УДК 517.555

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ON AN ATTEMPT TO INTRODUCE A NOTION OF BOUNDED INDEX FOR THE FUETER REGULAR FUNCTIONS OF THE QUATERNIONIC VARIABLE

V. P. Baksa, A. I. Bandura. *On an attempt to introduce a notion of bounded index for the Fueter regular functions of the quaternionic variable*, Mat. Stud. **60** (2023), 191–200.

There is introduced a concept of index for the Fueter regular function of the quaternionic variables. There are considered three approaches (Fueter, Sudbery and Mariconda) constructing the Fueter regular function from a holomorphic function of complex variable. Using Mariconda's approach there are constucted some analogs of such elementary functions as the exponent, the sine and the cosine. For the Mariconda analogs we proved that they have bounded index and their indices equal 1, 2, 2, respectively. Using recent results on sum of entire functions whose derivatives are of bounded index it is established that the Fueter regular function constructed by Mariconda's approach is of bounded index, if the derivatives of its addends have bounded index. Also there was examined a function of the form $H(q) = f_1(x_0 + ix_1) + jf_2(x_2 + ix_3)$, where f_1 and f_2 are entire functions of complex variable. For the function H it is proved its Fueter regularity and index boundedness if the first order derivatives of f_1 and f_2 have bounded index. Moreover, the index of the function H does not exceed the maximum of indices of the functions f_1' and f_2' increased by 1.

1. Introduction. In this paper we introduce a notion of index for regular functions of quaternionic variable. The notion was firstly appeared for entire functions of single complex variable [10]. These functions have applications in the analytic theory of differential equations [16] and value distribution theory [2, 6, 17]. Moreover, there are many papers on different multidimensional complex analogs of the notion [3–5, 13]. Although there are many papers and monographs [7–9, 18] on properties of regular functions, we do not know results on growth estimates, local behavior, value distribution and their applications to differential equations which are similar in the theory of bounded index for analytic functions of complex variable. The present paper is a starting point to develop theory of bounded index, value distribution theory and analytic theory of differential equations for these functions.

Introducing the notion of bounded index for entire functions of complex variable, B. Lepson [10] considers the notion as the central index of the power series. In the complex analysis, the notion is closely related to the concept of the complex derivative and power series. Unfortunately, the quaternionic regular functions have no expansion in power series. Moreover, even q^n is not regular function, if $n \geq 2$, and q is a quaternionic variable. These functions can be developed in some series by special type homogeneous polynomials whose coefficients

2020 *Mathematics Subject Classification:* 30D15, 30D20, 46S05.
Keywords: Fueter regular function; entire function; bounded index; partial derivative; quaternionic variable; Mariconda's approach.
doi:10.30970/ms.60.2.191-200

are calculated by partial derivatives in real variables. In view of these difficulties, Prof. Oleh Skaskiv (Lviv) posed the question: *How to introduce the concept of index boundedness for regular Fuether functions?* He assumed that such an approach would be successful if it is possible to deduce analogs of known results from theory of complex entire functions having bounded index. In the present paper, we make an attempt to introduce this concept and establish quaternionic analogs of the most recent statements on properties of functions having bounded index to test for applicability.

There are a lot of connections between general field theory and quaternionic analysis. For example, K. Guerlebeck and W. Sprossig [9, p.162] considered the application of a quaternionic operator calculus to the representation of the solution of Maxwell equations. It is known that these equations [19] are the cornerstone in electrodynamics. But the notion of bounded index allows to study properties of solutions of partial differential equations without explicitly looking for the solution itself.

2. Main notations and definitions. We will use standard notations and definitions from [8, 9, 18]. Let us remind some of them.

Let $\mathbb{H}$ be the real associative algebra of quaternions with the standard basis 1, i, j, k such that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We write an element $q \in \mathbb{H}$ in the form

$$q = x_0 + ix_1 + jx_2 + kx_3,$$

where $x_\ell \in \mathbb{R}$, for $\ell = 0, 1, 2, 3$ and we set $\operatorname{Re} q = x_0$, $\operatorname{Pu} q = ix_1 + jx_2 + kx_3$, $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. $\operatorname{Re} q$, $\operatorname{Pu} q$ and $|q|$ are called the real part, the imaginary part and the module of q, respectively. The quaternion $\bar{q} = \operatorname{Re} - \operatorname{Pu} q = x_0 - ix_1 - jx_2 - kx_3$ is called the conjugate of q and satisfies $|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q}$. Sometimes it will be useful to write a quaternion in a more compact way as $q = \sum_{\ell=0}^{3} \imath_\ell x_\ell$ where $x_\ell \in \mathbb{R}$ and $\imath_0 = 1$, $\imath_2 = i$, $\imath_2 = j$, $\imath_3 = k$.

Let us consider a function $f\colon \mathbb{H} \to \mathbb{H}$. One can extend the notion of holomorphicity to functions of one quaternionic variable, but while in the complex case there are several equivalent definitions, in the quaternionic case there is only one definition [8, 12, 18] which is meaningful, and it consists in defining a regular function (or quaternionic holomorphic function) as a function defined on an open set of the space of quaternions which is in the kernel of the so-called Cauchy-Fueter operator (a natural generalization of the Cauchy-Riemann operator). But in the last decade there were developed another approach to introduce the notion of holomorphicity for functions of quaternionic variable. F. Colombo, I. Sabadini, D. Struppa and others [7, 8, 12] examined the slice entire functions. These are functions of quaternionic variable which are entire functions at the every slice passing through origin and containing one point from the unit sphere of purely imaginary quaternions.

Let us now introduce two differential operators which generalize the Cauchy-Riemann operator to the quaternionic case:

$$\frac{\partial_l}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}, \quad \frac{\partial_r}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}i + \frac{\partial}{\partial x_2}j + \frac{\partial}{\partial x_3}k.$$

The two operators are called the left and right Cauchy-Fueter operators, respectively. We also define their conjugate operators

$$\frac{\partial_l}{\partial q} = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}, \quad \frac{\partial_r}{\partial q} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}i - \frac{\partial}{\partial x_2}j - \frac{\partial}{\partial x_3}k.$$

Definition 1. (see Definition 3.1.1. in [8]) Let $U \subseteq \mathbb{H}$ be an open set and let $f\colon U \to \mathbb{H}$ be a real differentiable function. We say that f is *left regular* on U

$$\frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0. \tag{1}$$

We say that f is *right regular* on U if

$$\frac{\partial_r f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = 0. \tag{2}$$

We denote by $\mathrm{Re}_\ell(U)$ the set of the left regular functions on U and by $\mathrm{Re}_r(U)$ the set of the right regular functions on U. The theory of the left regular functions is completely equivalent to the theory of the right regular functions so, classically, the theory is usually developed for the case of left regular functions.

Below we indicate three ways to construct the Fueter regular function of quaternionic variable from holomorphic functions of complex variable.

Proposition 1 (see Proposition 2.27 in [9], [11], the Mariconda Proposition)**.** *Let G_k, $(k = 1,2,3)$, be open sets in $\mathbb{C}$ and let $h_k\colon G_k \mapsto \mathbb{C}$ be complex holomorphic functions. Furthermore, let $h_k = \operatorname{Re} h_k + i \operatorname{Im} h_k$ and let $G := \{q = x_0 + \imath_1 x_1 + \imath_2 x_2 + \imath_3 x_3 \in \mathbb{H}\colon x_0 + \imath_k x_k \in G_k\}$. Then the function $H\colon G \to \mathbb{H}$ given by*

$$H(q) = \sum_{k=1}^{3} \operatorname{Re} h_k(x_0, x_k) + \sum_{k=1}^{3} \imath_k \operatorname{Im} h_k(x_0, x_k) \tag{3}$$

is quaternionic regular in G.

For each $q \in \mathbb{H}$, let $\eta_q\colon \mathbb{C} \to \mathbb{H}$ be the embedding of the complex numbers in the quaternions such that q is the image of a complex number $\zeta(q)$ lying in the upper half-plane; i.e.

$$\eta_q(x + iy) = x + \frac{\operatorname{Pu} q}{|\operatorname{Pu} q|}y, \quad \zeta(q) = \operatorname{Re} q + i|\operatorname{Pu} q|. \tag{4}$$

Theorem 1 (Theorem 3.1.6. in [8], the Fueter Theorem)**.** *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is analytic on open set $U \subseteq \mathbb{C}$, and define $\widetilde{f}\colon \mathbb{H} \to \mathbb{H}$ by $\widetilde{f} = \eta_q \circ f \circ \zeta(q)$. Then $\triangle \widetilde{f}$ is regular on the open set $\zeta_{-1}(U) \subseteq \mathbb{H}$, and its derivative is $\partial_l(\triangle \widetilde{f}) = \triangle \widetilde{f}'$, where f' is the derivative of the complex function f.*

Theorem 2 (see Theorem 4 in [18], the Sudbery Theorem)**.** *Let u be a real-valued function defined in $\mathbb{H}$. If the function u is harmonic and has continuous second derivatives, then there exists a regular function f, defined on H such that $\operatorname{Re} f = u$. In particular, the function can be obtained by the formula $f(q) = u(q) + 2\,Pu \int_0^1 s^2 \partial_l u(sq) q ds$.*

Every regular function $f\colon \mathbb{H} \to \mathbb{H}$ can be represented as a uniformly convergent series (see [8, 18])

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} p_\nu(q - q_0) a_\nu,$$

where $a_\nu = \frac{(-1)^n}{n!}\partial_\nu f(q_0)$, $\partial_\nu = \frac{\partial^n}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}}$, σ_n denotes the set of triples $\nu = (n_1, n_2, n_3)$, $n = n_1 + n_2 + n_3$, $q_0 \in \mathbb{H}$, and

$$p_\nu(q) = \sum_{1\le\lambda_1,\dots,\lambda_n\le 3} (x_0 \imath_{\lambda_1} - x_{\lambda_1})\dots(x_0 \imath_{\lambda_n} - x_{\lambda_n})$$

Here the sum is taken over the $\frac{n!}{n_1!n_2!n_3!}$ different alignments of n_i elements equal to i, with $i = 1, 2, 3$.

Below we introduce a notion of index for the regular function of quaternionic variable.

A regular function $f\colon \mathbb{H}\to\mathbb{H}$ is said to be of bounded index if there exists $n_0 \in \mathbb{Z}_+$ such that for all $q \in \mathbb{H}$ and for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$ the following inequality is true

$$\frac{|\partial_\alpha f(q)|}{(\alpha_1+\alpha_2+\alpha_3)!} \le$$
$$\le \max\left\{\frac{|\partial_\beta f(q)|}{(\beta_1+\beta_2+\beta_3)!}\colon\ 0\le \beta_1+\beta_2+\beta_3 \le n_0, \beta = (\beta_1,\beta_2,\beta_3)\in\mathbb{Z}_+^3\right\}. \tag{5}$$

The least such integer n_0 is called the *index of the function* f and is denoted by $N(f, \mathbb{H})$.

Remark 1. Since we use partial derivatives in the imaginary variables x_1, x_2, x_3 in (5), it is interesting to investigate how this definition relates to the definition of index boundedness for entire functions of complex variable. In the case, we rewrite (5) replacing $q = x_0 + ix_1 + jx_2 + kx_3$ by $z = x + iy$, ∂_α by $\frac{\partial^\alpha}{\partial y^\alpha}$ and ∂_β by $\frac{\partial^\beta}{\partial y^\beta}$. Then for any $\alpha \in \mathbb{Z}_+$ one has

$$\frac{1}{\alpha!}\left|\frac{\partial^\alpha f(z)}{\partial y^\alpha}\right| \le \max\left\{\frac{1}{\beta!}\left|\frac{\partial^\beta f(z)}{\partial y^\beta}\right|\colon\ 0\le\beta\le n_0, \beta\in\mathbb{Z}_+\right\}. \tag{6}$$

It is known that $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$ and $\frac{\partial f}{\partial \bar z} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$. Hence, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}$, and $\frac{\partial f}{\partial y} = i(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z})$. But the function f is entire. It means that $\frac{\partial f}{\partial \bar z} = 0$. Thus, $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z}$. Similarly, $\frac{\partial^n f}{\partial y^n} = i^n\frac{\partial^n f}{\partial z^n}$ for any $n\in\mathbb{N}$. Therefore, $|\frac{\partial^n f}{\partial y^n}| = |\frac{\partial^n f}{\partial z^n}|$ and inequality (6) is equivalent to

$$\frac{1}{\alpha!}\left|\frac{\partial^\alpha f(z)}{\partial z^\alpha}\right| \le \max\left\{\frac{1}{\beta!}\left|\frac{\partial^\beta f(z)}{\partial z^\beta}\right|\colon\ 0\le\beta\le n_0, \beta\in\mathbb{Z}_+\right\}.$$

Thus, our approach to quaternionic bounded index matches with the usual bounded index in the case entire function of complex variable.

3. Main results.

Recently, there was obtained such a theorem in [1]

Theorem 3 ([1])**.** *If $f_1(z_1)$, $f_2(z_2)$ are entire transcendental functions, and their derivatives $f_1'(z_1)$, $f_2'(z_2)$ are functions having bounded index, then the function $F(z_1, z_2) = f_1(z_1) + f_2(z_2)$ is also of bounded index in joint variables and $N(F) \le 1 + \max\{N(f_1'), N(f_2')\}$.*

In view of Theorem 3 and Proposition 1, we can prove such a proposition.

Theorem 4. *Let $h_k\colon \mathbb{C}\to\mathbb{C}$ and its the first order derivative be entire functions of bounded index, $k\in\{1,2,3\}$. Then the function $H(q)\colon \mathbb{H}\to\mathbb{H}$ from Proposition 1 is the Fueter-regular function of bounded index and its index does not exceed maximum of indices of the functions h_k' increased by one.*

Proof. Proof of the theorem is based on the proofs of Proposition 1 and Theorem 3 from [1]. Let $H\colon \mathbb{H}\to\mathbb{H}$ be defined in (3). Then for $\nu_1=(n_1,0,0)$ $(n_1>0)$ one has

$$\partial_{\nu_1}H(q)=\frac{\partial^{n_1}h_1(x_0+ix_1)}{\partial x_1^{n_1}}=i^{n_1}h_1^{(n_1)}(x_0+ix_1). \tag{7}$$

and

$$\frac{|\partial_{\nu_1}H(q)|}{n_1!}=\frac{|h_1^{(n_1)}(x_0+ix_1)|}{n_1!}=\frac{1}{n_1}\cdot\frac{|(h_1'(x_0+ix_1))^{(n_1-1)}|}{(n_1-1)!}\le$$
$$\le\frac{1}{n_1}\max\left\{\frac{|(h_1'(x_0+ix_1))^{(m)}|}{m!}\colon 0\le m\le N(h_1')\right\}=$$
$$=\frac{1}{n_1}\max\left\{(1+m)\frac{|h_1^{(1+m)}(x_0+ix_1)|}{(1+m)!}\colon 1\le 1+m\le 1+N(h_1')\right\}\le$$
$$\le\frac{1+N(h_1')}{n_1}\max\left\{\frac{|h_1^{(s)}(x_0+ix_1)|}{s!}\colon 1\le s\le 1+N(h_1')\right\},$$

because h_1' is of bounded index $N(h_1')$.

Choosing $n_1\ge 1+N(h_1')$ and using (7) we obtain for $\nu_1=(n_1,0,0)$

$$\frac{|\partial_{\nu_1}H(q)|}{n_1!}\le\max\left\{\frac{|h_1^{(s)}(x_0+ix_1)|}{s!}\colon 1\le s\le 1+N(h_1')\right\}=$$
$$=\max\left\{\frac{|\partial_{(s,0,0)}H(q)|}{s!}\colon 1\le s\le 1+N(h_1')\right\}.$$

Similarly for $\nu_2=(0,n_2,0)$ $(n_2\ge 1+N(h_2'))$ and $\nu_3=(0,0,n_3)$ $(n_3\ge 1+N(h_3'))$ we deduce

$$\frac{|\partial_{\nu_2}H(q)|}{n_2!}\le\max\left\{\frac{|\partial_{(0,s,0)}H(q)|}{s!}\colon 1\le s\le 1+N(h_2')\right\},$$
$$\frac{|\partial_{\nu_3}H(q)|}{n_3!}\le\max\left\{\frac{|\partial_{(0,0,s)}H(q)|}{s!}\colon 1\le s\le 1+N(h_3')\right\}.$$

If $\nu=(n_1,n_2,n_3)$ and at least two components (for example, n_1 and n_2) do not equal zero then

$$\partial_\nu H(q)=\frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}\frac{\partial^{n_1}}{\partial x_1^{n_1}}(h_1(x_0+ix_1)+h_2(x_0+jx_2)+h_3(x_0+kx_3))=$$
$$=\frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}(i^{n_1}h_1^{(n_1)}(x_0+ix_1))=0.$$

Combining all results, we deduce

$$\frac{|\partial_\nu H(q)|}{(n_1+n_2+n_3)!}\le\max\left\{|H(q)|,\max\left\{\frac{|\partial_{(s,0,0)}H(q)|}{s!}\colon 1\le s\le 1+N(h_1')\right\},\right.$$
$$\left.\max\left\{\frac{|\partial_{(0,s,0)}H(q)|}{s!}\colon 1\le s\le 1+N(h_2')\right\},\max\left\{\frac{|\partial_{(0,0,s)}H(q)|}{s!}\colon 1\le s\le 1+N(h_3')\right\}\right\}=$$

$$= \max\left\{\frac{|\partial_{(s_1,s_2,s_3)}H(q)|}{(s_1+s_2+s_3)!} : 0 \le s_1+s_2+s_3 \le 1+\max\{N(h_1'),N(h_2'),N(h_3'),\}\right\}.$$

Therefore, the function H has bounded index and its index $N(h)$ does not exceed $1+\max\{N(h_1'),N(h_2'),N(h_3'),\}$. □

Using the Mariconda Proposition (see above Proposition 1) it is possible to introduce the following Mariconda's analogs of the exponent, the sine and the cosine:

$$\begin{aligned}\exp_{\mathbb{M}}(q) &= e^{x_0}(\cos x_1+\cos x_2+\cos x_3)+e^{x_0}(i\sin x_1+j\sin x_2+k\sin x_3),\\ \cos_{\mathbb{M}}(q) &= \cos x_0(\operatorname{ch} x_1+\operatorname{ch} x_2+\operatorname{ch} x_3)-\sin x_0(i\operatorname{sh} x_1+j\operatorname{sh} x_2+k\operatorname{sh} x_3),\\ \sin_{\mathbb{M}}(q) &= \sin x_0(\operatorname{ch} x_1+\operatorname{ch} x_2+\operatorname{ch} x_3)+\cos x_0(i\operatorname{sh} x_1+j\operatorname{sh} x_2+k\operatorname{sh} x_3).\end{aligned}$$

In view of Proposition 1 they are Fueter regular. By Theorem 4 the regular functions $\exp_{\mathbb{M}}$, $\cos_{\mathbb{M}}$, $\sin_{\mathbb{M}}$ are functions of bounded index in the whole space $\mathbb{H}$ and their indices equal 1, 2, 2, respectively.

It is possible to prove a similar proposition for the sum of two holomorphic functions, which is not a consequence of the previous one.

Theorem 5. *Let $h_1(z)$, $h_2(z)$ and their derivatives be entire functions of bounded index. Then the function $H(q)\colon \mathbb{H}\to\mathbb{H}$ defined as*

$$H(q)=h_1(x_0+ix_1)+jh_2(x_2+ix_3)$$

is the Fueter-regular function of bounded index and its index does not exceed maximum of indexes of the functions h_k' increased by one.

Proof. At first, we will prove regularity of H. Since

$$\begin{aligned}H(q) &= \operatorname{Re} h_1(x_0,x_1)+i\operatorname{Im} h_1(x_0,x_1)+j(\operatorname{Re} h_2(x_2,x_3)+i\operatorname{Im} h_2(x_2,x_3)) =\\ &= \operatorname{Re} h_1(x_0,x_1)+i\operatorname{Im} h_1(x_0,x_1)+j\operatorname{Re} h_2(x_2,x_3)-k\operatorname{Im} h_2(x_2,x_3),\end{aligned}$$

we can calculate the operator $\frac{\partial_l}{\partial\overline{q}}$ of the function H using the Cauchy-Riemann equations for the entire functions h_1 and h_2

$$\begin{aligned}\frac{\partial_l H}{\partial\overline{q}} &= \sum_{k=0}^{3}\imath_k\frac{\partial H}{\partial x_k} = \frac{\operatorname{Re} h_1(x_0,x_1)}{\partial x_0}+i\frac{\partial\operatorname{Im} h_1(x_0,x_1)}{\partial x_0}+i\left(\frac{\operatorname{Re} h_1(x_0,x_1)}{\partial x_1}+i\frac{\partial\operatorname{Im} h_1(x_0,x_1)}{\partial x_1}\right)+\\ &+j\left(j\frac{\partial\operatorname{Re} h_2(x_2,x_3)}{\partial x_2}-k\frac{\partial\operatorname{Im} h_2(x_2,x_3)}{\partial x_2}\right)+k\left(j\frac{\partial\operatorname{Re} h_2(x_2,x_3)}{\partial x_3}-k\frac{\partial\operatorname{Im} h_2(x_2,x_3)}{\partial x_3}\right)=\\ &= \frac{\operatorname{Re} h_1(x_0,x_1)}{\partial x_0}-\frac{\operatorname{Im} h_1(x_0,x_1)}{\partial x_1}+i\left(\frac{\partial\operatorname{Im} h_1(x_0,x_1)}{\partial x_0}+\frac{\operatorname{Re} h_1(x_0,x_1)}{\partial x_1}\right)-\\ &-\left(\frac{\partial\operatorname{Re} h_2(x_2,x_3)}{\partial x_2}-\frac{\partial\operatorname{Im} h_2(x_2,x_3)}{\partial x_3}\right)-i\left(\frac{\partial\operatorname{Im} h_2(x_2,x_3)}{\partial x_2}+\frac{\partial\operatorname{Re} h_2(x_2,x_3)}{\partial x_3}\right)=0.\end{aligned}$$

Since $\frac{\partial_l H}{\partial\overline{q}}=0$ the function H is regular. Now will investigate the index boundedness. Then for $\nu_1=(n_1,0,0)$ $(n_1>0)$ one has

$$\partial_{\nu_1}H(q)=\frac{\partial^{n_1}h_1(x_0+ix_1)}{\partial x_1^{n_1}}=i^{n_1}h_1^{(n_1)}(x_0+ix_1), \tag{8}$$

$$\frac{|\partial_{\nu_1} H(q)|}{n_1!} = \frac{|h_1^{(n_1)}(x_0+ix_1)|}{n_1!} = \frac{1}{n_1}\cdot\frac{|(h_1'(x_0+ix_1))^{(n_1-1)}|}{(n_1-1)!} \leq$$
$$\leq \frac{1}{n_1}\max\left\{\frac{|(h_1'(x_0+ix_1))^{(m)}|}{m!} : 0\leq m\leq N(h_1')\right\} =$$
$$= \frac{1}{n_1}\max\left\{(1+m)\frac{|h_1^{(1+m)}(x_0+ix_1)|}{(1+m)!} : 1\leq 1+m\leq 1+N(h_1')\right\} \leq$$
$$\leq \frac{1+N(h_1')}{n_1}\max\left\{\frac{|h_1^{(s)}(x_0+ix_1)|}{s!} : 1\leq s\leq 1+N(h_1')\right\},$$

because h_1' is of bounded index $N(h_1')$.

Choosing $n_1 \geq 1+N(h_1')$ and using (8) we obtain for $\nu_1=(n_1,0,0)$

$$\frac{|\partial_{\nu_1} H(q)|}{n_1!} \leq \max\left\{\frac{|h_1^{(s)}(x_0+ix_1)|}{s!} : 1\leq s\leq 1+N(h_1')\right\} =$$
$$= \max\left\{\frac{|\partial_{(s,0,0)} H(q)|}{s!} : 1\leq s\leq 1+N(h_1')\right\}.$$

Then for $\nu_2=(0,n_2,0)$ $(n_1>0)$ one has

$$\partial_{\nu_2} H(q) = j\frac{\partial^{n_2} h_2(x_2+ix_3)}{\partial x_2^{n_2}} = jh_2^{(n_2)}(x_2+ix_3), \tag{9}$$
$$\frac{|\partial_{\nu_2} H(q)|}{n_2!} = \frac{|h_2^{(n_2)}(x_2+ix_3)|}{n_2!} = \frac{1}{n_2}\cdot\frac{|(h_2'(x_2+ix_3))^{(n_2-1)}|}{(n_2-1)!} \leq$$
$$\leq \frac{1}{n_2}\max\left\{\frac{|(h_2'(x_2+ix_3))^{(m)}|}{m!} : 0\leq m\leq N(h_2')\right\} =$$
$$= \frac{1}{n_2}\max\left\{(1+m)\frac{|h_2^{(1+m)}(x_2+ix_3)|}{(1+m)!} : 1\leq 1+m\leq 1+N(h_2')\right\} \leq$$
$$\leq \frac{1+N(h_2')}{n_2}\max\left\{\frac{|h_2^{(s)}(x_2+ix_3)|}{s!} : 1\leq s\leq 1+N(h_2')\right\},$$

because h_2' is of bounded index $N(h_1')$.

Choosing $n_2 \geq 1+N(h_2')$ and using (9) we obtain for $\nu_1=(0,n_1,0)$

$$\frac{|\partial_{\nu_2} H(q)|}{n_2!} \leq \max\left\{\frac{|h_2^{(s)}(x_2+ix_3)|}{s!} : 1\leq s\leq 1+N(h_2')\right\} =$$
$$= \max\left\{\frac{|\partial_{(0,s,0)} H(q)|}{s!} : 1\leq s\leq 1+N(h_2')\right\}.$$

By analogy for $\nu_3=(0,0,n_3)$ $(n_3\geq 1+N(h_2'))$ we deduce

$$\partial_{\nu_3} H(q) = j\frac{\partial^{n_3} h_2(x_2+ix_3)}{\partial x_3^{n_3}} = ji^{n_3}h_2^{(n_3)}(x_2+ix_3).$$

Then in view of (9) one has $ji^s/j\partial_{(0,s,0)}H(q) = \partial_{(0,0,s)}H(q)$ and

$$\frac{|\partial_{\nu_3}H(q)|}{n_3!} \le \max\left\{\frac{|\partial_{(0,s,0)}H(q)|}{s!} : 1 \le s \le 1 + N(h_2')\right\}.$$

Let $\nu = (n_1, n_2, 0)$, $n_1 \ge 1$, $n_2 \ge 1$. Then

$$\partial_\nu H(q) = \frac{\partial^{n_2}}{\partial x_2^{n_2}}\frac{\partial^{n_1}}{\partial x_1^{n_1}}(h_1(x_0 + ix_1) + jh_2(x_2 + jx_3)) = \frac{\partial^{n_2}}{\partial x_2^{n_2}}(i^{n_1}h_1^{(n_1)}(x_0 + ix_1)) = 0.$$

Let $\nu = (n_1, 0, n_3)$, $n_1 \ge 1$, $n_3 \ge 1$. Then

$$\partial_\nu H(q) = \frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_1}}{\partial x_1^{n_1}}(h_1(x_0 + ix_1) + jh_2(x_2 + jx_3)) = \frac{\partial^{n_3}}{\partial x_3^{n_3}}(i^{n_1}h_1^{(n_1)}(x_0 + ix_1)) = 0.$$

Let $\nu = (0, n_2, n_3)$, $n_2 \ge 1$, $n_3 \ge 1$. Then

$$\partial_\nu H(q) = \frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}(h_1(x_0 + ix_1) + jh_2(x_2 + jx_3)) = \frac{\partial^{n_3}}{\partial x_3^{n_3}}(jh_2^{(n_3)}(x_2 + ix_3)) =$$
$$= ji^{n_3}h_2^{(n_2+n_3)}(x_2 + ix_3). \qquad (10)$$

Therefore,

$$\frac{|\partial_{\nu_2}H(q)|}{(n_2+n_3)!} = \frac{|h_2^{(n_2+n_3)}(x_2 + ix_3)|}{(n_2+n_3)!} = \frac{1}{(n_2+n_3)} \cdot \frac{|(h_2'(x_2 + ix_3))^{(n_2+n_3-1)}|}{(n_2+n_3-1)!} \le$$
$$\le \frac{1}{n_2+n_3}\max\left\{\frac{|(h_2'(x_2 + ix_3))^{(m)}|}{m!} : 0 \le m \le N(h_2')\right\} =$$
$$= \frac{1}{n_2+n_3}\max\left\{(1+m)\frac{|h_2^{(1+m)}(x_2 + ix_3)|}{(1+m)!} : 1 \le 1+m \le 1 + N(h_2')\right\} \le$$
$$\le \frac{1+N(h_2')}{n_2+n_3}\max\left\{\frac{|h_2^{(s)}(x_2 + ix_3)|}{s!} : 1 \le s \le 1 + N(h_2')\right\},$$

because h_2' is of bounded index $N(h_2')$.

Choosing $n_2 + n_3 \ge 1 + N(h_2')$ and using (10) and (9) we obtain for $\nu = (0, n_2, n_3)$

$$\frac{|\partial_\nu H(q)|}{n_2!} \le \max\left\{\frac{|h_2^{(s)}(x_2 + ix_3)|}{s!} : 1 \le s \le 1 + N(h_2')\right\} =$$
$$= \max\left\{\frac{|\partial_{(0,s,0)}H(q)|}{s!} : 1 \le s \le 1 + N(h_2')\right\}.$$

If $\nu = (n_1, n_2, n_3)$, $n_1 \ge 1$, $n_2 \ge 1$, $n_3 \ge 1$, then

$$\partial_\nu H(q) = \frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}\frac{\partial^{n_1}}{\partial x_1^{n_1}}(h_1(x_0 + ix_1) + jh_2(x_0 + jx_2)) =$$
$$= \frac{\partial^{n_3}}{\partial x_3^{n_3}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}(i^{n_1}h_1^{(n_1)}(x_0 + ix_1)) = 0.$$

Combining all results, we deduce

$$\frac{|\partial_\nu H(q)|}{(n_1+n_2+n_3)!} \le \max\left\{|H(q)|, \max\left\{\frac{|\partial_{(s,0,0)}H(q)|}{s!} \colon 1 \le s \le 1+N(h_1')\right\},\right.$$
$$\left.\max\left\{\frac{|\partial_{(0,s,0)}H(q)|}{s!} \colon 1 \le s \le 1+N(h_2')\right\}\right\} =$$
$$= \max\left\{\frac{|\partial_{(s_1,s_2,s_3)}H(q)|}{(s_1+s_2+s_3)!} \colon 0 \le s_1+s_2+s_3 \le 1+\max\{N(h_1'),N(h_2')\}\right\}.$$

Therefore, the function H has bounded index and its index $N(h)$ does not exceed $1+\max\{N(h_1'),N(h_2')\}$. □

4. Open problems. The authors in the introduction of [1] claimed that Theorem 3 can become a base for the introduction of the concept of index for monogenic functions in a finite-dimensional commutative algebra [14, 15], regular functions of a quaternionic variable [12], or slice regular functions of a quaternionic variable [7]. In fact, Theorem 4 shows the validity of this statement for the Fueter regular functions of a quaternionic variable.

Fueter's Theorem and Sudbery's Theorem (Theorem 1 and Theorem 2) show another approach to construct Fueter regular functions of quaternionic variable. In this paper we thoroughly discussed the Mariconda approach within the theory of functions having bounded index. The obtained results (Theorem 4 and Theorem 5) lead to the following questions:

1) Are the analogs of the exponent, the sine and the cosine obtained by Fueter's Theorem or by Sudbery's Theorem of bounded index?

2) What are conditions on the function h in Fueter's Theorem and on the function u in Sudbery's Theorem providing the index boundedness of the corresponding Fueter regular function?

It is also posssible to construct the Fueter regular analogs of the exponent, the cosine and the sine using $\sin z$, $\cos z$, $\exp z$ as the function f in Theorem 1 and the real or imaginary parts of sin, cos, exp as the harmonic function u in Theorem 2. The investigation of index boundedness for these functions encounters technical difficulties related with calculation of partial derivatives. Therefore, we are not ready to give an exhaustive answer to the questions.

Acknowledgments. The research of the second author was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

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Received 07.01.2023
Revised 10.10.2023