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ESTIMATES OF MATRIX SOLUTIONS OF OPERATOR EQUATIONS WITH RANDOM PARAMETERS UNDER UNCERTAINTIES

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We investigate problems of estimating solutions of linear operator equations with random parameters under conditions of uncertainty. We establish that the guaranteed rms estimates of the matrices are found as solutions of special optimization problems under certain observations of the system state. As the output signals of the system, we have observations that are described by linear functions from the solutions of such equations with random right-hand sides, which have unknown second moments. Under the condition that the observation second moments of the right-hand parts and errors belong to certain sets, it is proved that the guaranteed estimates are expressed through solutions of operator equation systems. When the linear operator is given by the scalar product of rectangular matrices, a quasi-minimax estimate and its error are constructed. It is shown that the quasi-minimax estimation error tends to zero when the number of observations tends to infinity. An example of calculating the guaranteed rms estimate of the matrix's trace, which is a solution of a matrix equation with a random parameter, is given.

Introduction. Today, the applied relevance of problems under uncertainty is an established fact, so it becomes clear why similar problems have become the subject of numerous studies by both foreign and Ukrainian researchers. The papers [1–9] are devoted to the problems of estimating matrices based on observations. One of the most convenient tools for solving problems under conditions of uncertainty turned out to be the approach of guaranteed estimation of the solutions of the studied equations.

This approach demonstrates its effectiveness for various types of uncertainty, in particular for discrete observations [11], when the observation is nonlinear (has a superposition-type operator) [12], as well as for establishing sufficient conditions for the game termination in a finite time [13].

In our previous papers [14–17], the problem of linear estimation in the space of rectangular observation matrices was solved when the known matrices are perturbed. The operator equations for the coefficients of the vector linear estimate and the dependences of the linear estimates on small perturbations of the matrix coefficients of the linear regression were obtained. Formulas for guaranteed root-mean-square errors of estimates of linear operators are also substantiated under the assumption that unknown matrices are realizations of random

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matrices with a correlation operator, which is determined from a special operator relation and belongs to a certain bounded set.

In [18, 19], constructive mathematical methods are developed for finding linear guaranteed root-mean-square estimates of unknown non-stationary parameters of average values based on observations of realizations of a sequence of random matrices, it is shown that such guaranteed estimates are obtained either as solutions of boundary value problems for systems of linear differential equations or as solutions of the corresponding Cauchy problems. Explicit expressions of the guaranteed root-mean-square errors of estimates of linear operators acting from the space of rectangular matrices into some vector space are given.

In this paper, as output signals of the system, we consider observations of linear functions from the solutions of linear operator equations with random right-hand sides that have unknown second moments. With such observations, we establish that guaranteed rms estimates of matrices are found as solutions of special optimization problems.

1. Problem statement. Let the realizations of random scalar values be observed

$$y_k = SpXC_k^T + \eta_k, k = \overline{1, N}, \tag{1}$$

where $X \in H_{m\times n}$, $H_{m\times n}$ is the matrix space of $m \times n$ dimension; $C_k \in H_{m\times n}, k = \overline{1, N}$ are known matrices; T is a transposition symbol; $\eta_k, k = \overline{1, N}$ are realizations of random variables. The matrix X is the solution of the operator equation:

$$A(\omega) X = \sum_{i=1}^{p} B_i \xi_i + A^{(0)}, \tag{2}$$

where $A(\omega)$ is a random linear operator that acts from $H_{m\times n}$ space to $H_{m\times n}$ space; $B_i,\ i = \overline{1, p}$ are known matrices from $H_{m\times n}$; $\xi_i, i = \overline{1, p}$ are scalar random variables; $A^{(0)}$ is known matrix from $H_{m\times n}$. We also assume that equation (2) with probability one has a solution X such that $E\langle X, X\rangle \triangleq ESpXX^T < \infty$; $E\xi_i = 0$ for $i = \overline{1, p}$, (here E means the sign of mathematical expectation) and the vector $\xi = (\xi_1, ..., \xi_p)^T$ does not depend on the linear random operator $A(\omega)$.

Definition 1. We call the matrix $\widehat{LX}$ of the form

$$\widehat{LX} = \sum_{k=1}^{N} U_k y_k + W,$$

where $U_k, W \in H_{m_1\times n_1}$, the *linear estimate of the matrix LX according to observations* (1), where L is a linear operator that acts from the matrix space $H_{m\times n}$ into the matrix space $H_{m_1\times n_1}$.

We assume that the distribution of the operator $A(\omega)$ is known; $E\eta_k = 0$ at $k = \overline{1, N}$; correlation matrices R_ξ, R_η are unknown and belong to bounded closed sets G_1, G_2, respectively; η does not depend on $A(\omega)$ and ξ (here $\eta = (\eta_1, ..., \eta_N)^T$, $R_\xi \triangleq E\xi\xi^T$, $R_\eta \triangleq E\eta\eta^T$).

Definition 2. The value

$$\sigma(U, W) \triangleq \left\{\sup_{G_1, G_2} ESp\left(LX - \widehat{LX}\right)\left(LX - \widehat{LX}\right)^T\right\}^{\frac{1}{2}} \tag{3}$$

is called *guaranteed rms error of estimation of the matrix $\widehat{LX}$*, where $U = (U_1, ..., U_N)$.

Definition 3. We call the matrix $\overline{\overline{\overline{LX}}} = \sum_{k=1}^{N} \hat{U}_k y_k + \hat{W}$, where $\left(\hat{U}_1, ..., \hat{U}_N, \hat{W}\right) \in$ $\in Arg \min\limits_{U,W} \sigma\left(U, W\right)$, the *guaranteed rms estimate of the matrix* LX.

The problem is to find the guaranteed mean square estimate of the matrix LX and its error.

2. Solving the problem. Let $\Phi_k, k = \overline{1, r}$, $(r = m_1 \cdot n_1)$ denote an orthonormal basis in the matrix space $H_{m_1 \times n_1}$ with scalar product: $\left\langle \Psi_{(i)}, \Psi_{(j)} \right\rangle \triangleq Sp\Psi_{(i)} \Psi_{(j)}^T$, where $\Psi_{(i)}, \Psi_{(j)}$ are arbitrary matrices from the space $H_{m_1 \times n_1}$.

I. Let us introduce sequences of matrices from the space $H_{m \times n}$: $\hat{\Psi}_k, P_k, k = \overline{1, r}$, which are solutions of systems of equations:

$$\begin{cases} A^*\left(\omega\right) \hat{\Psi}_k = L^* \Phi_k - \rho\left(C\right) \hat{u}^{(k)}, \\ A\left(\omega\right) P_k = Q \hat{\Psi}_k, \end{cases} \tag{4}$$

where

$$\hat{u}^{(k)} = R_\eta^+ \rho^*\left(C\right) \left(E P_k + \bar{A} A^{(0)} \left\langle E\hat{\Psi}_k, A^{(0)} \right\rangle\right), \ \bar{A} = E A^{-1}\left(\omega\right)$$

and operators $\rho\left(\cdot\right)$ act from the space of vectors to the space of matrices according to the rule: $\rho\left(Z\right) x = \sum_{i=1}^{s} Z_i x_i$, $x = \left(x_1, ..., x_s\right)^T$, where $Z_i, i = \overline{1, s}$ is a sequence of matrices of the same dimension, $\rho^*\left(Z\right)$ is the conjugate operator to $\rho\left(Z\right)$, $Q = \rho\left(B\right) R_\xi \rho^*\left(B\right) + A^{(0)} \otimes A^{(0)}$, R_η^+ is the pseudoinverse matrix of the correlation matrix R_η, $A^{(0)} \otimes A^{(0)}$ is the tensor product of matrices.

Lemma 1. *Let S be an arbitrary matrix from the space $H_{m_1 \times n_1}$. Then from equality*

$$\min_{u,\alpha} E\left(\left\langle S, LX \right\rangle - \left(u, y\right) - \alpha\right)^2 = E\left(\left\langle S, LX \right\rangle - \left\langle S, \hat{X}_1\left(y\right) \right\rangle\right)^2,$$

where the matrix $\hat{X}_1\left(y\right)$ does not depend on the matrix S, the equalities follow

$$\min_{U,W} \sigma^2\left(U, W\right) = E\left\langle LX - \hat{X}_1\left(y\right), LX - \hat{X}_1\left(y\right) \right\rangle = \sum_{k=1}^{r} E\left\langle \Phi_k, LX - \hat{X}_1\left(y\right) \right\rangle^2.$$

Proof. From obvious equalities

$$\widehat{LX} = \sum_{k=1}^{N} U_k y_k + W = \rho\left(U\right) y + W,$$

$$\left\langle LX - \widehat{LX}, LX - \widehat{LX} \right\rangle = \sum_{k=1}^{r} \left\langle \Phi_k, LX - \widehat{LX} \right\rangle^2 =$$

$$= \sum_{k=1}^{r} \left(\left\langle \Phi_k, LX \right\rangle - \left(\rho^*\left(U\right) \Phi_k, y\right) - \left\langle W, \Phi_k \right\rangle\right)^2$$

we have

$$E\left\langle LX - \widehat{LX}, LX - \widehat{LX} \right\rangle \geq \sum_{k=1}^{r} \min_{U,W} E\left(\left\langle \Phi_k, LX - \widehat{LX} \right\rangle\right)^2 \geq$$

$$\geq \sum_{k=1}^{r} \min_{u,\alpha} E\left(\langle \Phi_k, LX\rangle - (u,y) - \alpha\right)^2 =$$

$$= \sum_{k=1}^{r} E\left(\langle \Phi_k, LX\rangle - \left\langle \Phi_k, \hat{X}_1(y)\right\rangle\right)^2 = E\left\langle LX - \hat{X}_1(y), LX - \hat{X}_1(y)\right\rangle.$$

Thus, we can write

$$\min_{U,W} \sigma^2(U,W) = E\left\langle LX - \hat{X}_1(y), LX - \hat{X}_1(y)\right\rangle,$$

which completes the proof of the lemma. □

Proposition 1. *The equality*

$$\sigma_k^2(R_\xi, R_\eta) \triangleq E\left\langle Q\hat{\Psi}_k, \hat{\Psi}_k\right\rangle + E\left(\left\langle \hat{\Psi}_k - E\hat{\Psi}_k, A^{(0)}\right\rangle\right) + \left(R_\eta \hat{u}^{(k)}, \hat{u}^{(k)}\right)$$

holds for the guaranteed mean squared error of estimation

$$\min_{U,W} \sigma(U,W) = \left\{\max_{R_\xi, R_\eta} \sigma_{(1)}(R_\xi, R_\eta)\right\}^{\frac{1}{2}},$$

where $\sigma_{(1)}(R_\xi, R_\eta) \triangleq \sum_{k=1}^{r} \sigma_k^2(R_\xi, R_\eta)$.

Proof. Note that the function

$$\bar{\sigma}^2(U,W,R_\xi,R_\eta) \triangleq E\left\langle LX - \widehat{LX}, LX - \widehat{LX}\right\rangle$$

has the form $\bar{\sigma}^2(U,W,R_\xi,R_\eta) = \sum_{k=1}^{r} E\left\langle LX - \widehat{LX}, \Phi_k\right\rangle^2$. We obtain

$$\left\langle LX - \widehat{LX}, \Phi_k\right\rangle = \langle L^*\Phi_k, X\rangle - \langle \rho(C)\rho^*(U)\Phi_k, X\rangle - \langle \Phi_k, \rho(U)\eta\rangle$$

from the equalities

$$\left\langle LX - \widehat{LX}, \Phi_k\right\rangle = \langle LX, \Phi_k\rangle - \langle \Phi_k, \rho(U)y\rangle - \langle \Phi_k, W\rangle, \quad y = \rho^*(C)X + \eta.$$

Note that the matrix X can be written in the form $X = \sum_{i=1}^{p} X_i\xi_i + X_0$, where the matrices $X_i, i = \overline{1,p}$ are solutions of the equations $A(\omega)X_i = B_i,\ i = \overline{1,p},\ A(\omega)X_0 = A^{(0)}$.

Thus, we get

$$\left\langle LX - \widehat{LX}, \Phi_k\right\rangle = \sum_{i=1}^{p} \langle D_k(U), X_i\rangle \xi_i + \langle D_k(U), X_0\rangle - \langle \rho^*(U)\Phi_k, \eta\rangle - \langle \Phi_k, W\rangle,$$

where $D_k(U) = L^*\Phi_k - \rho(C)\rho^*(U)\Phi_k$.

For the squared error of the estimation the representation holds

$$\bar{\sigma}^2(U,W,R_\xi,R_\eta) = \sum_{k=1}^{r} \bar{\sigma}_k^2(U,W,R_\xi,R_\eta),$$

where

$$\bar{\sigma}_k^2(U,W,R_\xi,R_\eta) \triangleq \sum_{j_1,j_2=1}^{p} E\langle D_k(U), X_{j_1}\rangle \langle D_k(U), X_{j_2}\rangle E\xi_{j_1}\xi_{j_2} +$$

$$+\left(R_\eta \rho^*(U)\Phi_k, \rho^*(U)\Phi_k\right) + \left(\langle \Phi_k, W\rangle - \langle D_k(U), EX_0\rangle\right)^2 + E\left\langle D_k(U), X_0 - EX_0\right\rangle^2.$$

Since the function $\bar{\sigma}^2(U, W, R_\xi, R_\eta)$ is convex in the variables U, W and linear in the arguments R_ξ, R_η, then it has a saddle point, and therefore the condition

$$\min_{U,W}\max_{R_\xi,R_\eta}\bar{\sigma}^2(U,W,R_\xi,R_\eta) = \max_{R_\xi,R_\eta}\min_{U,W}\bar{\sigma}^2(U,W,R_\xi,R_\eta)$$

is satisfied.

First we find the value $\min\limits_{U,W}\bar{\sigma}^2(U,W,R_\xi,R_\eta)$. To do this, we prove the equality

$$\forall k \in \overline{1,r} : \min_{u,\alpha} E\left(\langle \Phi_k, LX\rangle - (u,y) - \alpha\right)^2 = E\left\langle \Phi_k, LX - \overline{\overline{LX}}\right\rangle,$$

where $\overline{\overline{LX}} = \sum_{k=1}^N \hat{U}_k y_k + \hat{W}$. Let us introduce matrices $\bar{\Psi}_k$ as solutions of equations $A^*(\omega)\bar{\Psi}_k = L^*\Phi_k - \rho(C)u$. Then we get

$$\langle \Phi_k, LX\rangle - (u,y) = \left\langle \bar{\Psi}_k, \rho(B)\xi\right\rangle + \left\langle \bar{\Psi}_k, A^{(0)}\right\rangle - (u,\eta) \Rightarrow$$
$$\Rightarrow E\left(\langle \Phi_k, LX\rangle - (u,y) - \alpha\right)^2 = E\left\langle \bar{\Psi}_k, \rho(B)\xi\right\rangle^2 +$$
$$+E\left\langle \bar{\Psi}_k - E\bar{\Psi}_k, A^{(0)}\right\rangle^2 + (R_\eta u, u) + \left(\alpha - \left\langle A^{(0)}, E\bar{\Psi}_k\right\rangle\right)^2.$$

The function $\tilde{\sigma}_k^2(u,\alpha) \triangleq E\left(\langle \Phi_k, LX\rangle - (u,y) - \alpha\right)^2$ is convex in variables u and α, which means that there are values of $\hat{u}^{(k)}$ and $\hat{\alpha}_k$ such that $\min\limits_{u,\alpha}\tilde{\sigma}_k^2(u,\alpha) = \tilde{\sigma}_k^2\left(\hat{u}^{(k)}, \hat{\alpha}_k\right)$.

It is obvious that

$$\tilde{\sigma}_k^2(u,\alpha) \geq \delta_k^2(u) \geq \min_u \delta_k^2(u) = \delta_k^2\left(\hat{u}^{(k)}\right),$$

where $\delta_k^2(u) \triangleq \tilde{\sigma}_k^2(u,\alpha_k)$, $\alpha_k = \left\langle A^{(0)}, E\bar{\Psi}_k\right\rangle$.

Let us find the optimal vector $\hat{u}^{(k)}$ from the condition $\frac{d}{dt}\delta_k^2(\hat{u}+tv)\,|_{t=0} \equiv 0$. Since the equality

$$\frac{1}{2}\frac{d}{dt}\delta_k^2(\hat{u}+tv)\,|_{t=0} = E\left\langle Q\hat{\Psi}_k, \tilde{\Psi}_k\right\rangle - \left\langle E\hat{\Psi}_k, A^{(0)}\right\rangle\left\langle E\tilde{\Psi}_k, A^{(0)}\right\rangle + \left(R_\eta \hat{u}^{(k)}, v\right)$$

holds, where $\tilde{\Psi}_k$ is the solution of the equation $A^*(\omega)\tilde{\Psi}_k = -\rho(C)v$, then we get

$$E\left\langle Q\hat{\Psi}_k, \tilde{\Psi}_k\right\rangle - E\left\langle \hat{\Psi}_k, A^{(0)}\right\rangle\left\langle E\tilde{\Psi}_k, A^{(0)}\right\rangle + \left(R_\eta \hat{u}^{(k)}, v\right) =$$
$$= -\left(\rho^*(C)\left(EP_k + E\left\langle \hat{\Psi}_k, A^{(0)}\right\rangle \bar{A}A^{(0)}\right), v\right) + \left(R_\eta \hat{u}^{(k)}, v\right) \equiv 0.$$

Hence the expression $\hat{u}^{(k)} = R_\eta^+\rho^*(C)\left(EP_k + \bar{A}A^{(0)}\left\langle E\hat{\Psi}_k, A^{(0)}\right\rangle\right)$ follows.

Taking into account the above expressions, we conclude that Proposition 1 is valid. □

Corollary 1. *Let the matrix* $A^{(0)} = 0$ *in formula* (2). *Then the equality*

$$\sigma_k^2(R_\xi, R_\eta) = \langle LEP_k, \Phi_k\rangle, \tag{5}$$

is fulfilled.

Proof. From the equality

$$\sigma_k^2(R_\xi, R_\eta) = E\left\langle \tilde{Q}\hat{\Psi}_k, \hat{\Psi}_k\right\rangle + \left(R_\eta \hat{u}^{(k)}, \hat{u}^{(k)}\right),$$

where $\hat{u}^{(k)} = R_\eta^+ \rho^* (C) EP_k$, and considering that $\tilde{Q}_k \hat{\Psi}_k = A(\omega) P_k$, where $\tilde{Q} = \rho(B) R_\xi \rho^*(B)$ we get

$$\left\langle \tilde{Q}\hat{\Psi}_k, \hat{\Psi}_k \right\rangle = \left\langle A(\omega) P_k, \hat{\Psi}_k \right\rangle = \left\langle P_k, A^*(\omega) \hat{\Psi}_k \right\rangle = \langle P_k, L^* \Phi_k \rangle - \left\langle P_k, \rho(C) \hat{u}^{(k)} \right\rangle;$$

$$\left\langle R_\eta \hat{u}^{(k)}, \hat{u}^{(k)} \right\rangle = \left\langle EP_k, \rho(C) \hat{u}^{(k)} \right\rangle.$$

Now we can claim that equality (5) holds. □

II. In the following, we will assume that there are positive definite matrices $Q_1^-, Q_1^+, Q_2^-, Q_2^+$ such that: $G_1^- \subseteq G_1 \subseteq G_1^+, G_2^- \subseteq G_2 \subseteq G_2^+$, where

$$\begin{aligned} G_1^- &= \left\{ R_\xi : SpQ_1^- R_\xi \le 1 \right\}, \ \ G_1^+ = \left\{ R_\xi : SpQ_1^+ R_\xi \le 1 \right\}, \\ G_2^- &= \left\{ R_\eta : SpQ_2^- R_\eta \le 1 \right\}, \ \ G_2^+ = \left\{ R_\eta : SpQ_2^+ R_\eta \le 1 \right\}. \end{aligned} \tag{6}$$

Proposition 2. *Let the sets $G_i^-, G_i^+, i = \overline{1,2}$ be determined by formula* (6)*. Then, for the square of the rms error of estimation, the inequalities*

$$f_1(U) + \left\langle W - E\Gamma A^{(0)}, W - E\Gamma A^{(0)} \right\rangle \le \max_{G_1, G_2} E \left\langle LX - \widehat{LX}, LX - \widehat{LX} \right\rangle \le$$
$$\le f_2(U) + \left\langle W - E\Gamma A^{(0)}, W - E\Gamma A^{(0)} \right\rangle,$$

are fulfilled, where

$$\begin{gathered} f_1(U) = \lambda_{\max}\left(V_1^-(U)\right) + \lambda_{\max}\left(V_2^-(U)\right) + f_3(U), \\ f_2(U) = \lambda_{\max}\left(V_1^+(U)\right) + \lambda_{\max}\left(V_2^+(U)\right) + f_3(U), \\ V_1^-(U) = \left(Q_1^-\right)^{-\frac{1}{2}} \tilde{R}_\xi \left(Q_1^-\right)^{-\frac{1}{2}}, V_1^+(U) = \left(Q_1^+\right)^{-\frac{1}{2}} \tilde{R}_\xi \left(Q_1^+\right)^{-\frac{1}{2}}, \\ V_2^-(U) = \left(Q_2^-\right)^{-\frac{1}{2}} \tilde{R}_\eta \left(Q_2^-\right)^{-\frac{1}{2}}, V_2^-(U) = \left(Q_2^+\right)^{-\frac{1}{2}} \tilde{R}_\eta \left(Q_2^+\right)^{-\frac{1}{2}}, \\ \tilde{R}_\xi = \rho^*(B) E\Gamma^*\Gamma\rho(B), \tilde{R}_\eta = \rho^*(U)\rho(U), \\ f_3(U) = E\left\langle (\Gamma - E\Gamma) A^{(0)}, (\Gamma - E\Gamma) A^{(0)} \right\rangle, \end{gathered}$$

and Γ is a linear operator that is the solution of the equation $\Gamma A(\omega) = L - \rho(U)\rho^*(C)$.

Proof. Let us show only he validity of the estimation from above. For the square of the error, the inequality

$$\sigma^2(U, W) \le \max_{G_1^+, G_2^+} E\left\langle LX - \widehat{LX}, LX - \widehat{LX} \right\rangle \tag{7}$$

holds. Now, taking into account equalities

$$LX - \widehat{LX} = \Gamma\rho(B)\xi + \Gamma A^{(0)} - \rho(U)\eta, \quad \max_{G_1^+} E\left(V_1^+\xi, \xi\right) = \lambda_{\max}\left(V_1^+(U)\right),$$
$$\max_{G_2^+} E\left(V_2^+\eta, \eta\right) = \lambda_{\max}\left(V_2^+(U)\right)$$

we conclude that proposition 2 is valid. □

In the following, we will assume that the sets G_1 and G_2 have the form

$$G_1 = \{R_\xi : SpQ_1 R_\xi \le 1\}, \ G_2 = \{R_\eta : SpQ_2 R_\eta \le 1\},$$

where Q_1 and Q_2 are positive definite matrices. Let us introduce a sequence of matrices $\hat{\Psi}_k$ and P_k, which are solutions of the system of equations (4) for $\hat{u}^{(k)} = Q_2\rho^*(C) EP_k$.

Proposition 3. *Let the error $\sigma(U,W)$ of estimation be determined by formula* (3). *Then the inequality* $\min\limits_{U,W} \sigma^2(U,W) \le \sum_{k=1}^{r} \langle \Phi_k, LEP_k\rangle$ *holds.*

Proof. Since the inequalities $\lambda_{\max}\left(V_i^+(U)\right) \le SpV_i^+(U)$, $i=\overline{1,2}$ holds, then for the squared error we have the estimation from above

$$\sigma^2(U,W) \le SpQ_1^{-1}\tilde{R}_\xi + SpQ_2^{-1}\tilde{R}_\eta + E\left\langle \Gamma A^{(0)}, \Gamma A^{(0)}\right\rangle + \\ + \left\langle W - E\Gamma A^{(0)}, W - E\Gamma A^{(0)}\right\rangle \triangleq f(U,W).$$

Note that the inequality

$$\min_{U,W} \sigma^2(U,W) \le \min_{U,W} f(U,W)$$

is fulfilled and the function $f(U,W)$ can also be represented in the form

$$f(U,W) = E\left\langle LX_2 - \widehat{LX_2}, LX_2 - \widehat{LX_2}\right\rangle,$$

where the matrix X_2 is the solution of the equation $A(\omega)X_2 = \rho(B)\bar{\xi} + A^{(0)}$, and the estimation $\widehat{LX_2}$ is based on the formula $\widehat{LX_2} = \rho(U)\rho^*(C)X_2 + \rho(U)\bar{\eta}$ (here the random vectors $\bar{\xi}$ and $\bar{\eta}$ are uncorrelated and such that $E\bar{\xi}\cdot\bar{\xi}^T = Q_1^{-1}$, $E\bar{\eta}\cdot\bar{\eta}^T = Q_2^{-1}$).

Next, we apply Lemma 1 to find $\min\limits_{U,W} f(U,W)$ and the inequality

$$\min_{U,W} f(U,W) \le \sum_{k=1}^{r} \langle \Phi_k, LEP_k\rangle,$$

which completes the proof of Proposition 3. □

III. Next, let us consider the case, when the linear operator is given in the form $LX = \langle L, X\rangle$, where the matrix L belongs to the space $H_{m\times n}$.

Proposition 4. *The following equality*

$$\tilde{\sigma}^2(u,\alpha) \triangleq \max_{G_1,G_2} E\left(\langle L,X\rangle - (u,y) - \alpha\right)^2 = \lambda_{\max}(K(u)) + E\left(\left\langle \Psi, A^{(0)}\right\rangle - \left\langle E\Psi, A^{(0)}\right\rangle\right)^2 + \\ + \left(Q_2^{-1}u, u\right) + \left(\alpha - \left\langle E\Psi, A^{(0)}\right\rangle\right)^2$$

holds, where the matrix Ψ is the solution of the equation $A^*(\omega)\Psi = L - \rho(C)u$, and the matrix $K(u)$ is found according to the formulas $K(u) = Q_1^{-\frac{1}{2}} E\Phi\Phi^T Q_1^{-\frac{1}{2}}$, $\Phi = \rho^*(B)\Psi$.

Proof. From the equality $\langle L,X\rangle - \widehat{\langle L,X\rangle} = \langle \Psi, \rho(B)\xi\rangle + \left\langle \Psi, A^{(0)}\right\rangle - (u,\eta) - \alpha$ it follows that

$$E\left(\langle L,X\rangle - \widehat{\langle L,X\rangle}\right)^2 = E\left(E\Phi\Phi^T\xi,\xi\right) + E(u,\eta)^2 + E\left\langle \Psi - E\Psi, A^{(0)}\right\rangle^2 + \left(\alpha - \left\langle E\Psi, A^{(0)}\right\rangle\right)^2.$$

Now, taking into account the equalities

$$\max_{G_2} E(u,\eta)^2 = \left(Q_2^{-1}u,u\right), \quad \max_{G_1} E\left(E\Phi\Phi^T\xi,\xi\right) = \lambda_{\max}(K(u)),$$

we make sure that Proposition 4 is valid. □

Corollary 2. *The following equality* $\min_{u,\alpha} \tilde{\sigma}^2(u,\alpha) = \tilde{\sigma}^2(\hat{u},\hat{\alpha})$ *holds, where*

$$\tilde{\sigma}^2(u,\alpha) = \lambda_{\max}(K(\hat{u})) + E\left\langle \hat{\Psi} - E\hat{\Psi}, A^{(0)}\right\rangle + \left(Q_2^{-1}\hat{u},\hat{u}\right),\ \hat{\alpha} = \left\langle E\hat{\Psi}, A^{(0)}\right\rangle,\ \hat{\Psi} = \Psi\,|_{u=\hat{u}}$$

Corollary 3. *For the value* $\langle L, X\rangle$ *there exists a unique guaranteed rms estimate.*

The validity of this result follows from the strong convexity and continuity of the function $\tilde{\sigma}^2(u,\alpha)$.

Let $e^k, k=\overline{1,N}$ be an orthonormal basis in the space $\mathbb{R}^N$. We denote by $\Psi\left(e^k\right)$ the solution of the equation $A(\omega)\Psi\left(e^k\right) = \rho^*(C)\,e^k$, and by $\rho(E\Psi(e))$ the linear operator that acts from the space $\mathbb{R}^N$ to the space of matrices $H_{m\times n}$ according to the rule

$$\rho(E\Psi(e))\,v = \sum_{k=1}^{N} E\Psi\left(e^k\right)\left(v,e^k\right),\ \ \forall v\in\mathbb{R}^N.$$

Proposition 5. *Let* $\lambda_m \triangleq \lambda_{\max}(K(\hat{u}))$ *be a maximum eigenvalue with multiplicity one of the matrix* $K(\hat{u})$*, and let* φ_λ *be an eigenvector corresponding to this eigenvalue, moreover the following condition is fulfilled* $(\varphi_\lambda,\varphi_\lambda)=1$*. Then the equality*

$$\hat{u} = Q_2\left(\rho^*(C)\,EP^1 - \rho^*(E\Psi(e))\,A^{(0)}\left\langle E\hat{\Psi}^1, A^{(0)}\right\rangle\right)$$

holds, where the matrices $E\hat{\Psi}^1,\ EP^1$ *are determined from the system of equations*

$$\begin{cases} A^*(\omega)\,\hat{\Psi}^1 = L - \rho(C)\,\hat{u}, \\ A(\omega)\,P^1 = \left\langle \hat{\Psi}^1, \rho(B)\,Q^{-\frac{1}{2}}\varphi_\lambda\right\rangle \rho(B)\,Q^{-\frac{1}{2}}\varphi_\lambda + \left\langle A^{(0)}, \hat{\Psi}^1\right\rangle A^{(0)}, \\ K(\hat{u})\,\varphi_\lambda = \lambda_m\varphi_\lambda. \end{cases}$$

Proof. Note that the identity $\frac{dg(t)}{dt}\,|_{t=0} \equiv 0,\ \forall v\in\mathbb{R}^N$ holds, where $g(t) = J(\hat{u}+tv)$,

$$J(u) = \lambda_{\max}(K(u)) + E\left\langle \Psi - E\Psi, A^{(0)}\right\rangle^2 + \left(Q_2^{-1}u,u\right).$$

In our case, the function $\lambda_{\max}(K(\hat{u}+tv))$ is differentiable with respect to t and the equality

$$\frac{d}{dt}\lambda_{\max}(K(\hat{u}+tv)) = \left(\frac{d}{dt}K(\hat{u}+tv)\,\varphi_\lambda,\varphi_\lambda\right)$$

is fulfilled. Thus, we can write the equality

$$\frac{dg(t)}{dt}\,|_{t=0} = \left(\frac{d}{dt}K(\hat{u}+tv)\,\varphi_\lambda,\varphi_\lambda\right) + 2E\left\langle \hat{\Psi}^1, A^{(0)}\right\rangle\cdot\left\langle \tilde{\Psi}^1, A^{(0)}\right\rangle - $$
$$-2\left\langle E\hat{\Psi}^1, A^{(0)}\right\rangle\cdot\left\langle E\tilde{\Psi}^1, A^{(0)}\right\rangle + 2\left(Q_2^{-1}\hat{u},v\right),$$

where $\tilde{\Psi}^1$ is the solution of the equation $A^*(\omega)\,\tilde{\Psi}^1 = -\rho^*(C)\,v$. From the equality

$$\left(\frac{d}{dt}K(\hat{u}+tv)\,|_{t=0}\varphi_\lambda,\varphi_\lambda\right) = 2\left\langle \hat{\Psi}^1, \rho(B)\,Q^{-\frac{1}{2}}\varphi_\lambda\right\rangle\left\langle \tilde{\Psi}^1, \rho(B)\,Q^{-\frac{1}{2}}\varphi_\lambda\right\rangle$$

we obtain the identity

$$\frac{1}{2}\frac{d}{dt}g(t)\,|_{t=0}\left(Q_2^{-1}\hat{u} - \rho^*(C)\,EP^1 + \rho^*(E\Psi(e))\,A^{(0)}\left\langle E\hat{\Psi}^1, A^{(0)}\right\rangle, v\right) \equiv 0\ \ \forall v\in\mathbb{R}^N,$$

from which the necessary expression for the vector $\hat{u}$ follows. □

Now consider the case, when the eigenvalue λ_m is a multiple. Then the function $\lambda_{\max}(K(\hat{u}+tv))$ is not differentiable and it has a subdifferential and a unique vector $\hat{u}$, that satisfies the condition $0 \in \partial J(\hat{u})$, where $\partial J(\hat{u})$ is a subdifferential at the point $\hat{u}$.

Let $O(u)$ denote an orthogonal matrix such that

$$O^*(u) K(u) O(u) = \operatorname{diag}(\lambda_1(u), ..., \lambda_r(u)),$$

where $\lambda_i(u)$ are the eigenvalues of the matrix $K(u)$.

Proposition 6. *Let the maximum eigenvalue of the matrix $K(u)$ have multiplicity μ, $1 \le \mu \le r$. Then, there exist numbers $p_j \ge 0, j = \overline{1,\mu}$ such that the vector $\hat{u}$ is the solution of the system of equations*

$$\begin{cases} A^*(\omega)\hat{\Psi}^{(2)} = L - \rho(c)\hat{u}, \\ A(\omega)P^{(2)} = J_1\left(\hat{\Psi}^{(2)}\right), \\ \hat{u} = Q_2\left(\rho^*(C)EP^{(2)} - \rho^*(E\Psi(C))A^{(0)}\left\langle E\hat{\Psi}^{(2)}, A^{(0)}\right\rangle\right). \end{cases}$$

Here the function $J_1\left(\hat{\Psi}^{(2)}\right)$ has the form

$$J_1\left(\hat{\Psi}^{(2)}\right) = \sum_{j=1}^{\mu} p_j \left\langle \hat{\Psi}^{(2)}, \rho(B)\bar{q}_k(\hat{u})\right\rangle \rho(B)\bar{q}_k(\hat{u}) + \left\langle A^{(0)}, \hat{\Psi}^{(2)}\right\rangle A^{(0)},$$

where $\bar{q}_j(\hat{u}) = O(\hat{u})w^j, j = \overline{1,\mu}$ and vectors w^j belong to the set $W^{(1)}$ of unit vectors from the space $\mathbb{R}^\mu$ such that if $w \in W^{(1)}$, then $(w, e^i) = 0$, $i = \overline{\mu+1, r}$.

Proof. Note that the condition $0 \in \partial J(\hat{u})$ is written in the form $0 \in \left\{\partial\lambda_{\max}(K(\hat{u})) + J_2'(\hat{u})\right\}$, where $J_2(u) = E\left\langle \Psi - E\Psi, A^{(0)}\right\rangle^2 + \left(Q_2^{-1}u, u\right)$. Since [10] the equality $\partial\lambda_{\max}(K(\hat{u})) = \operatorname{co}\left\{(K'(\hat{u})O(\hat{u})w, O(\hat{u})w), w \in W^{(1)}\right\}$ is fulfilled, then we can write the condition $0 \in \partial J(\hat{u})$ as follows

$$\sum_{j=1}^{\mu} p_j (K'(\hat{u})\bar{g}_j(\hat{u}), \bar{g}_j(\hat{u})) + J_2'(\hat{u}) \equiv 0,$$

where $p_j, j = \overline{1,\mu}$ are real numbers such that $0 \le p_j \le 1$, $\sum_{j=1}^{\mu} p_j = 1$.

From the equality $\sum_{j=1}^{\mu} p_j (K'(\hat{u})\bar{g}_j(\hat{u}), \bar{g}_j(\hat{u})) = \operatorname{sp} K'(\hat{u})\bar{Q}(\hat{u})$, where $\bar{Q}(\hat{u}) = \sum_{j=1}^{\mu} p_j \bar{q}_j(\hat{u})\bar{q}_j^T(\hat{u})$ the identity $\operatorname{sp} K'(\hat{u})\bar{Q}(\hat{u}) + J_2'(\hat{u}) \equiv 0$ follows for $\hat{u}$. Now, if we take into account the expressions for $K'(\hat{u})$ and $J_2'(\hat{u})$, then, similarly to how it was proved in Proposition 5, we obtain the system of equations for finding the vector $\hat{u}$. □

Definition 4. Linear estimate of the value $\langle L, X\rangle$ of the form

$$\widetilde{\langle L, X\rangle} = (\hat{u}, y) + \hat{\alpha},$$

where $(\hat{u}, \hat{\alpha}) \in Arg \min\limits_{(u,\alpha)} \tilde{\sigma}_1^2(u, \alpha)$, is called a *quasi-minimax estimate*. Here $\tilde{\sigma}_1^2(u,\alpha) = E \max\limits_{G_1,G_2} E\left(\langle L, X\rangle - \widehat{\langle L, X\rangle}\right)^2 |_{A(\omega)}$ and $E\left(\langle L, X\rangle - \widehat{\langle L, X\rangle}\right)^2 |_{A(\omega)}$ is the conditional mathematical expectation with the fixed operator $A(\omega)$.

Proposition 7. *Let the matrices $\hat{\Psi}$ and $\hat{P}$ be solutions of the system of equations*

$$\begin{cases} A^*(\omega)\,\hat{\Psi} = L - \rho(C)\,\hat{u}, \\ A(\omega)\,\hat{P} = Q\hat{\Psi}, \end{cases} \tag{8}$$

where the vector $\hat{u}$ is calculated by the formula $\hat{u} = Q_2\rho^*(C)\left(E\hat{P} + \left\langle E\hat{\Psi}, A^{(0)}\right\rangle \bar{A}A^{(0)}\right)$. Then the unique quasi-minimax estimate has the form $\left\langle \widetilde{L, X}\right\rangle = (\hat{u}, y) + \hat{\alpha}$, where $\hat{\alpha} = \left\langle E\hat{\Psi}, A^{(0)}\right\rangle$ and at the same time, for the squared error, the following applies

$$\tilde{\sigma}_1^2(\hat{u}, \hat{\alpha}) = E\left\langle Q\hat{\Psi}, \hat{\Psi}\right\rangle - \left\langle E\hat{\Psi}, A^{(0)}\right\rangle^2 + \left(Q_2^{-1}\hat{u}, \hat{u}\right).$$

Proof. Since equalities are satisfied

$$\max_{G_1,G_2} E\left(\langle L, X\rangle - (u, y) - \alpha\right)^2 \big|_{A(\omega)} = \max_{G_1}\langle\Psi, \rho(B)\,\xi\rangle + \left(\langle\Psi, B\rangle - \alpha\right)^2 + \max_{G_2} E(u, \eta)^2 =$$
$$= \left\langle \rho(B)\,Q_1^{-1}\rho^*(B)\,\Psi, \Psi\right\rangle + \left(Q_2^{-1}u, u\right) + \left(\alpha - \left\langle\Psi, A^{(0)}\right\rangle\right)^2$$

where the matrix Ψ is the solution of the equation $A^*(\omega)\,\Psi = L - \rho(C)\,u$, then for the squared error we have

$$\tilde{\sigma}_1^2(u, \alpha) = E\left\langle \rho(B)\,Q_1^{-1}\rho^*(B)\,\Psi, \Psi\right\rangle + E\left(\alpha - \left\langle\Psi, A^{(0)}\right\rangle\right)^2 + \left(Q_2^{-1}u, u\right).$$

This implies the equality $\min\limits_{u,\alpha}\tilde{\sigma}_1^2(u, \alpha) = \min\limits_{u}\bar{J}(u)$, where the function $\bar{J}(u)$ is calculated according to the formula

$$\bar{J}(u) \triangleq E\left\langle \rho(B)\,Q_1^{-1}\rho^*(B)\,\Psi, \Psi\right\rangle + E\left(\left\langle\Psi, A^{(0)}\right\rangle - \left\langle E\Psi, A^{(0)}\right\rangle\right)^2 + \left(Q_2^{-1}u, u\right) =$$
$$= E\langle Q\Psi, \Psi\rangle + \left(Q_2^{-1}u, u\right) - \left\langle E\Psi, A^{(0)}\right\rangle^2.$$

Now we can write down the equality $\min\limits_{u}\bar{J}(u) = E\left\langle Q\hat{\Psi}, \hat{\Psi}\right\rangle + \left(Q_2^{-1}\hat{u}, \hat{u}\right) - \left(E\hat{\Psi}, A^{(0)}\right)^2$, which we had to prove. □

Corollary 4. *Let the matrix $A^{(0)} = 0$. Then the equalities hold $\hat{u} = Q_2\rho^*(C)\,E\hat{P}$, $\hat{\alpha} = 0$, $\tilde{\sigma}_1^2(\hat{u}, \hat{\alpha}) = \left\langle L, \hat{P}\right\rangle$ hold.*

Proof. Since at $A^{(0)} = 0$ the function $\bar{J}$ at $u = \hat{u}$ takes the value $\bar{J}(\hat{u}) = E\left\langle Q\hat{\Psi}, \hat{\Psi}\right\rangle + \left\langle Q_2^{-1}\hat{u}, \hat{u}\right\rangle$, then, taking into account that the equalities

$$Q\hat{\Psi} = A(\omega)\,\hat{P}, \quad \left\langle Q\hat{\Psi}, \hat{\Psi}\right\rangle = \left\langle L, \hat{P}\right\rangle - \left(Q_2^{-1}\hat{u}, \hat{u}\right),$$

are fulfilled, we obtain the required equality for $\tilde{\sigma}_1^2(\hat{u}, 0)$. □

Corollary 5. *The inequality $\min\limits_{u,\alpha}\max\limits_{G_1,G_2} E\left(\langle L, X\rangle - (u, y) - \alpha\right)^2 \le \bar{J}(\hat{u})$ is valid.*

Proposition 8. *Suppose that $\lim\limits_{N\to\infty}\lambda_{\min}(Q) = \infty$. Then for the squared estimation error, the equality $\lim\limits_{N\to\infty}\min\limits_{u,\alpha}\tilde{\sigma}^2(u, \alpha) = 0$ holds.*

Proof. Since the inequalities

$$\min_{u,\alpha} \tilde{\sigma}^2(u,\alpha) \le \tilde{\sigma}^2(u,\alpha) \le \tilde{\sigma}_1^2(u,\alpha) \tag{9}$$

hold for arbitrary values of u and α, then we put

$$u = \hat{u} = Q_2 \rho^*(C) E\hat{P}; \quad \alpha = \hat{\alpha} = \left\langle E\hat{\Psi}, A^{(0)} \right\rangle .$$

For the estimation error $\tilde{\sigma}_1(\hat{u},\hat{\alpha})$, conditions

$$\tilde{\sigma}_1^2(\hat{u},\hat{\alpha}) = E\left\langle Q\hat{\Psi},\hat{\Psi}\right\rangle + \left(Q_2^{-1}\hat{u},\hat{u}\right) - \left\langle E\hat{\Psi}, A^{(0)}\right\rangle \le$$
$$\le E\left\langle Q\hat{\Psi},\hat{\Psi}\right\rangle + \left(Q_2^{-1}\hat{u},\hat{u}\right) = E\left\langle L,\hat{P}\right\rangle = \left\langle L, E\hat{P}\right\rangle,$$

are fulfilled, then considering (9), we obtain the inequality $\min\limits_{u,\alpha} \tilde{\sigma}^2(u,\alpha) \le \left\langle L, E\hat{P}\right\rangle$.

Now note that the inequality $\left\langle L, E\hat{P}\right\rangle \le \left\langle L, E\hat{P}(\varepsilon)\right\rangle$ holds, where the scalar parameter $\varepsilon > 0$, $\left\langle L, E\hat{P}(\varepsilon)\right\rangle = E\left\langle Q(\varepsilon)\hat{P}(\varepsilon), \hat{P}(\varepsilon)\right\rangle + \left(Q_2^{-1}\hat{u}(\varepsilon), \hat{u}(\varepsilon)\right)$, $Q(\varepsilon) = Q + \varepsilon^2 I$, I is the unary operator, $\hat{\Psi}(\varepsilon)$ and $\hat{P}(\varepsilon)$ are the solutions of system (8) for $Q = Q(\varepsilon)$.

Since there exists an inverse matrix for $Q(\varepsilon)$, so we obtain the following equalities

$$A^*(\omega) Q^{-1}(\varepsilon) A(\omega) \hat{P}(\varepsilon) = L - \rho(C) Q_2 \rho^*(C) E\hat{P}(\varepsilon),$$
$$A^*(\omega) Q^{-1}(\varepsilon) A(\omega) = D(\omega)$$

and for $E\hat{P}(\varepsilon)$ we get the following equation $\left(\left(ED^{-1}(\omega)\right)^{-1} + \rho(C) Q_2 \rho^*(C)\right) E\hat{P}(\varepsilon) = L$.

For $\lambda_{\min}\left(\tilde{Q}_2\right)$ we have

$$\lambda_{\min}\left(\tilde{Q}_2\right) = \left\langle E\hat{P}(\varepsilon), E\hat{P}(\varepsilon)\right\rangle \le \left\langle E\left(\left(D^{-1}(\omega)\right)^{-1} + \tilde{Q}_2\right) E\hat{P}(\varepsilon), E\hat{P}(\varepsilon)\right\rangle = \left\langle L, E\hat{P}(\varepsilon)\right\rangle,$$

where $\tilde{Q}_2 = \rho(C) Q_2 \rho^*(C)$. Taking into account the inequality

$$\left\langle L, E\hat{P}(\varepsilon)\right\rangle \le \langle L, L\rangle^{\frac{1}{2}} \cdot \left\langle E\hat{P}(\varepsilon), E\hat{P}(\varepsilon)\right\rangle^{\frac{1}{2}},$$

we obtain that $\left\langle E\hat{P}(\varepsilon), E\hat{P}(\varepsilon)\right\rangle^{\frac{1}{2}} \le \langle L, L\rangle^{\frac{1}{2}} \cdot \lambda_{\min}^{-\frac{1}{2}}\left(\tilde{Q}_2\right)$.

As a result, we obtain the inequality $\min\limits_{u,\alpha} \tilde{\sigma}^2(u,\alpha) \le \langle L, L\rangle \cdot \lambda_{\min}\left(\tilde{Q}_2\right)$, which completes the proof of Proposition 8. □

Remark. Let $E_s, s = 1,2,...$ be the basis matrices in the space $H_{m\times n}$ such that $\langle E_s, E_p\rangle = \delta_{sp}$, where δ_{sp} is the Kronecker symbol. Let $\left\langle \widehat{E_s.X}\right\rangle_k$ denote the quasi-minimax estimate of the scalar product $\langle E_s, X\rangle$. Then, by the quasi-minimax estimate of the matrix X according to observations (1) we mean the expression $\hat{X} = \sum_s \left\langle \widehat{E_s, X}\right\rangle_k E_s$.

IV. Next, consider an example for calculating the guaranteed estimate and its error in a partial case. Let equation (2) have the form $(A + \omega I) X = \xi$, where $A = (a_{ij})_{i,j=\overline{1,n}}$ is a symmetric positive definite matrix with eigenvalues $\lambda_i, i = \overline{1,n}$, I is a unit matrix, ω is a random variable uniformly distributed on a given segment $(0,a)$, ξ is a random matrix independent of ω, for which the conditions $E\xi = 0$, $E\langle \xi, \xi\rangle \le q^2$ are fulfilled, q^2 is a given positive number.

The squared errors of the minimax and quasiminimax estimates according to Propositions 4, 7 are calculated, respectively, by the formulas

$$\sigma^2_{minm}(u) = \left(1 - \sum_{k=1}^{N} u_k\right)^2 q^2 \lambda_{\max}(E\Psi_1 \otimes \Psi_1) + \left(Q_2^{-1}u, u\right),$$
$$\sigma^2_{kminm}(u) = \left(1 - \sum_{k=1}^{N} u_k\right)^2 q^2 E\langle\Psi_1, \Psi_1\rangle + \left(Q_2^{-1}u, u\right). \tag{10}$$

The inequality $\sigma^2_{minm}(u) \le \sigma^2_{kminm}(u) \Leftrightarrow \frac{\sigma^2_{minm}(u)}{\sigma^2_{kminm}(u)} \le 1$ holds for arbitrary vectors u. If we put $Q_2 = q_2^2 I$, where I is the unit matrix of dimension N, then (10) takes the form

$$\sigma^2_{minm}(u) = \left(1 - \sum_{k=1}^{N} u_k\right)^2 q^2 \lambda_{\max}(E\Psi_1 \otimes \Psi_1) + q_2^{-2}(u, u),$$
$$\sigma^2_{kminm}(u) = \left(1 - \sum_{k=1}^{N} u_k\right)^2 q^2 E\langle\Psi_1, \Psi_1\rangle + q_2^{-2}(u, u).$$

Thus, the right-hand sides for the errors of minimax and quasi-minimax estimations can be represented as

$$J(u) = \left(1 - \sum_{k=1}^{N} u_k\right)^2 d^2 q^2 + q_2^{-2}(u, u), \tag{11}$$

where the parameter $d^2 = \lambda_{\max}(E\Psi_1 \otimes \Psi_1)$ for the minimax estimation error and $d^2 = \sum_{k=1}^{n} E(\lambda_k + \omega)^{-2}$ for the quasi-minimax estimation error. By differentiating the function (11) we find the minimum point $\frac{1}{2}J'(u) = -(1 - (u, e))eq^2d^2 + q_2^{-2}u \equiv 0 \Rightarrow \hat{u} = (1 - (\hat{u}, e))eq_1^2d^2$, where $q_1 = (qq_2)$, $e \in \mathbb{R}^N$ and all the components of the vector e are equal to one. Let us multiply the left and right hand sides of the last equality by the vector e: $(\hat{u}, e) = (1 - (\hat{u}, e))Nq_1^2d^2$. Hence we find the scalar product $(\hat{u}, e)$: $(\hat{u}, e) = \frac{Nq_1^2d^2}{1+Nq_1^2d^2}$.

Therefore, the optimal value of the vector $\hat{u}$ has the form $\hat{u} = \frac{q_1^2d^2e}{1+Nq_1^2d^2} = \frac{e}{q_1^{-2}d^{-2}+N}$, and the minimax and quasiminimax estimates are as follows $(\hat{u}, y) = \frac{(e,y)}{q_1^{-2}d^{-2}+N}$.

Let us find the value of the function (11) at the point $\hat{u}$:

$$J(\hat{u}) = \frac{q^2d^2}{(1 + Nq_1^2d^2)^2} + q_2^{-2}\frac{N}{\left(q_1^{-2}d^{-2} + N\right)^2} = \frac{q^2d^2(1 + q^2q_2^2d^2N)}{(1 + q^2q_2^2d^2N)^2} = \frac{1}{q^{-2}d^{-2} + Nq_2^2}.$$

Thus, the ratio of the squared minimax and quasi-minimax estimation errors is as follows

$$\frac{\sigma^2_{minm}}{\sigma^2_{kminm}} = \frac{q^{-2}d_2^{-2} + Nq_2^2}{q^{-2}d_1^{-2} + Nq_2^2}, \tag{12}$$

where

$$d_1^2 = \lambda_{\max}(M),\ M = (\mu_{ij})_{i,j=\overline{1,n}},\ \mu_{ij} = E(\lambda_i + \omega)^{-1}(\lambda_j + \omega)^{-1},\ d_2^2 = \sum_{i=1}^{n} E(\lambda_i + \omega)^{-2}.$$

Note that $\lim\limits_{N\to\infty} \frac{\sigma^2_{minm}}{\sigma^2_{kminm}} = 1$, i.e. when $N \gg 1$, the condition $\sigma^2_{minm} \approx \sigma^2_{kminm}$ is fulfilled.

Let us find the lower bound for the relation $\frac{\sigma^2_{minm}}{\sigma^2_{kminm}}$. From the formula

$$d_2^2 = \sum_{i=1}^{n} E\left(\lambda_i + \omega\right)^{-2}$$

there follows the inequality $d_2^2 \leq m \max_i E\left(\lambda_i + \omega\right)^{-2}$, from which we obtain

$$d_2^2 \leq mE \max_i \left(\lambda_i + \omega\right)^{-2} = mE\left(\min_i \lambda_i + \omega\right)^{-2} = mE\left(\lambda_1 + \omega\right)^{-2},$$

where λ_1 is the minimal eigenvalue of matrix A. Therefore, the inequality holds

$$d_2^{-2} \geq \left(mE\left(\lambda_1 + \omega\right)^{-2}\right)^{-1}. \tag{13}$$

The relation (12) under the inequality (13) takes the form

$$\frac{\sigma^2_{minm}}{\sigma^2_{kminm}} \geq \frac{q^{-2}\left(nE\left(\lambda_1 + \omega\right)^{-2}\right)^{-1} + Nq_2^2}{q^{-2}d_1^{-2} + Nq_2^2}. \tag{14}$$

Now let us put $n = 2, q^2 = 1, q_2^2 = 1$ and define the matrix A in the form

$$A = \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 2, \lambda_2 = 6$. Thus, from (12) we obtain a specific formula for calculating the ratio of the squared minimax and quasi-minimax estimation errors

$$\frac{\sigma^2_{minm}}{\sigma^2_{kminm}} = \frac{d_2^{-2} + N}{d_1^{-2} + N}, \tag{15}$$

and from inequality (14) we obtain the formula for calculating the lower bound of the ratio of the squared minimax and quasi-minimax estimation errors

$$\frac{\sigma^2_{minm}}{\sigma^2_{kminm}} \geq \frac{\left(2E\left(2 + \omega\right)^{-2}\right)^{-1} + N}{d_1^{-2} + N} = \frac{2 + a + N}{d_1^{-2} + N}.$$

Let us find the elements of the matrix M:

$$\mu_{11} = E\left(\lambda_1 + \omega\right)^{-2} = \frac{1}{a}\int_0^a \frac{dx}{\left(2 + x\right)^2} = \frac{1}{2\left(2 + a\right)};$$

$$\mu_{22} = E\left(\lambda_2 + \omega\right)^{-2} = \frac{1}{a}\int_0^a \frac{dx}{\left(6 + x\right)^2} = \frac{1}{6\left(6 + a\right)};$$

$$\mu_{12} = E\left(\lambda_1 + \omega\right)^{-1}\left(\lambda_2 + \omega\right)^{-1} = \frac{1}{a}\int_0^a \frac{dx}{\left(2 + x\right)\left(6 + x\right)} = \frac{1}{4a}\ln\frac{3\left(2 + a\right)}{\left(6 + a\right)}.$$

The maximum eigenvalue of the matrix M is as follows

$$\lambda_{\max}\left(M\right) = \frac{1}{2}\left(\frac{2\left(a + 5\right)}{3\left(a + 2\right)\left(a + 6\right)} + \sqrt{\left(\frac{a + 8}{3\left(a + 2\right)\left(a + 6\right)}\right)^2 + \left(\frac{1}{2a}\ln\frac{3\left(a + 2\right)}{\left(a + 6\right)}\right)^2}\right) = d_1^2.$$

The parameter d_2^2 is calculated according to the formula

$$d_2^2 = \sum_{i=1}^{2} E\left(\lambda_i + \omega\right)^{-2} = \frac{1}{a}\sum_{i=1}^{2}\int_0^a \left(\lambda_i + x\right)^{-2} dx = \frac{2\left(a+5\right)}{3\left(a+2\right)\left(a+6\right)}.$$

Now we present a table that characterizes the ratio of the squared minimax and quasi-minimax estimation errors depending on the values of the a and N parameters (by formula (15)):

N		20	100	1000	10000
$\frac{\sigma^2_{minm}}{\sigma^2_{kminm}}$	$a=2$	0.9995739	0.9998929	0.9999886	0.9999989
	$a=10$	0.9957887	0.9986112	0.9998374	0.9999835
	$a=50$	0.8217023	0.8926699	0.9804064	0.9978643

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