\,t}; L^2(Q_{0,\tau})\|\Big)^2 \leq$$

$$\leq nE^{00}\Big(n \max_{1\leq l\leq n} \|\omega_{l\,t}; L^2(Q_{0,\tau})\|\Big)^2 = n^3 E^{00} \max_{1\leq l\leq n} \int\limits_{Q_{0,\tau}} |\omega_{l\,t}|^2 \, dxdt \leq$$

$$\leq n^3 E^{00} \int\limits_{Q_{0,\tau}} \sum_{l=1}^n |\omega_{l\,t}|^2 \, dxdt = n^3 E^{00} \int\limits_{Q_{0,\tau}} |\omega_t|^2 \, dxdt.$$

Thus, from (33) we get the equality

$$\frac{1}{2}\int\limits_{\Omega_\tau} \Big[|\omega_t|^2 + a_{00}\sum_{i=1}^n |\omega_{x_i}|^2\Big] \, dx \leq C_1 \int\limits_{Q_{0,\tau}} \Big[|\omega_t|^2 + \sum_{i=1}^n |\omega_{x_i}|^2\Big] dxdt, \quad \tau \in (0,T], \tag{34}$$

where $C_1 > 0$ is independent of τ. Let $y(\tau) = \int_{\Omega_\tau} \Big[|\omega_t|^2 + \sum\limits_{i=1}^n |\omega_{x_i}|^2\Big] dx, \quad \tau \in (0,T]$.

Then from (34) it follows that $y(\tau) \leq 2\,C_1 \int_0^\tau y(t)dt, \ \tau \in (0,T]$. Then the Gronwall-Bellman Lemma yields that $y(\tau) \leq 0$ for $\tau \in [0,T]$, hence $u = \widetilde{u}$.

From (32) we get equality $\nabla(\pi - \widetilde{\pi}) = 0$ in a sense of space $[D^*(Q_{0,T})]^n$. Then, using condition (3), we get that $\pi = \widetilde{\pi}$ and the Theorem 1 is proved. □

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