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ANALYTIC IN THE UNIT POLYDISC FUNCTIONS OF BOUNDED L-INDEX IN DIRECTION

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The concept of bounded L-index in a direction $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ is generalized for a class of analytic functions in the unit polydisc, where L is some continuous function such that for every $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$ one has $L(z) > \beta \max_{1\le j\le n} \frac{|b_j|}{1-|z_j|}$, $\beta = \mathrm{const} > 1$, $\mathbb{D}^n$ is the unit polydisc, i.e. $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| \le 1, j \in \{1, \ldots, n\}\}$. For functions from this class we obtain sufficient and necessary conditions providing boundedness of L-index in the direction. They describe local behavior of maximum modulus of derivatives for the analytic function F on every slice circle $\{z + t\mathbf{b} : |t| = r/L(z)\}$ by their values at the center of the circle, where $t \in \mathbb{C}$. Other criterion describes similar local behavior of the minimum modulus via the maximum modulus for these functions. We proved an analog of the logarithmic criterion desribing estimate of logarithmic derivative outside some exceptional set by the function L. The set is generated by the union of all slice discs $\{z^0 + t\mathbf{b} : |t| \le r/L(z^0)\}$, where z^0 is a zero point of the function F. The analog also indicates the zero distribution of the function F is uniform over all slice discs. In one-dimensional case, the assertion has many applications to analytic theory of differential equations and infinite products, i.e. the Blaschke product, Naftalevich-Tsuji product. Analog of Hayman's Theorem is also deduced for the analytic functions in the unit polydisc. It indicates that in the definition of bounded L-index in direction it is possible to remove the factorials in the denominators. This allows to investigate properties of analytic solutions of directional differential equations.

1. Introduction. A notion of the index for entire functions was firstly appeared in papers of J. Mac-Donnell [19] and B. Lepson [18]. They considered the hyper-Dirichlet series and studied its convergence domain and possible application to infinite order linear differential equation. But the functions having bounded index [11,21,26] belong to the class of functions of exponential type. Therefore, M. Sheremeta and A. Kuzyk [17] introduced the l-index for entire functions with a continuous function $l : \mathbb{C} \to \mathbb{R}_+$. Their approach proved to be quite productive in the scientific sense because it allows to find the l-index for any entire function with bounded multiplicities of zeros [12]. Moreover, the functions of bounded l-index (and bounded index, if $l \equiv 1$) have applications in the analytic theory of differential equations [15,28–30] and the value distribution theory [17,22,24]. One-dimensional Sheremeta-Kuzyk's approach developed in two multidimensional subapproaches: bounded L-index in direction [9] and bounded $\mathbf{L}$-index in joint variables [10]. A notion of bounded index for bivariate entire functions [23, 25]) matches with the notion of bounded $\mathbf{L}$-index in joint variables, if

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$\mathbf{L} \equiv (1, \ldots, 1)$. These approaches allow to deduce many multidimensional analogs for known properties of entire functions of single varible. Moreover, they are applicable completely or partially not only to entire functions of several complex variables [7], but also to analytic functions in a ball [4], in a polydisc [2], in the Cartesian product of a disc and a complex plane [3], to slice entire functions [5] and to slice analytic functions in the unit ball [9]. Nevertheless some important assertions have not full analogs for the bounded $\mathbf{L}$-index in joint variables. For example, in the case of the logatithmic criterion [6, 13, 27] we know only sufficient conditions for the bounded $\mathbf{L}$-index in joint variables [4, 8]. The notion of L-index in direction is more flexible and admits more direct generalizations. Therefore, it leads to the following question: what is the bounded L-index in a direction for functions analytic in some multidimensional complex domain?

For analytic functions in the unit ball there is an exhaustive answer to the question [1]. In addition to the unit ball, there is another interesting multidimensional complex domain. This is the unit polydisc. It is known that these domains are not biholomorphic equivalent. At the same time, there is constructed theory of bounded $\mathbf{L}$-index in joint variables for analytic functions in the unit polydisc [2], but the question of contructing theory of bounded L-index in a direction is still open for these functions.

In view of this, the paper is the first attempt to fill this gap and develop a theory of bounded directional index for the polydisc.

2. Main definitions and notations. Let $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| < 1, j \in \{1, 2, \ldots, n\}\}$ be the unit polydisc, $L\colon\ \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function such that for all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{D}^n$

$$L(z) > \beta \max_{1 \le j \le n} \frac{|b_j|}{1 - |z_j|}, \ \beta = \text{const} > 1. \tag{1}$$

Remark 1. Note that if $\eta \in [0, \beta]$, $z \in \mathbb{D}^n$ and $|t| \le \frac{\eta}{L(z)}$ then $z + t\mathbf{b} \in \mathbb{D}^n$. Indeed, using (1) we have

$$|z_j + tb_j| \le |z_j| + |tb_j| \le |z_j| + \frac{\eta|b_j|}{L(z)} < |z_j| + \frac{\beta|b_j|}{\beta \max_{1 \le s \le n} \frac{|b_s|}{1 - |z_s|}} \le |z_j| + \frac{|b_j|}{\frac{|b_j|}{1 - |z_j|}} = 1.$$

Since for each $j \in \{1, \ldots, n\}$ one has $|z_j + tb_j| < 1$, the point $z + t\mathbf{b}$ is contained in the unit polydisc.

An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{D}^n$ the following inequality is valid

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \le k \le m_0 \right\}, \tag{2}$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\Big(\partial_{\mathbf{b}}^{k-1} F(z)\Big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by* $N_{\mathbf{b}}(F, L) = m_0$. If $n = 1$, $\mathbf{b} = 1$, $L = l$, $F = f$, then $N(f, l) \equiv N_1(f, l)$ is called the l-index of the function f. In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an analytic function in the unit disc of bounded l-index [31].

The positivity and continuity of the function L and condition (1) are not sufficient to explore the behavior of analytic function of bounded L-index in direction. Below we impose an extra condition on behavior of the function L.

For a given $z \in \mathbb{D}^n$ we denote $D_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{D}^n\}$. In other words, $D_z = \{t \in \mathbb{C} : |t| < \min_{1\le j\le n} \frac{1-|z_j|}{|b_j|}\}$. Here if $b_j = 0$ then we suppose $\frac{1-|z_j|}{|b_j|} = +\infty$. Denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{D}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{D}^n)$ stands for a class of positive continuous functions $L : \mathbb{D}^n \to \mathbb{R}_+$, satisfying (1) and

$$(\forall \eta \in [0, \beta]) : \ \lambda_{\mathbf{b}}(\eta) < +\infty. \tag{3}$$

Let $\mathbb{D} \equiv \mathbb{D}^1$, $Q_\beta(\mathbb{D}) \equiv Q_1(\mathbb{D})$. Using definition of $Q_{\mathbf{b}}(\mathbb{D}^n)$ it is not difficult to prove that if $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n : |z_j| \le 1, j \in \{1, 2, \dots, n\}\}$, $L : \overline{\mathbb{D}}^n \to \mathbb{R}_+$ is a continuous function, $m = \min\{L(z) : z \in \overline{\mathbb{D}}^n\}$ then $\widetilde{L}(z) = \frac{\beta}{m} L(z) \cdot \max_{1\le j\le n} \frac{|b_j|}{(1-|z_j|)^\alpha} \in Q_{\mathbf{b}}(\mathbb{D}^n)$ for every $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $\alpha \ge 1$.

3. Criteria of L-index boundedness in direction, which describe local behavior of the function F.

Theorem 1. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if for every $\eta \in (0, \beta]$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for each $z \in \mathbb{D}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$ with $0 \le k_0 \le n_0$ and the following inequality holds*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \eta/L(z)\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{4}$$

Proof. **Necessity.** Let F be of bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F; L) \equiv N < +\infty$. We denote

$$q(\eta) = [2\eta(N+1)(\lambda^{\mathbf{b}}(\eta))^{2N+1}] + 1,$$

where $[a]$ stands for the integer part of the number $a \in \mathbb{R}$. For $z \in \mathbb{D}^n$ and $p \in \{0, 1, \dots, q(\eta)\}$ we put

$$R_p^{\mathbf{b}}(z, \eta) = \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z+t\mathbf{b})} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\},$$

$$\widetilde{R}_p^{\mathbf{b}}(z, \eta) = \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k! L^k(z)} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\}.$$

However, $|t| \le \frac{p\eta}{q(\eta)L(z)} \le \frac{\eta}{L(z)}$, then $\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \le \lambda_{\mathbf{b}}(\eta)$. It is clear that $R_p^{\mathbf{b}}(z, \eta)$, $\widetilde{R}_p^{\mathbf{b}}(z, \eta)$ are well-defined. Moreover,

$$\begin{aligned} R_p^{\mathbf{b}}(z,\eta) = & \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\frac{L(z)}{L(z+t\mathbf{b})}\right)^k : 0 \le k \le N, |t| \le \frac{p\eta}{q(\eta)L(z)} \right\} \le \\ \le & \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right)\right)^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le & \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} (\lambda_{\mathbf{b}}(\eta))^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le (\lambda_{\mathbf{b}}(\eta))^N & \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} = \widetilde{R}_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N, \end{aligned} \tag{5}$$

and

$$\begin{gathered}\widetilde{R}_p^{\mathbf{b}}(z,\eta)=\max\left\{\tfrac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})}\left(\tfrac{L(z+t\mathbf{b})}{L(z)}\right)^k : |t|\le\tfrac{p\eta}{q(\eta)L(z)}, 0\le k\le N\right\}\le\\ \le\max\left\{\tfrac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})}\left(\lambda_{\mathbf{b}}\left(\tfrac{p\eta}{q(\eta)}\right)\right)^k : |t|\le\tfrac{p\eta}{q(\eta)L(z)}, 0\le k\le N\right\}\le\\ \le\max\left\{(\lambda_{\mathbf{b}}(\eta))^k\tfrac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t|\le\tfrac{p\eta}{q(\eta)L(z)}, 0\le k\le N\right\}\le\\ \le(\lambda_{\mathbf{b}}(\eta))^N\max\left\{\tfrac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t|\le\tfrac{p\eta}{q(\eta)L(z)},\ 0\le k\le N\right\}=R_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N.\end{gathered}\tag{6}$$

Let $k_p^z\in\mathbb{Z}$, $0\le k_p^z\le N$, and $t_p^z\in\mathbb{C}$, $|t_p^z|\le\dfrac{p\eta}{q(\eta)L(z)}$, be such that

$$\frac{|\partial_{\mathbf{b}}^{k_p^z}F(z+t_p^z\mathbf{b})|}{k_p^z!L^{k_p^z}(z)}=\widetilde{R}_p^{\mathbf{b}}(z,\eta).\tag{7}$$

For every given $z\in\mathbb{D}^n$ the function $F(z+t\mathbf{b})$ and its directional derivatives are analytic functions in variable $t\in D_z$. By the maximum modulus principle, the equality (7) holds for such t_p^z that $|t_p^z|=\dfrac{p\eta}{q(\eta)L(z)}$. We set $\widetilde{t_p^z}=\frac{p-1}{p}t_p^z$. Then

$$|\widetilde{t_p^z}|=\frac{(p-1)\eta}{q(\eta)L(z)},\quad |\widetilde{t_p^z}-t_p^z|=\frac{|t_p^z|}{p}=\frac{\eta}{q(\eta)L(z)}.\tag{8}$$

It follows from (8) and the definition of $\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$ that $\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)\ge\frac{|\partial_{\mathbf{b}}^{k_p^z}F(z+\widetilde{t_p^z}\mathbf{b})|}{k_p^z!L^{k_p^z}(z)}$. Therefore,

$$\begin{gathered}0\le\widetilde{R}_p^{\mathbf{b}}(z,\eta)-\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)\le\frac{\left|\partial_{\mathbf{b}}^{k_p^z}F(z+t_p^z\mathbf{b})\right|-\left|\partial_{\mathbf{b}}^{k_p^z}F(z+\widetilde{t_p^z}\mathbf{b})\right|}{k_p^z!L^{k_p^z}(z)}=\\ =\frac{1}{k_p^z!L^{k_p^z}(z)}\int_0^1\frac{d}{ds}\left|\partial_{\mathbf{b}}^{k_p^z}F(z+(\widetilde{t_p^z}+s(t_p^z-\widetilde{t_p^z}))\mathbf{b})\right|ds.\end{gathered}\tag{9}$$

For every analytic complex-valued function of real variable $\varphi(s)$, $s\in\mathbb{R}$, the inequality $\frac{d}{ds}|\varphi(s)|\le\left|\frac{d}{ds}\varphi(s)\right|$ holds where $\varphi(s)\ne0$. Applying this inequality to (9) and using the mean value theorem we obtain

$$\begin{gathered}\widetilde{R}_p^{\mathbf{b}}(z,\eta)-\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)\le\frac{|t_p^z-\widetilde{t_p^z}|}{k_p^z!L^{k_p^z}(z)}\int_0^1\left|\partial_{\mathbf{b}}^{k_p^z+1}F(z+(\widetilde{t_p^z}+s(t_p^z-\widetilde{t_p^z}))\mathbf{b})\right|ds=\\ =\frac{|t_p^z-\widetilde{t_p^z}|}{k_n^p!L^{k_p^z}(z)}\left|\partial_{\mathbf{b}}^{k_p^z+1}F(z+(\widetilde{t_p^z}+s^*(t_p^z-\widetilde{t_p^z}))\mathbf{b})\right|=\\ =\frac{1}{(k_p^z+1)!L^{k_p^z+1}(z)}\left|\partial_{\mathbf{b}}^{k_p^z+1}F(z+(\widetilde{t_p^z}+s^*(t_p^z-\widetilde{t_p^z}))\mathbf{b})\right|L(z)(k_p^z+1)|t_p^z-\widetilde{t_p^z}|,\end{gathered}$$

where $s^*\in[0,1]$.

The point $\widetilde{t_p^z}+s^*(t_p^z-\widetilde{t_p^z})$ belongs to the set $\left\{t\in\mathbb{C}:|t|\le\dfrac{p\eta}{q(\eta)L(z)}\right\}$. Using the definition of bounded L-index in the direction $\mathbf{b}$, the definition of $q(\eta)$, inequality (5) and (8), for

$k_p^z \leq N$ we have

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta)-\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)\leq \frac{|\partial_{\mathbf{b}}^{k_p^z+1}F(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})|}{(k_p^z+1)!L^{k_p^z+1}(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})}\times$$
$$\times\left(\frac{L(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})}{L(z)}\right)^{k_p^z+1}L(z)(k_p^z+1)|t_p^z-\widetilde{t}_p^z|\leq\eta\frac{N+1}{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N+1}\times$$
$$\times\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})|}{k!L^{k}(z+(\widetilde{t}_p^z+s^*(t_p^z-\widetilde{t}_p^z))\mathbf{b})}:0\leq k\leq N\right\}\leq\eta\frac{N+1}{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N+1}R_p^{\mathbf{b}}(z,\eta)\leq$$
$$\leq\frac{\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}}{[2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}]+1}\widetilde{R}_p^{\mathbf{b}}(z,\eta)\leq\frac{1}{2}\widetilde{R}_p^{\mathbf{b}}(z,\eta).$$

It follows that $\widetilde{R}_p^{\mathbf{b}}(z,\eta)\leq 2\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$. Using inequalities (5) and (6), we deduce for $R_p^{\mathbf{b}}(z,\eta)$

$$R_p^{\mathbf{b}}(z,\eta)\leq 2(\lambda_{\mathbf{b}}(\eta))^N\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)\leq 2(\lambda_{\mathbf{b}}(\eta))^{2N}R_{p-1}^{\mathbf{b}}(z,\eta).$$

Hence,

$$\begin{aligned}&\max\left\{\tfrac{|\partial_{\mathbf{b}}^kF(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})}:|t|\leq\tfrac{\eta}{L(z)},0\leq k\leq N\right\}=R_{q(\eta)}^{\mathbf{b}}(z,\eta)\leq\\ &\leq 2(\lambda_{\mathbf{b}}(\eta))^{2N}R_{q(\eta)-1}^{\mathbf{b}}(z,\eta)\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^2R_{q(\eta)-2}^{\mathbf{b}}(z,\eta)\leq\\ &\leq\cdots\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}R_0^{\mathbf{b}}(z,\eta)=\\ &=(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\max\left\{\tfrac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}:0\leq k\leq N\right\}.\end{aligned}\tag{10}$$

Let $k_z\in\mathbb{Z}$, $0\leq k_z\leq N$, and $\widetilde{t}_z\in\mathbb{C}$, $|\widetilde{t}_z|=\dfrac{\eta}{L(z)}$, be such that

$$\frac{|\partial_{\mathbf{b}}^{k_z}F(z)|}{k_z!L^{k_z}(z)}=\max_{0\leq k\leq N}\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)},$$

and

$$|\partial_{\mathbf{b}}^{k_z}F(z+\widetilde{t}_z\mathbf{b})|=\max\{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|:|t|\leq\eta/L(z)\}.$$

Inequality (10) implies

$$\frac{|\partial_{\mathbf{b}}^{k_z}F(z+\widetilde{t}_z\mathbf{b})|}{k_z!L^{k_z}(z+\widetilde{t}_z\mathbf{b})}\leq\max\left\{\frac{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|}{k_z!L^{k_z}(z+t\mathbf{b})}:|t|=\frac{\eta}{L(z)}\right\}\leq$$
$$\leq\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z+t\mathbf{b})|}{k!L^{k}(z+t\mathbf{b})}:|t|=\frac{\eta}{L(z)},0\leq k\leq N\right\}\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\frac{|\partial_{\mathbf{b}}^{k_z}F(z)|}{k_z!L^{k_z}(z)}.$$

Hence, we get

$$\max\left\{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|:|t|\leq\eta/L(z)\right\}\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\frac{L^{k_z}(z+\widetilde{t}_z\mathbf{b})}{L^{k_z}(z)}|\partial_{\mathbf{b}}^{k_z}F(z)|\leq$$
$$\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^N|\partial_{\mathbf{b}}^{k_z}F(z)|\leq(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^N|\partial_{\mathbf{b}}^{k_z}F(z)|.$$

We conclude (4) with $n_0=N_{\mathbf{b}}(F,L)$ and

$$P_1(\eta)=(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^N>1.$$

Sufficiency. We suppose that for every $\eta \in (0,\beta]$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $z \in \mathbb{D}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, for which inequality (4) holds. We choose $\eta > 1$ and $j_0 \in \mathbb{N}$ satisfying $P_1 \leq \eta^{j_0}$. For given $z \in \mathbb{D}^n$, $k_0 = k_0(z)$ and $j \geq j_0$ by Cauchy's formula for $F(z + t\mathbf{b})$ as a function of variable t

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int_{|t|=\eta/L(z)} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

In view of (4) we have

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \leq \frac{L^j(z)}{\eta^j} \max\left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| = \frac{\eta}{L(z)} \right\} \leq P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

that is

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \leq \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}$$

for all $j \geq j_0$.

Since $k_0 \leq n_0$, $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z, this inequality means that the function F is of bounded L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$. Theorem 1 is proved. □

Theorem 2. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\frac{1}{\beta} < \theta_1 \leq \theta_2 < +\infty$, $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L^*-index in the direction $\mathbf{b}$ if and only if F is of bounded L-index in the direction $\mathbf{b}$.*

Proof. Obviously, if $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$, then $L^* \in Q_{\mathbf{b},\beta^*}(\mathbb{D}^n)$, $\beta^* \in [\theta_1\beta; \theta_2\beta]$ and $\beta^* > 1$.

Let $N_{\mathbf{b}}(F, L^*) < +\infty$. Therefore, by Theorem 1 for each η^*, $0 < \eta^* < \beta\theta_2$, there exist $n_0(\eta^*) \in \mathbb{Z}_+$ and $P_1(\eta^*) \geq 1$ such that for every $z \in \mathbb{D}^n$, $t_0 \in S_z$ and some k_0, $0 \leq k_0 \leq n_0$, inequality (4) is valid with L^* and η^* instead of L and η. Taking $\eta* = \theta_2\eta$ we obtain

$$P_1 |\partial_{\mathbf{b}}^{k_0} F(z)| \geq \max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta^*/L^*(z)\} \geq \max\left\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : \ |t| \leq \eta/L(z)\right\}.$$

Therefore, by Theorem 1 the function $F(z)$ is of bounded L-index in the direction $\mathbf{b}$. The converse assertion is obtained by replacing L on L^*. □

Theorem 3. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $m \in \mathbb{C} \setminus \{0\}$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if $F(z)$ is of bounded L-index in the direction $m\mathbf{b}$.*

Proof. Let $F(z)$ be an analytic function in $\mathbb{D}^n$ of bounded L-index in the direction $\mathbf{b}$. By Theorem 1 $(\forall \eta > 0)$ $(\exists n_0(\eta) \in \mathbb{Z}_+)$ $(\exists P_1(\eta) \geq 1)$ $(\forall z \in \mathbb{D}^n)$ $(\exists k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0)$, and the following inequality is valid

$$\max\left\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{11}$$

Since $\partial_{m\mathbf{b}}^k F = m^k \partial_{\mathbf{b}}^k F$, inequality (11) is equivalent to the inequality

$$\max\left\{|m|^{k_0} |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1 |m|^{k_0} |\partial_{\mathbf{b}}^{k_0} F(z)|$$

as well as to the inequality

$$\max\left\{\left|\partial^{k_0}_{m\mathbf{b}}F\big(z+\frac{t}{m}m\mathbf{b}\big)\right| : |t/m|\le \eta/(|m|L(z))\right\}\le P_1|\partial^{k_0}_{m\mathbf{b}}F(z)|.$$

Denoting $t^*=\frac{t}{m}$, $\eta^*=\frac{\eta}{|m|}$, we obtain

$$\max\left\{|\partial^{k_0}_{m\mathbf{b}}F(z+t^*m\mathbf{b})| : |t^*|\le \eta^*/L(z)\right\}\le P_1|\partial^{k_0}_{\mathbf{b}}F(z)|.$$

By Theorem 1 the function $F(z)$ is of bounded L-index in the direction $\mathbf{b}$. The converse assertion can be proved similarly. □

Using Fricke's idea [14], we deduce a modification of Theorem 1.

Theorem 4. *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$. If there exist $\eta\in(0,\beta]$, $n_0=n_0(\eta)\in\mathbb{Z}_+$ and $P_1=P_1(\eta)\ge 1$ such that for any $z\in\mathbb{D}^n$ there exists $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, and*

$$\max\{|\partial^{k_0}_{\mathbf{b}}F(z+t\mathbf{b})| : |t|\le \eta/L(z)\}\le P_1|\partial^{k_0}_{\mathbf{b}}F(z)|,$$

then the analytic function $F:\mathbb{D}^n\to\mathbb{C}$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$.

Proof. Assume that there exist $\eta\in(0,\beta]$, $n_0=n_0(\eta)\in\mathbb{Z}_+$ and $P_1=P_1(\eta)\ge 1$ such that for any $z\in\mathbb{D}^n$ there exists $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, and

$$\max\{|\partial^{k_0}_{\mathbf{b}}F(z+t\mathbf{b})| : |t|\le \frac{\eta}{L(z)}\}\le P_1|\partial^{k_0}_{\mathbf{b}}F(z)|. \tag{12}$$

If $\eta\in(1,\beta]$, then we choose $j_0\in\mathbb{N}$ such that $P_1\le\eta^{j_0}$. And for $\eta\in(0;1]$ we choose $j_0\in\mathbb{N}$ such that $\frac{j_0!k_0!}{(j_0+k_0)!}P_1<1$. The j_0 is well-defined because

$$\frac{j_0!k_0!}{(j_0+k_0)!}P_1=\frac{k_0!}{(j_0+1)(j_0+2)\cdot\ldots\cdot(j_0+k_0)}P_1\to 0,\ j_0\to\infty.$$

Applying integral Cauchy's formula to the function $F(z+t\mathbf{b})$ as analytic function of one complex variable t for $j\ge j_0$ we obtain that for every $z\in\mathbb{D}^n$ there exists $k_0=k_0(z)$, $0\le k_0\le n_0$, and

$$\partial^{k_0+j}_{\mathbf{b}}F(z)=\frac{j!}{2\pi i}\int\limits_{|t|=\frac{\eta}{L(z)}}\frac{\partial^{k_0}_{\mathbf{b}}F(z+t\mathbf{b})}{t^{j+1}}dt.$$

Taking into account (12), we deduce

$$\frac{|\partial^{k_0+j}_{\mathbf{b}}F(z)|}{j!}\le\frac{L^j(z)}{\eta^j}\max\left\{|\partial^{k_0}_{\mathbf{b}}F(z+t\mathbf{b})| :\ |t|=\frac{\eta}{L(z)}\right\}\le P_1\frac{L^j(z)}{\eta^j}|\partial^{k_0}_{\mathbf{b}}F(z)|. \tag{13}$$

In view of choice j_0 with $\eta\in(1,\beta]$, for all $j\ge j_0$ one has

$$\frac{|\partial^{k_0+j}_{\mathbf{b}}F(z)|}{(k_0+j)!L^{k_0+j}(z)}\le\frac{j!k_0!}{(j+k_0)!}\frac{P_1}{\eta^j}\frac{|\partial^{k_0}_{\mathbf{b}}F(z)|}{k_0!L^{k_0}(z+t_0\mathbf{b})}\le\eta^{j_0-j}\frac{|\partial^{k_0}_{\mathbf{b}}F(z)|}{k_0!L^{k_0}(z)}\le\frac{|\partial^{k_0}_{\mathbf{b}}F(z)|}{k_0!L^{k_0}(z)}.$$

Since $k_0\le n_0$, the numbers $n_0=n_0(\eta)$ and $j_0=j_0(\eta)$ do not depend on z, and $z\in\mathbb{D}^n$ is arbitrary, the last inequality is equivalent to the assertion that F has boudned L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F,L)\le n_0+j_0$.

If $\eta \in (0,1)$, then from (13) it follows that for all $j \geq j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j}F(z)|}{(k_0+j)!L^{k_0+j}(z)} \leq \frac{j!k_0!P_1}{(j+k_0)!}\frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{\eta^j k_0!L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{\eta^j k_0!L^{k_0}(z)}$$

or in view of the choice j_0

$$\frac{|\partial_{\mathbf{b}}^{k_0+j}F(z)|}{(k_0+j)!}\frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0}F(z)|}{k_0!}\frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function F is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then by Theorem 2 the function F has bounded L-index in the direction $\mathbf{b}$, if $\eta\beta > 1$. When $\eta \leq \frac{1}{\beta}$, we choose an arbitrary $\gamma > \frac{1}{\eta\beta}$. By Theorem 2 the function F is of bounded L_1-index in the direction $\mathbf{b}$, where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then be Theorem 3 the function F has bounded L_1-index in the direction $\gamma\mathbf{b}$. Since $\partial_{\gamma\mathbf{b}}^k F = \gamma^k \partial_{\mathbf{b}}^k F$ and $L_1^k(z) = \gamma^k L^k(z)$, in inequality (2) with the definition of L-index boundedness in direction the corresponding multiplier γ is reduced. Hence, the function F is of bounded L-index in the direction $\mathbf{b}$. The theorem is proved. □

The following propostion is directly deduced from the definition of L-index boundedness in direction.

Proposition 1. *Let $L : \mathbb{D}^n \to \mathbb{C}$ be a positive continuous function. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if the function $G(z) = F(\mathbf{a}z + \mathbf{c})$ has bounded L_*-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ for any $\mathbf{c} \in \mathbb{D}^n$ and $\mathbf{a} \in \mathbb{D}^n$ such that $|c_j| < 1 - |a_j|$, $a_j \neq 0$ $(\forall j \in \{1,\dots,n\})$, where $\mathbf{a}z + \mathbf{c} = (a_1z_1 + c_1, \dots, a_nz_n + c_n)$, $\frac{\mathbf{b}}{\mathbf{a}} = (\frac{b_1}{a_1}, \dots, \frac{b_n}{a_n})$, $L_*(z) = L(\mathbf{a}z + \mathbf{c})$.*

Proof. Let an analytic function F in $\mathbb{D}^n$ be of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$. One should observe that

$$\partial_{\mathbf{b}/\mathbf{a}}G(z) = \sum_{j=1}^{n}\frac{\partial G(z)}{\partial z_j}\frac{b_j}{a_j} = \sum_{j=1}^{n}\frac{\partial F(\mathbf{a}z+\mathbf{c})}{\partial z_j}a_j\frac{b_j}{a_j} = \partial_{\mathbf{b}}F(\mathbf{a}z+\mathbf{c}).$$

By the mathematical induction it is easy to prove that $\partial_{\mathbf{b}/\mathbf{a}}^k G(z) = \partial_{\mathbf{b}}^k F(\mathbf{a}z+\mathbf{c})$ for all $k \in \mathbb{N}$. From inequality (2) with $\mathbf{a}z + \mathbf{c}$ instead of z it follows

$$\frac{|\partial_{\mathbf{b}/\mathbf{a}}^m G(z)|}{m!L_*^m(z)} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(\mathbf{a}z+\mathbf{c})|}{k!L^k(\mathbf{a}z+\mathbf{c})} : 0 \leq k \leq m_0\right\} = \max\left\{\frac{|\partial_{\mathbf{b}/\mathbf{a}}^k G(z)|}{k!L_*^k(z)} : 0 \leq k \leq m_0\right\}.$$

The last inequality yields that the function $G(z)$ is of bounded L_*-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ and vice versa. □

4. Estimate of maximum modulus on a larger circle via maximum modulus on a smaller circle and via minimum modulus. Now we consider the behavior of analytic functions in the unit polydisc of bounded L-index in direction. Using Theorem 1, we prove a criterion of L-index boundedness in direction.

Theorem 5. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for any r_1 and any r_2 with $0 < r_1 < r_2 \le \beta$, there exists a number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{D}^n$*

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})| : \ |t| = \frac{r_1}{L(z_0)}\right\}. \tag{14}$$

Proof. **Necessity.** Let $N_{\mathbf{b}}(F, L) < +\infty$. On the contrary, we assume that there exist numbers r_1 and r_2, $0 < r_1 < r_2 \le \beta$, such that for every $P_* \ge 1$ there exists $z^* = z^*(P_*) \in \mathbb{D}^n$, for which the following inequality is valid

$$\max\left\{|F(z^* + t\mathbf{b})| : |t| = \frac{r_2}{L(z^*)}\right\} > P_* \max\left\{|F(z^* + t\mathbf{b})| : \ |t| = \frac{r_1}{L(z^*)}\right\}. \tag{15}$$

By Theorem 1 there exist $n_0 = n_0(r_2) \in \mathbb{Z}_+$ and $P_0 = P_0(r_2) \ge 1$ such that for every $z^* \in \mathbb{D}^n$ and some $k_0 = k_0(z^*) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, one has

$$\max\left\{\left|\partial_{\mathbf{b}}^{k_0} F(z^* + t\mathbf{b})\right| : |t| = r_2/L(z^*)\right\} \le P_0 |\partial_{\mathbf{b}}^{k_0} F(z^*)|. \tag{16}$$

We remark that for $k_0 = 0$ the proof of necessity is obvious because (16) yields $\max\left\{|F(z^* + t\mathbf{b})| : \ |t| = r_2/L(z^*)\right\} \le P_0 |F(z^*)| \le P_0 \max\left\{|F(z^* + t\mathbf{b})| : \ |t| = r_1/L(z^*)\right\}$.

Suppose that $k_0 > 0$. Put

$$P_* = n_0! \left(\frac{r_2}{r_1}\right)^{n_0} \left(P_0 + \frac{r_1}{r_2 - r_1}\right) + 1. \tag{17}$$

We assume $t_0 \in D_{z^*}$ is such that $|t_0| = r_1/L(z^*)$ and

$$|F(z^* + t_0\mathbf{b})| = \max\{|F(z^* + t\mathbf{b})| : |t| = r_1/L(z^*)\} > 0,$$

and $t_{0j} \in D_{z^*}$, $|t_{0j}| = r_2/L(z^*)$, is such that

$$|\partial_{\mathbf{b}}^j F(z^* + t_{0j}\mathbf{b})| = \max\{|\partial_{\mathbf{b}}^j F(z^* + t\mathbf{b})| : |t| = r_2/L(z^*)\},$$

$j \in \mathbb{Z}_+$. In the case $|F(z^* + t_0\mathbf{b})| = 0$ by the uniqueness theorem for all $t \in D_{z^*}$ we obtain $F(z^* + t\mathbf{b}) = 0$. However, it contradicts inequality (15). By Cauchy's inequality we have

$$\frac{|\partial_{\mathbf{b}}^j F(z^*)|}{j!} \le \left(\frac{L(z^*)}{r_1}\right)^j |F(z^* + t_0\mathbf{b})|, j \in \mathbb{Z}_+ \tag{18}$$

$$\left|\partial_{\mathbf{b}}^j F(z^* + t_{0j}\mathbf{b}) - \partial_{\mathbf{b}}^j F(z^*)\right| = \left|\int_0^{t_{0j}} \partial_{\mathbf{b}}^{j+1} F(z^* + t\mathbf{b})\, dt\right| \le \left|\partial_{\mathbf{b}}^{j+1} F(z^* + t_{0(j+1)}\mathbf{b})\right| \frac{r_2}{L(z^*)}. \tag{19}$$

From (18) and (19) we have

$$|\partial_{\mathbf{b}}^{j+1} F(z^* + t_{0(j+1)}\mathbf{b})| \ge \frac{L(z^*)}{r_2} \left\{|\partial_{\mathbf{b}}^j F(z^* + t_{0j}\mathbf{b})| - |\partial_{\mathbf{b}}^j F(z^*)|\right\} \ge$$
$$\ge \frac{L(z^* + t^*\mathbf{b})}{r_2} \left|\partial_{\mathbf{b}}^j F(z^* + t_{0j}\mathbf{b})\right| - \frac{j! L^{j+1}(z^*)}{r_2 (r_1)^j} |F(z^* + t_0\mathbf{b})|,$$

where $j\in\mathbb{Z}_+$. Hence, for $k_0\geq 1$ we get

$$|\partial_{\mathbf{b}}^{k_0}F(z^*+t_{0k_0}\mathbf{b})|\geq\frac{L(z^*)}{r_2}|\partial_{\mathbf{b}}^{k_0-1}F(z^*+t_{0(k_0-1)}\mathbf{b})|-$$
$$-\frac{(k_0-1)!L^{k_0}(z^*)}{r_2(r_1)^{k_0-1}}|F(z^*+t_0\mathbf{b})|\geq\ldots\geq\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}|F(z^*+t_{00}\mathbf{b})|-$$
$$-\left(\frac{0!}{(r_2)^{k_0}}+\frac{1!}{(r_2)^{k_0-1}r_1}+\ldots+\frac{(k_0-1)!}{r_2(r_1)^{k_0-1}}\right)L^{k_0}(z^*)|F(z^*+t_0\mathbf{b})|=$$
$$=\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}|F(z^*+t_0\mathbf{b})|\left(\frac{|F(z^*+t_{00}\mathbf{b})|}{|F(z^*+t_0\mathbf{b})|}-\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\right).\tag{20}$$

In view of (15) we have $|F(z^*+t_{00}\mathbf{b})|/|F(z^*+t_0\mathbf{b})|>P_*$. Besides, this inequality holds

$$\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\leq k_0!\left(\frac{(r_2/r_1)^{k_0}-1}{r_2/r_1-1}\right)\leq n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}.$$

Applying (17), we obtain

$$\frac{|F(z^*+t_{00}\mathbf{b})|}{|F(z^*+t_0\mathbf{b})|}-\sum_{j=0}^{k_0-1}j!\frac{r_2^j}{r_1^j}>P_*-\frac{n_0!r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}=n_0!\left(\frac{r_2}{r_1}\right)^{n_0}P_0+1.$$

It follows from (20), (16) and (18) that

$$\left|\partial_{\mathbf{b}}^{k_0}F(z^*+t_{0k_0}\mathbf{b})\right|>\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}\left(P_*-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\left(\frac{r_1}{L(z^*)}\right)^{k_0}\times$$
$$\times\frac{|\partial_{\mathbf{b}}^{k_0}F(z^*)|}{k_0!}\geq\left(\frac{r_1}{r_2}\right)^{n_0}\left(P_*-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\frac{|\partial_{\mathbf{b}}^{k_0}F(z^*+t_{0k_0}\mathbf{b})|}{n_0!P_0}.$$

Hence, $P_*<n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_0+\frac{r_1}{r_2-r_1}\right)$ which contradicts (17).

Sufficiency. We choose any two numbers $r_1\in(0,1)$ and $r_2\in(1,\beta)$. For given $z^0\in\mathbb{D}^n$ we expand the function $F(z^0+t\mathbf{b})$ in a power series by powers of t

$$F(z^0+t\mathbf{b})=\sum_{m=0}^{\infty}b_m(z^0)t^m,\ b_m(z^0)=\frac{\partial_{\mathbf{b}}^{m}F(z^0)}{m!}$$

in the disc $\left\{t:|t|\leq\dfrac{\beta}{L(z^0)}\right\}\subset D_{z^0}$. For $r\leq\dfrac{\beta}{L(z^0)}$ we denote

$$M_{\mathbf{b}}(r,z^0,F)=\max\{|F(z^0+t\mathbf{b})|:|t|=r\},\quad \mu_{\mathbf{b}}(r,z^0,F)=\max\{|b_m(z^0)|r^m:m\geq 0\},$$
$$\nu_{\mathbf{b}}(r,z^0,F)=\max\{|b_m(z^0)|r^m:|b_m(z^0)|r^m=\mu_{\mathbf{b}}(r,z^0,F)\}.$$

By Cauchy's inequality $\mu_{\mathbf{b}}(r,z^0,F)\leq M_{\mathbf{b}}(r,z^0,F)$. But for $r=1/L(z^0)$ we have

$$M_{\mathbf{b}}(r_1r,z^0,F)\leq\sum_{m=0}^{\infty}|b_m(z^0)|r^mr_1^m\leq\mu_{\mathbf{b}}(r,z^0,F)\sum_{m=0}^{\infty}r_1^m=\frac{\mu_{\mathbf{b}}(r,z^0,F)}{1-r_1}$$

and since $\nu_{\mathbf{b}}(r, z^0, F)$ is monotone in r, we deduce

$$\ln \mu_{\mathbf{b}}(r_2 r, z^0, F) - \ln \mu_{\mathbf{b}}(r, z^0, F) = \int_r^{r_2 r} \frac{\nu_{\mathbf{b}}(t, z^0, F)}{t} dt \geq \nu_{\mathbf{b}}(r, z^0, F) \ln r_2.$$

Hence,

$$\nu_{\mathbf{b}}(r, z^0, F) \leq \frac{1}{\ln r_2}(\ln \mu_{\mathbf{b}}(r_2 r, z^0, F) - \ln \mu_{\mathbf{b}}(r, z^0, F)) \leq$$
$$\leq \frac{1}{\ln r_2}\{\ln M_{\mathbf{b}}(r_2 r, z^0, F) - \ln((1-r_1)M_{\mathbf{b}}(r_1 r, z^0, F))\} =$$
$$= -\frac{\ln(1-r_1)}{\ln r_2} + \frac{1}{\ln r_2}\{\ln M_{\mathbf{b}}(r_2 r, z^0, F) - \ln M_{\mathbf{b}}(r_1 r, z^0, F))\} \qquad (21)$$

Let $N_{\mathbf{b}}(z^0, L, F)$ be the L-index in the direction $\mathbf{b}$ of the function F at the point z^0, i. e. $N_{\mathbf{b}}(z^0, L, F)$ is the smallest number m_0 for which inequality (2) holds with $z = z^0$. It is obvious that $N_{\mathbf{b}}(z^0, L, F) \leq \nu_{\mathbf{b}}(1/L(z^0, z^0, F) = \nu_{\mathbf{b}}(r, z^0, F)$. However, inequality (14) can be written in the following form $M_{\mathbf{b}}\left(\frac{r_2}{L(z^0)}, z^0, F\right) \leq P_1(r_1, r_2) M_{\mathbf{b}}\left(\frac{r_1}{L(z^0)}, z^0, F\right)$. Thus, from (21) we obtain $N_{\mathbf{b}}(z^0, L, F) \leq -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1, r_2)}{\ln r_2}$ for every $z^0 \in \mathbb{D}^n$, i.e. $N_{\mathbf{b}}(F, L) \leq -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1, r_2)}{\ln r_2}$. Theorem 5 is proved. □

In view of the proof of Theorem 5 the following theorem is true.

Theorem 6. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist numbers r_1 and r_2, $0 < r_1 < 1 < r_2 \leq \beta$, and $P_1 \geq 1$ such that for every $z^0 \in \mathbb{D}^n$ and $t_0 \in D_{z^0}$ inequality (14) holds.*

Theorem 7. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, F be a function analytic in $\mathbb{D}^n$. If there exist r_1 and r_2, $0 < r_1 < r_2 \leq \beta$, and $P_1 \geq 1$ such that for all $z^0 \in \mathbb{D}^n$ inequality* (14) *is satisfied, then the function F is of bounded L-index in the direction* $\mathbf{b}$.

Proof. Inequality (14) for $0 < r_1 < r_2 < \beta$ implies

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{2r_2}{r_1 + r_2}\frac{r_1 + r_2}{2L(z^0)}\right\} \leq P_1 \max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{2r_1}{r_1 + r_2}\frac{r_1 + r_2}{2L(z_0)}\right\}.$$

Putting $L^*(z) = \frac{2L(z)}{r_1 + r_2}$, we obtain

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{2r_2}{(r_1 + r_2)L^*(z^0)}\right\} \leq P_1 \max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{2r_1}{(r_1 + r_2)L^*(z^0)}\right\}, \qquad (22)$$

where $0 < \frac{2r_1}{r_1+r_2} < 1 < \frac{2r_2}{r_1+r_2} < \frac{2\beta}{r_1+r_2}$. Clearly, $L^*(z) = \frac{2L(z)}{r_1+r_2} > \frac{2\beta}{(r_1+r_2)} \max_{1\leq j\leq n} \frac{|b_j|}{(1-|z_j|)}$, i.e., L^* satisfies (1) and belongs to the class $Q_{\mathbf{b}}(\mathbb{D}^n)$ with $\frac{2\beta}{r_1+r_2}$ instead β. From validity of inequality (22) we get that by Theorem 6 the function F has bounded L^*-index in the direction $\mathbf{b}$. And by Theorem 2 the function F has bounded L-index in the direction $\mathbf{b}$. □

The following theorem gives an estimate of the maximum modulus by the minimum modulus.

Theorem 8. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if for every R, $0 < R \leq \beta$, there exist numbers $P_2(R) \geq 1$ and $\eta(R) \in (0, R)$ such that for each $z^0 \in \mathbb{D}^n$ and some $r = r(z^0) \in [\eta(R), R]$*

$$\max\Big\{|F(z^0 + t\mathbf{b})| : |t| = r/L(z^0)\Big\} \leq P_2 \min\Big\{|F(z^0 + t\mathbf{b})| : \ |t| = r/L(z^0)\Big\}. \tag{23}$$

Proof. **Necessity.** Let $N_{\mathbf{b}}(F, L) = N < +\infty$ and $R \geq 0$. We put

$$R_0 = 1, r_0 = \frac{R}{8(R+1)}, \ R_j = \frac{R_{j-1}}{4N} r_{j-1}^N, \ r_j = \frac{1}{8} R_j (j = 1, 2, \ldots, N).$$

Let $z^0 \in \mathbb{D}^n$, and $N_0 = N_{\mathbf{b}}(z^0, L, F)$ be the L-index in the direction $\mathbf{b}$ of the function F at the point z^0, i.e. $N_{\mathbf{b}}(z^0, L, F)$ be the least number m_0, for which inequality (2) holds with $z = z^0$. The maximum on the right-hand side of (2) is attained at m_0. Obviously, $0 \leq N_0 \leq N$. For $z^0 \in \mathbb{D}^n$ we develop $F(z^0 + t\mathbf{b})$ in a series by powers t

$$F(z^0 + t\mathbf{b}) = \sum_{m=0}^{\infty} b_m(z^0) t^m, \ b_m(z^0) = \frac{\partial_{\mathbf{b}}^m F(z^0)}{m!}.$$

We put $a_m(z^0) = \frac{|b_m(z^0)|}{L^m(z^0)} = \frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m! L^m(z^0)}$. For any $m \in \mathbb{Z}_+$ the inequality $a_{N_0}(z^0) \geq a_m(z^0) = R_0 a_m(z^0)$ holds. There exists the least number $n_0 \in \{0, 1, \ldots, N_0\}$ such that for all $m \in \mathbb{Z}_+$ $a_{n_0}(z^0) \geq a_m(z^0) R_{N_0 - n_0}$. Thus, $a_{n_0}(z^0) \geq a_{N_0}(z^0) R_{N_0 - n_0}$ and $a_j(z^0) < a_{N_0}(z^0) R_{N_0 - j}$ for $j < n_0$, because if $a_{j_0}(z^0) \geq a_{N_0}(z^0) R_{N_0 - j_0}$ for some $j_0 < n_0$, then $a_{j_0}(z^0) \geq a_m(z^0) R_{N_0 - j_0}$ for all $m \in \mathbb{Z}_+$ and it contradicts the choice of n_0. In view of $a_j(z^0) < a_{N_0}(z^0) R_{N_0 - j}$ $(j < n_0)$ and $a_m(z^0) \leq a_{N_0}(z^0)$ $(m > n_0)$ for $t \in D_{z^0}$ and $|t| = \frac{1}{L(z^0)} r_{N_0 - n_0}$ we have

$$\begin{aligned}
|F(z^0 + t\mathbf{b})| &= \Big| b_{n_0}(z^0) t^{n_0} + \sum_{m \neq n_0} b_m(z^0) t^m \Big| \geq |b_{n_0}(z^0)| |t|^{n_0} - \sum_{m \neq n_0} |b_m(z^0)| |t|^m = \\
&= a_{n_0}(z^0) r_{N_0 - n_0}^{n_0} - \sum_{m \neq 0} a_m(z^0) r_{N_0 - n_0}^m = a_{n_0}(z^0) r_{N_0 - n_0}^{n_0} - \sum_{j < n_0} a_j(z^0) r_{N_0 - n_0}^j - \sum_{m > n_0} a_m(z^0) r_{N_0 - n_0}^m \geq \\
&\geq a_{N_0}(z^0) R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \sum_{j < n_0} a_{N_0}(z^0) R_{N_0 - j} r_{N_0 - n_0}^j - \sum_{m > n_0} a_{N_0}(z^0) r_{N_0 - n_0}^m \geq \\
&\geq a_{N_0}(z^0) R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - n_0 a_{N_0}(z^0) R_{N_0 - n_0 + 1} - a_{N_0}(z^0) r_{N_0 - n_0}^{n_0 + 1} \frac{1}{1 - r_{N_0 - n_0}} = \\
&= a_{N_0}(z^0) \left(R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \frac{n_0}{4N} R_{N_0 - n_0} r_{N_0 - n_0}^N - r_{N_0 - n_0}^{n_0} \frac{r_{N_0 - n_0}}{1 - r_{N_0 - n_0}} \right) \geq \\
&\geq a_{N_0}(z^0) \left(R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \frac{1}{4} R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} - \frac{1}{4} R_{N_0 - n_0} r_{N_0 - n_0}^{n_0} \right) = \frac{1}{2} a_{N_0}(z^0) R_{N_0 - n_0} r_{N_0 - n_0}^{n_0}.
\end{aligned} \tag{24}$$

For $t \in D_{z^0}$ we also have

$$|F(z^0 + t\mathbf{b})| \leq \sum_{m=0}^{+\infty} |b_m(z^0)| |t|^m = \sum_{m=0}^{\infty} a_m(z^0) r_{N_0 - n_0}^m \leq a_{N_0}(z^0) \sum_{m=0}^{+\infty} r_{N_0 - n_0}^m = \frac{a_{N_0}(z^0)}{1 - r_{N_0 - n_0}} \leq \frac{8}{7} a_{N_0}(z^0). \tag{25}$$

From (24) and (25) we obtain

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r_{N_0-n_0}}{L(z^0)}\right\}\leq\frac{8}{7}a_{N_0}(z^0)\leq\frac{16/7}{R_{N_0-n_0}r_{N_0-n_0}^{n_0}}\times$$
$$\times\min\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r_{N_0-n_0}}{L(z^0)}\right\}\leq\frac{16}{7}\frac{1}{R_N}r_N^{-N}\min\left\{|F(z^0+t\mathbf{b})|:|t|=r_{N_0-n_0}/L(z^0)\right\},$$

i.e. (23) holds with $P_2(R)=\dfrac{16}{7R_Nr_N^N}$, $\eta(R)=r_N=\dfrac{1}{8R_N}$ and $r=r_{N_0-n_0}$.

Sufficiency. In view of Theorem 6 it is sufficient to prove that there exists number P_1 such that for every $z^0\in\mathbb{D}^n$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta+1}{2L(z^0)}\right\}\quad\leq\quad P_1\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta-1}{4\beta L(z^0)}\right\}. \quad (26)$$

Let $\widetilde{R}=\frac{\beta-1}{4\beta}$. Then there exist $P_2^*=P_2(\widetilde{R})$ and $\eta=\eta(\widetilde{R})\in(0,\widetilde{R})$ that for every $z^0\in\mathbb{D}^n$ and some $r\in[\eta,\widetilde{R}]$ the following inequality is valid

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r}{L(z^0)}\right\}\leq P_2^*\min\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r}{L(z^0)}\right\}.$$

Put $L^*=\max\{L(z^0+t\mathbf{b}):|t|\leq\beta/L(z^0)\}$, $\rho_0=(\beta-1)/(4\beta L(z^0))$, $\rho_k=\rho_0+k\eta/L^*$, $k\in\mathbb{Z}_+$. Hence, $\frac{\eta}{L^*}<\frac{\beta-1}{4\beta L(z^0)}<\frac{\beta}{L(z^0)}-\frac{\beta+1}{2L(z^0)}$. Therefore, there exists $n^*\in\mathbb{N}$ independent of z^0 and t_0 such that $\rho_{p-1}<\frac{\beta+1}{2L(z^0)}\leq\rho_p\leq\frac{\beta}{L(z^0)}$ for some $p=p(z^0)\leq n^*$.

Let $c_k=\{t\in\mathbb{C}:|t|=\rho_k\}$, $|F(z^0+t_k^{**}\mathbf{b})|=\max\{|F(z^0+t\mathbf{b})|:t\in c_k\}$ and t_k^* be the intersection point of the segment $[0,t_k^{**}]$ with the circle c_{k-1}. Then for every $r>\eta$ one has $|t_k^{**}-t_k^*|=\eta/L^*\leq r/L(z^0+t_k^*\mathbf{b})$. Hence, for some $r\in[\eta,\widetilde{R}]$ we deduce

$$|F(z^0+t_k^{**}\mathbf{b})|\leq\max\left\{|F(z^0+t\mathbf{b})|:|t-t_k^*|=r/L(z^0+t_k^*\mathbf{b})\right\}\leq$$
$$\leq P_2^*\min\left\{|F(z^0+t\mathbf{b})|:|t-t_k^*|=r/L(z^0+t_k^*\mathbf{b})\right\}\leq P_2^*\max\{|F(z^0+t\mathbf{b})|:t\in c_{k-1}\}.$$

Therefore, we get inequality (26) with $P_1^*=(P_2^*)^{n^*}$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta+1}{2L(z^0)}\right\}\leq$$
$$\leq\max\{|F(z^0+t\mathbf{b})|:t\in c_p\}\leq P_2^*\max\{|F(z^0+t\mathbf{b})|:t\in c_{p-1}\}\leq\ldots\leq$$
$$\leq(P_2^*)^p\max\{|F(z^0+t\mathbf{b})|:t\in c_0\}\leq(P_2^*)^{n^*}\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta-1}{4\beta L(z^0)}\right\}.$$

Theorem 8 is proved. □

Theorem 9. *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F:\mathbb{D}^n\to\mathbb{C}$ be an analytic function. If there exists $R\in(0,\beta/2)$ (or if there exists $R\in[\beta/2,\beta)$ and $(\forall z\in\mathbb{D}^n):L(z)>2\beta\max_{1\leq j\leq n}\frac{|b_j|}{1-|z_j|}$) and there exist $P_2\geq1$, $\eta\in(0,R)$ such that for all $z^0\in\mathbb{D}^n$ and some $r=r(z^0)\in[\eta,R]$ inequality* (23) *holds, then the function F has bounded L-index in the direction* $\mathbf{b}$.

Proof. In view of Theorem 7 we need to show existence P_1 such that for all $z^0\in\mathbb{D}^n$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=(\beta-R)/L(z^0)\right\}\leq P_1\max\left\{|F(z^0+t\mathbf{b})|:|t|=R/L(z^0)\right\}. \quad (27)$$

Assume that there exist $R \in (0, \beta/2)$, $P_2 \geq 1$ and $\eta \in \big(0, R\big)$ such that for every $z^0 \in \mathbb{D}^n$ and some $r = r(z^0) \in \big[\eta, R\big]$ we have

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\}.$$

Denote $L^* = \max\left\{L(z^0+t\mathbf{b}) : |t| \leq \beta/L(z^0)\right\}$, $\rho_0 = R/L(z^0)$, $\rho_k = \rho_0 + k\eta/L^*$, $k \in \mathbb{Z}_+$. We obtain $\frac{\eta}{L^*} < \frac{R}{L^*} \leq \frac{R}{L(z^0)} = \frac{\beta}{L(z^0)} - \frac{\beta-R}{L(z^0)}$. Therefore, there exists $n^* \in \mathbb{N}$, independent of z^0 and such that $\rho_{p-1} < \frac{\beta-R}{L(z^0)} \leq \rho_p \leq \frac{\beta}{L(z^0)}$, for some $p = p(z^0) \leq n^*$. It is possible because $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. At first, one has

$$\left(\frac{\beta}{L(z^0)} - \rho_0\right) \Big/ \left(\frac{\eta}{L^*}\right) = \frac{(\beta-R)L^*}{\eta L(z^0)} = \frac{\beta-R}{\eta} \max\left\{\frac{L(z^0+t\mathbf{b})}{L(z^0)} : |t| \leq \frac{\beta}{L(z^0)}\right\} \leq \frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta).$$

Therefore, $n^* = \left[\frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta)\right]$, where $[a]$ is the entire part of number $a \in \mathbb{R}$. Let $|F(z^0 + t_k^{**}\mathbf{b})| = \max\{|F(z^0+t\mathbf{b})| : t \in c_k\}$, $c_k = \{t \in \mathbb{C} : |t| = \rho_k\}$, and t_k^* be the intersection point of the segment $[0, t_k^{**}]$ with the circle c_{k-1}. Hence, for every $r > \eta$ and for each $k \leq n^*$ we get the inequality $|t_k^{**} - t_k^*| = \frac{\eta}{L^*} \leq \frac{r}{L(z^0+t_k^*\mathbf{b})}$. Thus, for some $r = r(z^0 + t_k^*\mathbf{b}) \in [\eta, R]$ we deduce

$$\begin{gathered}|F(z^0+t_k^{**}\mathbf{b})| \leq \max\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b})\right\} \leq \\ \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b})\right\} \leq \\ \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b}), |t-t_0| \leq \rho_{k-1}\right\} \leq \\ \leq P_2 \max\{|F(z^0+t\mathbf{b})| : t \in c_{k-1}\}.\end{gathered}$$

Hence,

$$\begin{gathered}\max\left\{|F(z^0+t\mathbf{b})| : |t| = (\beta-R)/L(z^0)\right\} \leq \\ \leq \max\{|F(z^0+t\mathbf{b})| : t \in c_p\} \leq P_2 \max\{|F(z^0+t\mathbf{b})| : t \in c_{p-1}\} \leq \\ \leq \ldots \leq (P_2)^p \max\{|F(z^0+t\mathbf{b})| : t \in c_0\} \leq (P_2)^{n^*} \max\left\{|F(z^0+t\mathbf{b})| : |t| = R/L(z^0)\right\}.\end{gathered}$$

We get (27) with $P_1 = (P_2)^{n^*}$. Thus, for $R \in (0, \beta/2)$ Theorem 9 is proved.

Now, suppose that $R \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{D}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$. Then inequality (23) can be rewritten as

$$\max\left\{|F(z^0 + \frac{t}{2}\cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0 + \frac{t}{2}\cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\}.$$

Denoting $t' = t/2$, one has

$$\max\left\{|F(z^0 + t'\cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0 + t'\cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\}.$$

Since $r \leq R \in [\beta/2, \beta)$, we have $r/2 \leq R \in [\beta/4, \beta/2) \subset (0, \beta/2)$. Therefore, as shown above the function F has bounded L-index in the direction $2\mathbf{b}$, but by Theorem 3 the function is also of bounded L-index in the direction $\mathbf{b}$. □

5. Logarithmic derivative and zeros. Below we prove another criterion of L-index boundedness in direction that describes behavior of the directional logarithmic derivative and distribution of zeros.

We need some additional denotations.

Denote $g_{z^0}(t) := F(z^0 + t\mathbf{b})$ for a given $z^0 \in \mathbb{D}^n$. If one has $g_{z^0}(t) \neq 0$ for all $t \in D_{z^0}$, then $G_r^{\mathbf{b}}(F, z^0) := \varnothing$; if $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} \colon t \in D_{z^0}\}$. If f $g_{z^0}(t) \not\equiv 0$ and a_k^0 are zeros of $g_{z^0}(t)$, then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{ z^0 + t\mathbf{b} \colon |t - a_k^0| \leq \frac{r}{L(z^0 + a_k^0\mathbf{b})} \right\}, \quad r > 0.$$

Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{D}^n} G_r^{\mathbf{b}}(F, z^0). \tag{28}$$

By $n\big(r, z^0, 1/F\big) = \sum_{|a_k^0| \leq r} 1$ we denote the counting function of zeros (a_k^0) of the function $F(z^0 + t\mathbf{b})$ in the disk $\{t \in \mathbb{C} : \ |t| \leq r\}$.

Theorem 10. *Let $F(z)$ be an analytic function in $\mathbb{D}^n$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $\mathbb{D}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \varnothing$. $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if*

1) for every $r \in (0, \beta]$ there exists $P = P(r) > 0$ that for each $z \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$

$$\left| \frac{\partial_{\mathbf{b}} F(z)}{F(z)} \right| \leq P L(z); \tag{29}$$

2) for every $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for each $z^0 \in \mathbb{D}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$,

$$n\left(r/L(z^0), z^0, 1/F\right) \leq \widetilde{n}(r). \tag{30}$$

Proof. **Necessity.** First, we prove that if the function $F(z)$ is of bounded L-index in the direction $\mathbf{b}$, then for every $z^0 \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$ $(r \in (0, \beta])$ and for every $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}$ the following inequality

$$|z^0 - \widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_2^{\mathbf{b}}(z^0, r)} \tag{31}$$

holds. On the contrary, we assume that there exist $z^0 \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$ and $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}$ such that $|z^0 - \widetilde{a}_k| \leq \dfrac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_2^{\mathbf{b}}(z^0, r)} \leq \dfrac{r|\mathbf{b}|}{2L(z^0)} < \dfrac{r|\mathbf{b}|}{L(z^0)}$. Hence, $|a_k^0| < \frac{r}{L(z^0)}$. But for $\lambda_2^{\mathbf{b}}$ the following estimate $L(\widetilde{a}^k) \leq \lambda_2^{\mathbf{b}}(z^0, r)\, L(z^0)$ holds and $|z^0 - \widetilde{a}^k| = |\mathbf{b}| \cdot |a_k^0| \leq \frac{r|\mathbf{b}|}{2L(\widetilde{a}^k)}$, i.e. $|a_k^0| \leq \dfrac{r}{2L(\widetilde{a}^k)}$. It contradicts $z^0 \in \mathbb{C}^n \backslash G_r^{\mathbf{b}}(F)$. In fact, in (31) instead of $\lambda_2^{\mathbf{b}}(z^0, r)$ we can take $\lambda_2^{\mathbf{b}}(r)$.

We choose in Theorem 8 $R = \dfrac{r}{2\lambda_2^{\mathbf{b}}(r)}$. Then there exist $P_2 \geq 1$ and $\eta \in (0, R)$ such that for every $z^0 \in \mathbb{D}^n$ and some $r^* \in [\eta, R]$ inequality (23) holds with r^* instead of r. Therefore, by Cauchy's inequality

$$|\partial_{\mathbf{b}} F(z^0)| \leq \frac{L(z^0)}{r^*} \max\left\{ |F(z^0 + t\mathbf{b}) : |t| = \frac{r^*}{L(z^0)} \right\} \leq \frac{P_2 L(z^0)}{\eta} \min\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\} \tag{32}$$

In view of (31) the set $\left\{z^0+t\mathbf{b}:|t|\le\frac{r}{2\lambda_2^{\mathbf{b}}(r)L(z^0)}\right\}$ does not contain zeros of the function $F(z^0+t\mathbf{b})$ for every $z^0\in\mathbb{D}^n\backslash G_r^{\mathbf{b}}(F)$. Therefore, applying the maximum principle to $1/F$, as a function of t, we have

$$|F(z^0)|\ge\min\left\{|F(z^0+t\mathbf{b})|:|t|=r^*/L(z^0)\right\}. \tag{33}$$

Inequalities (32) and (33) imply (29) with $P=P_2/\eta$.

Now we prove that if F is of bounded L-index in the direction $\mathbf{b}$, then there exists $P_3>0$ such that for every $z^0\in\mathbb{D}^n$ $(F(z^0+t\mathbf{b})\not\equiv 0)$, $r\in(0,1]$

$$n(r/L(z^0),z^0,1/F)\min\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r}{L(z^0)}\right\}\le P_3\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{1}{L(z^0)}\right\}. \tag{34}$$

By Cauchy's inequality and Theorem 5 for all $t\in D_{z^0}$ such that $|t|=1/L(z^0)$ we have

$$\begin{gathered}\left|\partial_{\mathbf{b}}F(z^0+t\mathbf{b})\right|\le\frac{L(z^0)}{\beta-1}\max\left\{|F(z^0)|:|\theta-t|=\frac{\beta-1}{L(z^0)}\right\}\le\\ \le\frac{L(z^0)}{\beta-1}\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta}{L(z^0)}\right\}\le\frac{P_1(1,\beta)}{\beta-1}L(z^0)\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{1}{L(z^0)}\right\}.\end{gathered} \tag{35}$$

If $F(z^0+t\mathbf{b})\ne 0$ on a circle $\{t\in D_{z^0}:|t|=r/L(z^0)\}$, then

$$\begin{gathered}n\left(\frac{r}{L(z^0)},z^0,\frac{1}{F}\right)=\left|\frac{1}{2\pi i}\int\limits_{|t|=\frac{r}{L(z^0)}}\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}dt\right|\le\\ \le\frac{\max\left\{\left|\partial_{\mathbf{b}}F(z^0+t\mathbf{b})\right|:|t|=r/L(z^0)\right\}}{\min\left\{|F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\right\}}\frac{r}{L(z^0)}.\end{gathered} \tag{36}$$

From (35) and (36) we have

$$\begin{gathered}n\left(r/L(z^0),z^0,1/F\right)\min\left\{|F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\right\}\le\\ \le\frac{r}{L(z^0)}\max\left\{|\partial_{\mathbf{b}}F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\right\}\le\\ \le\frac{1}{L(z^0)}\max\left\{|\partial_{\mathbf{b}}F(z^0+t\mathbf{b})|:|t|=1/L(z^0)\right\}\le\\ \le P_1(1,\beta)/(\beta-1)\max\left\{|F(z^0+t\mathbf{b})|:|t|=1/L(z^0)\right\}.\end{gathered}$$

Thus, we obtain (34) with $P_3=P_1(1,\beta)/(\beta-1)$. If the function $F(z^0+t\mathbf{b})$ has zeros on the circle $\left\{t\in D_R^{z^0}:\ |t|=r/L(z^0)\right\}$ then inequality (34) is obvious.

Now we put $R=1$ in Theorem 8. Then there exists $P_2=P_2(1)\ge 1$ and $\eta\in(0,1)$ such that for each $z^0\in\mathbb{D}^n$ and some $r^*=r^*(z^0)\in[\eta,1]$

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r^*}{L(z^0)}\right\}\le P_2\min\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{r^*}{L(z^0)}\right\}.$$

Moreover, by Theorem 5 there exists $P_1 \geq 1$ such that for all $z^0 \in \mathbb{D}^n$

$$\begin{gathered}\max\left\{|F(z^0+t\mathbf{b})| : |t| = 1/L(z^0)\right\} \leq \\ \leq P_1(1,\eta)\max\left\{|F(z^0+t\mathbf{b})| : \ |t| = \eta/L(z^0)\right\} \leq \\ \leq P_1(1,\eta)\max\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\} \leq \\ \leq P_1(1,\eta)P_2\min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\}.\end{gathered}$$

Taking into account (34), we have

$$\begin{gathered}n\left(r^*/L(z^0), z^0, 1/F\right)\min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\} \leq \\ \leq P_3P_1(1,\eta)P_2\min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\},\end{gathered}$$

i.e. $n\left(\frac{r^*}{L(z^0)}, z^0, \frac{1}{F}\right) \leq P_1(1,\eta)P_2P_3$. Hence,

$$n\left(\frac{r^*}{L(z^0)}, z^0, \frac{1}{F}\right) \leq P_4 = P_1(1,\eta)P_2P_3 = \frac{P_1(1,\eta)P_2(1)P_1(1,r+1)}{r}.$$

If $r \in (0,\eta]$ then property (30) is proved.

Let $r \in (\eta,\beta]$ and $L^* = \max\left\{L(z^0+t\mathbf{b}) : |t| = \frac{r}{L(z^0)}\right\}$. Using properties of $Q^n_{\mathbf{b}}$, we have $L^* \leq \lambda_2^{\mathbf{b}}(r)L(z^0)$. Put $\rho = \frac{\eta}{L(z^0)\lambda_2^{\mathbf{b}}(r)}$, $R = \frac{r}{L(z^0)}$. We can cover every set $\overline{K} = \{z^0+t\mathbf{b} : |t| \leq R\}$ by a finite number $m = m(r)$ of closed sets $\overline{K}_j = \{z^0+t\mathbf{b} : |t-t_j| \leq \rho\}$, where $t_j \in \overline{K}$. Since $\frac{\eta}{\lambda_2^{\mathbf{b}}(r)L(z^0)} \leq \frac{\eta}{L^*} \leq \frac{\eta}{L(z^0+t_j\mathbf{b})}$ in each $\overline{K}_j$ there are at most $[P_4]$ zeros of function $F(z^0+t\mathbf{b})$. Thus, $n\left(\frac{r}{L(z^0)}, z^0, 1/F\right) \leq \widetilde{n}(r) = [P_4]\,m(r)$ and property (30) is proved.

Sufficiency. On the contrary, suppose that conditions (29) and (30) hold. By condition (30) for every $R \in (0,\beta]$ there exists $\widetilde{n}(R) \in \mathbb{Z}_+$ such that in each set $\overline{K} = \left\{z^0+t\mathbf{b} : |t| \leq \frac{R}{L(z^0)}\right\}$ the number of zeros of $F(z^0+t\mathbf{b})$ does not exceed $\widetilde{n}(r)$.

We put $a = a(R) = \frac{R\lambda_1^{\mathbf{b}}(R)}{2(\widetilde{n}(R)+1)}$. By condition (29) there exists $P = P(a) = \widetilde{P}(R) \geq 1$ such that $\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \leq PL(z)$ for all $z \in \mathbb{D}^n\backslash G_a^{\mathbf{b}}$, that is for all $z \in \overline{K}$ lying outside the sets

$$b_k^0 = \left\{z^0+t\mathbf{b} : \ |t-a_k^0| < a(R)/L(z^0+a_k^0\mathbf{b})\right\},$$

where $a_k^0 \in \overline{K}$ are zeros of the function $F(z^0+t\mathbf{b}) \not\equiv 0$. By the definition of $\lambda_1^{\mathbf{b}}$ we have $\lambda_1^{\mathbf{b}}(R)L(z^0) \leq \lambda_1^{\mathbf{b}}(R,z^0)L(z^0) \leq L(z^0+a_k^0\mathbf{b})$. Therefore, $|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}| \leq PL(z)$ for all $z \in \mathbb{D}^n$, lying outside the sets

$$c_k^0 = \left\{z^0+t\mathbf{b} : |t-a_k^0| \leq \frac{a(R)}{\lambda_1^{\mathbf{b}}(R)L(z^0+t_0\mathbf{b})} = \frac{R}{2(\widetilde{n}(R)+1)L(z^0+t_0\mathbf{b})}\right\}.$$

Obviously, the sum of diameters of sets c_k^0 does not exceed $\frac{R\widetilde{n}(R)}{(\widetilde{n}(R)+1)L(z^0)} < \frac{R}{L(z^0)}$. Therefore, there exist a set $\widetilde{c}^0 = \left\{z^0+t\mathbf{b} : |t| = \frac{r}{L(z^0)}\right\}$, where $\frac{R}{2(\widetilde{n}(R)+1)} = \eta(R) < r < R$, such that for all $z \in \widetilde{c}^0$ the following inequality is valid

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \leq PL(z) \leq P\lambda_2^{\mathbf{b}}(r)L(z^0) \leq P\lambda_2^{\mathbf{b}}(R)\,L(z^0).$$

For any points $z_1 = z^0 + t_1\mathbf{b}$ and $z_2 = z^0 + t_2\mathbf{b}$ from $\widetilde{c}^0$ we have

$$\ln\left|\frac{F(z^0+t_1\mathbf{b})}{F(z^0+t_2\mathbf{b})}\right| \le \int_{t_1}^{t_2}\left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right||dt| \le P\lambda_2^{\mathbf{b}}(R)\,L(z^0)\frac{2r}{L(z^0)} \le 2R\,P(R)\lambda_2^{\mathbf{b}}(R).$$

Hence, we get

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\} \le P_2\min\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\},$$

where $P_2 = \exp\left\{2R\,P(R)\lambda_2^{\mathbf{b}}(R)\right\}$. Thus, by Theorem 8 the function $F(z)$ is of bounded L-index in the direction $\mathbf{b}$. Theorem 10 is proved. □

Theorem 11. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\mathbb{D}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \varnothing$, $F : \mathbb{D}^n \to \mathbb{C}$ be an analytic function. If the following conditions are satisfied*

1) *there exists $r_1 \in (0, \beta/2)$ (or there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{D}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1; \infty)$;*
2) *there exist $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{D}^n \setminus G_{r_2}(F)$ inequality (29) is true;*

then the function F has bounded L-index in the direction $\mathbf{b}$.

Proof. Suppose that conditions 1) and 2) are true.

At first, we consider the case $n(r_1) \in \{-1; 0\}$. Then in the best case the function F can only identically equals zero on the complex line $z^* + t\mathbf{b}$ for some $z^* \in \mathbb{D}^n$, i.e., $F(z^* + t\mathbf{b}) \equiv 0$. For all points lying on such complex lines inequality (23) is obvious.

Let $z^0 \in \mathbb{D}^n \setminus G_{r_2}$. For any points t_1 and t_2 such that $|t_j| = \frac{r_2}{L(z_0)}$, $j \in \{1, 2\}$, one has

$$\ln\left|\frac{F(z^0+t_2\mathbf{b})}{F(z^0+t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2}\left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right||dt| \le P\int_{t_1}^{t_2} L(z^0+t\mathbf{b})|dt| \le \pi r_2 P\lambda_{\mathbf{b}}(r_2)$$

(we also use that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$). Hence,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_2\min\left\{|F(z^0+t\mathbf{b})| : \ |t| = \frac{r_1}{L(z^0)}\right\},$$

where $P_2 = \exp\left\{\pi r_2\, P\lambda_2(r_2)\right\}$. Therefore, by Theorem 9 the function F has bounded L-index in the direction $\mathbf{b}$.

Let $r_1 > 0$ be a such that $n(r_1) \in [1; \infty)$ and $2n(r_1)r_2 < r_1/\lambda_{\mathbf{b}}(r_1)$. Put $c = \frac{r_1}{2r_2\lambda_{\mathbf{b}}(r_1)} - n(r_1) > 0$. Clearly, $r_2 = r_1/(2(n(r_1)+c)\lambda_{\mathbf{b}}(r_1))$.

Under condition 1) each set $\overline{K} = \left\{z^0 + t\mathbf{b} : |t| \le \frac{r_1}{L(z^0)}\right\}$ has at most $n(r_1)$ zeros of the function F, where $F(z^0 + t\mathbf{b}) \not\equiv 0$.

Under condition 2) there exists $P > 0$ such that $|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}| \le PL(z)$ for every $z \in \mathbb{D}^n \setminus G_{r_2}$, i.e., for all $z \in \overline{K}$, lying outside the sets $\left\{z^0 + t\mathbf{b} : \ |t - a_k^0| < \frac{r_2}{L(z^0+a_k^0\mathbf{b})}\right\}$, where $a_k^0 \in \overline{K}$ are zeros of the slice function $F(z^0 + t\mathbf{b}) \not\equiv 0$. By the definition of $\lambda_{\mathbf{b}}$ we obtain $L(z^0)/\lambda_{\mathbf{b}}(r_1) \le L(z^0 + a_k^0\mathbf{b})$. Then $|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}| \le PL(z)$ for every point $z \in \mathbb{D}^n$, lying outside the union of the sets

$$c_k^0 = \left\{z^0 + t\mathbf{b} : |t - a_k^0| \le \frac{r_2\lambda_{\mathbf{b}}(r_1)}{L(z^0)} = \frac{r_1}{2(n(r_1)+c)L(z^0)}\right\}.$$

The total sum of diameters of the sets c_k^0 does not exceed the value $\frac{r_1 n(r_1)}{(n(r_1)+c)L(z^0)} < \frac{r_1}{L(z^0)}$. Hence, there exists a set $\widetilde{c}^0 = \left\{z^0 + t\mathbf{b} : |t| = \frac{r}{L(z^0)}\right\}$, where $\frac{r_1 \min\{1,c\}}{2(n(r_1)+c)} = \eta < r < r_1$, such that for all $z \in \widetilde{c}^0$

$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \leq PL(z) \leq P\lambda_{\mathbf{b}}(r)L(z^0) \leq P\lambda_{\mathbf{b}}(r_1)\, L(z^0).$$

For any points $z_1 = z^0 + t_1\mathbf{b}$ and $z_2 = z^0 + t_2\mathbf{b}$ with $\widetilde{c}^0$ one has

$$\ln\left|\frac{F(z^0+t_2\mathbf{b})}{F(z^0+t_1\mathbf{b})}\right| \leq \int_{t_1}^{t_2}\left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right||dt| \leq P\lambda_2(r_1)\, L(z^0)\frac{\pi r}{L(z^0)} \leq \pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1)\,.$$

Therefore,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| :\ |t| = \frac{r}{L(z^0)}\right\}, \tag{37}$$

where $P_2 = \exp\{\pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1)\}$. If $F(z^0+t\mathbf{b}) \equiv 0$, then inequality (37) is obvious. By Theorem 9 the function $F(z)$ has bounded L-index in the direction $\mathbf{b}$. Theorem 11 is proved. □

6. Hayman's Theorem. It is an analog of Hayman's Theorem [16]. The theorem helps to investigate boundedness L-index in direction of analytic solutions of differential equations. At the end of the paper, we will present a scheme of this application.

Theorem 12. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for every $z \in \mathbb{D}^n$*

$$\left|\frac{\partial_{\mathbf{b}}^{p+1}F(z)}{L^{p+1}(z)}\right| \leq C\max\left\{\left|\frac{\partial_{\mathbf{b}}^k F(z)}{L^k(z)}\right| :\ 0 \leq k \leq p\right\}. \tag{38}$$

Proof. Using some additional propositions, we will prove the theorem. The auxiliary statements are proved in the next sections. They describe local behavior of analytic function of bounded L-index in direction.

Necessity. If $N_{\mathbf{b}}(F,L) < +\infty$, then by the definition of L-index boundedness in direction we obtain inequality (38) with $p = N_{\mathbf{b}}(F,L)$ and $C = (N_{\mathbf{b}}(F,L)+1)!$

Sufficiency. Let inequality (38) holds, $z^0 \in \mathbb{D}^n$ and $K = \{t \in \mathbb{C} : |t| \leq 1/L(z^0)\}$. Thus, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and (38) imply that for every $t \in K$

$$\begin{aligned}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0)} &\leq \left(\frac{L(z^0+t\mathbf{b})}{L(z^0)}\right)^{p+1}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq (\lambda_2^{\mathbf{b}}(1))^{p+1}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq \\ &\leq C(\lambda_2^{\mathbf{b}}(1))^{p+1}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{L^k(z^0+t\mathbf{b})} :\ 0 \leq k \leq p\right\} \leq \\ &\leq C(\lambda_2^{\mathbf{b}}(1))^{p+1}\max\left\{\left(\frac{L(z^0)}{L(z^0+t\mathbf{b})}\right)^k\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{L^k(z^0)} : 0 \leq k \leq p\right\} \leq \\ &\leq C(\lambda_2^{\mathbf{b}}(1))^{p+1}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{L^k(z^0)}(\lambda_1^{\mathbf{b}}(1))^{-k} : 0 \leq k \leq p\right\} \leq Bg_{z^0}(t),\end{aligned} \tag{39}$$

where $B = C(\lambda_2^{\mathbf{b}}(1))^{p+1}(\lambda_1^{\mathbf{b}}(1))^{-p}$ and

$$g_{z^0}(t)=\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{L^k(z^0)}:0\le k\le p\right\}.$$

We write $\gamma_1 = \left\{t\in\mathbb{C}:|t|=\frac{1}{2\beta L(z^0)}\right\}$, $\gamma_2=\left\{t\in\mathbb{C}:|t|=\frac{\beta}{L(z^0)}\right\}$. We choose arbitrary points $t_1\in\gamma_1$, $t_2\in\gamma_2$ and join them by an analytic curve $\gamma=\{t=t(s):0\le s\le T\}$, such that $g_{z^0}(t)\ne 0$ for $t\in\gamma$. We choose the curve γ such that its length $|\gamma|$ does not exceed $\dfrac{2\beta^2+1}{\beta L(z^0)}$.

Clearly, the function $g_{z^0}(t(s))$ is continuous on $[0,T]$. Without loss of generality, we may assume that the function $t=t(s)$ is analytic on $[0,T]$. First, we prove that the function $g_{z^0}(t(s))$ is continuously differentiable on $[0,T]$ except, perhaps, a finite set of points. For arbitrary $k_1,k_2, 0\le k_1\le k_2\le p$, either $\frac{|\partial_{\mathbf{b}}^{k_1}F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)}\equiv\frac{|\partial_{\mathbf{b}}^{k_2}F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ for $s\in[0,T]$ or the equality $\frac{|\partial_{\mathbf{b}}^{k_1}F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)}=\frac{|\partial_{\mathbf{b}}^{k_2}F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ holds only for a finite set of points $s_k\in[0,T]$. Thus, we can split the segment $[0,T]$ on a finite number of segments that on each partial segment

$$g_{z^0}(t(s))\equiv\frac{|\partial_{\mathbf{b}}^k F(z^0+t(z)\mathbf{b})|}{L^k(z^0)}$$

for some k, $0\le k\le p$. It means that a function $g_{z^0}(t(s))$ is continuously differentiable except, perhaps, a finite set of points. In view of (39) we obtain

$$\frac{dg_{z^0}(t(s))}{ds}\le\max\left\{\frac{d}{ds}\left(\frac{|\partial_{\mathbf{b}}^k F(z^0+t(s)\mathbf{b})|}{L^k(z^0)}\right):0\le k\le p\right\}\le$$
$$\le\max\left\{|\partial_{\mathbf{b}}^{k+1}F(z^0+t(s)\mathbf{b})||t'(s)|/L^k(z^0):0\le k\le p\right\}=$$
$$=L(z^0)|t'(s)|\max\left\{|\partial_{\mathbf{b}}^{k+1}F(z^0+t(s)\mathbf{b})|/L^{k+1}(z^0):\ 0\le k\le p\right\}\le Bg_{z^0}(t(s))|t'(s)|L(z^0).$$

Hence,

$$\left|\ln\frac{g_{z^0}(t_2)}{g_{z^0}(t_1)}\right|=\left|\int_0^T\frac{dg_{z^0}(t(s))}{g_{z^0}(t(s))}\right|\le BL(z^0)\int_0^T|t'(s)|ds=BL(z^0)|\gamma|\le 2B(\beta^2+1)/(\beta).$$

If we choose a point $t_2\in\gamma_2$ such that

$$|F(z^0+t_2\mathbf{b})|=\max\left\{|F(z^0+t\mathbf{b})|:|t|=\beta/L(z^0)\right\},$$

then we obtain

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{\beta}{L(z^0)}\right\}\le g_{z^0}(t_2)\le g_{z^0}(t_1)\exp\left\{2B\frac{\beta^2+1}{\beta}\right\}. \qquad (40)$$

Applying Cauchy's inequality and using $t_1\in\gamma_1$, for all $j=1,\dots,p$ we get

$$|\partial_{\mathbf{b}}^j F(z^0+t_1\mathbf{b})|\le j!(2\beta L(z^0))^j\max\left\{|F(z^0+t\mathbf{b}):|t-t_1|=\frac{1}{2\beta L(z^0)}\right\}\le$$
$$\le j!(2\beta L(z^0))^j\max\left\{|F(z^0+t\mathbf{b}):|t-t_0|=\frac{1}{\beta L(z^0)}\right\},$$

that is

$$g_{z^0}(t_1) \le p!(2\beta)^p \max\left\{|F(z^0+t\mathbf{b}) : |t-t_0| = \frac{1}{\beta L(z^0)}\right\}.$$

Thus, (40) implies

$$|F(z^0+t_2\mathbf{b})| = \max\left\{|F(z^0+t\mathbf{b})| : |t| = \beta/L(z^0)\right\} \le g_{z^0}(t_2) \le$$
$$\le g_{z^0}(t_1)\exp\left\{2B\frac{\beta^2+1}{\beta}\right\} \le p!(2\beta)^p \exp\left\{2B\frac{\beta^2+1}{\beta}\right\}\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{1}{\beta L(z^0)}\right\}.$$

By Theorem 6 we conclude that the function F is of bounded L-index in the direction $\mathbf{b}$. Theorem 12 is proved. □

7. Analytic functions in the unit polydisc of bounded value L-distribution in a direction.

Definition 1. Analytic function $F : \mathbb{D}^n \to \mathbb{C}$, is said to be of bounded value L-distribution in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$, if there exists $p \in \mathbb{Z}_+$ such that for every $w \in \mathbb{C}$ and for all $z_0 \in \mathbb{D}^n$ with $F(z^0+t\mathbf{b}) \not\equiv w$ the equation $F(z^0+t\mathbf{b}) = w$ has in the disc $\{t : \ |t| \le \frac{1}{L(z^0)}\}$ at most p solutions. In other words, the function $F(z^0+t\mathbf{b})$ is p-valent in $\{t : \ |t| \le \frac{1}{L(z^0)}\}$.

Theorem 13. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is a function of bounded value L-distribution in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ if and only if its directional derivative $\partial_{\mathbf{b}}F$ has bounded L-index in the same direction $\mathbf{b}$.*

Proof. Assume that the function F is of bounded value L-distribution in a direction $\mathbf{b}$, i.e. for every $z^0 \in \mathbb{D}^n$ such that $F(z^0+t\mathbf{b}) \not\equiv const$ the function $F(z^0+t\mathbf{b})$ is p-valent in the disc $\{t : \ |t| \le \frac{1}{L(z^0)}\}$.

To prove the theorem we need the following proposition [27, p. 48, Theorem 2.8].

Theorem 14. *[[27]] Let $D_0 = \{t : \ |t-t_0| < R\}$, $0 < R < \infty$. If an analytic function in D_0 is p-valent in D_0 then for $j > p$*

$$\frac{|f^{(j)}(t_0)|}{j!}R^j \le (Aj)^{2p}\max\left\{\frac{|f^{(k)}(t_0)|}{k!}R^k : \ 1 \le k \le p\right\}, \tag{41}$$

where $A = \sqrt[2p]{\frac{p+2}{2}}\sqrt{8e^{\pi^2}}$.

By Theorem 14 inequality (41) holds with $R = \frac{1}{L(z^0)}$ for the function $F(z^0+t\mathbf{b})$ as a function of one variable $t \in \mathbb{C}$ for every given $z^0 \in \mathbb{D}^n$. For convenience we will use the notation $g_{z^0}(t) = F(z^0+t\mathbf{b})$. Then it is easy to prove that for every $m \in \mathbb{N}$ the following equality $g_{z^0}^{(p)}(t) = \frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}$ is valid. Put $j = p+1$ and $t_0 = 0$ in Theorem 14. From (41) it follows

$$\frac{1}{(p+1)!L^{p+1}(z_0)}\left|\partial_{\mathbf{b}}^{p+1}F(z^0)\right| \le (A(p+1))^{2p}\max\left\{\frac{1}{k!L^k(z_0)}\left|\partial_{\mathbf{b}}^k F(z^0)\right| : \ 1 \le k \le p\right\} \Rightarrow$$
$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0)|}{L^{p+1}(z_0)} \le (p+1)!(A(p+1))^{2p}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0)|}{L^k(z_0)} : \ 1 \le k \le p\right\}\max\left\{\frac{1}{k!} : 1 \le k \le p\right\} \Rightarrow$$
$$\frac{|\partial_{\mathbf{b}}^p \partial_{\mathbf{b}}F(z^0)|}{L^p(z_0)} \le L(z^0)\cdot(p+1)!A^{2p}(p+1)^{2p}\max\left\{\frac{|\partial_{\mathbf{b}}^{k-1}\partial_{\mathbf{b}}F(z^0)|}{L^k(z_0)} : \ 0 \le k-1 \le p-1\right\} \Rightarrow$$

$$\frac{|\partial_{\mathbf{b}}^{p}\partial_{\mathbf{b}}F(z^0)|}{L^p(z_0)}\leq (p+1)!A^{2p}(p+1)^{2p}\max\left\{\frac{1}{L^{k-1}(z_0)}\left|\partial_{\mathbf{b}}^{k-1}\partial_{\mathbf{b}}F(z^0)\right|:0\leq k-1\leq p-1\right\}$$

Thus, for the function $\frac{\partial F}{\partial \mathbf{b}}$ inequality (38) in Theorem 12 is fulfilled with $p-1$ instead of p and with $C=(p+1)!A^{2p}(p+1)^{2p}$. In Theorem 14 the constant $A\geq \max\limits_{j>p}\frac{p+2}{2}(8e^{\pi^2})^p(1-\frac{1}{j})^j$ does not depend on z^0, because the parameter p is independent of z^0. Hence, the quantity $C=(p+1)!A^{2p}(p+1)^{2p}$ also does not depend on z^0. Then by Theorem 12 the function $\frac{\partial F}{\partial \mathbf{b}}$ has bounded L-index in the direction $\mathbf{b}$.

On the contrary, let $\frac{\partial F}{\partial \mathbf{b}}$ be an analytic function in the unit polydisc of bounded L-index in the direction $\mathbf{b}$. By Theorem 12 there exist $p\in\mathbb{Z}_+$ and $C\geq 1$ such that for every $z\in\mathbb{D}^n$ the following inequality is true

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\leq C\max\left\{\frac{1}{L^{k}(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right|:\ 1\leq k\leq p\right\}. \tag{42}$$

Let us consider the disc $K_0=\left\{t\in\mathbb{C}:|t|\leq\frac{1}{L(z^0)}\right\}$, $z^0\in\mathbb{D}^n$.

Observe that if $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, then for all $z^0\in\mathbb{D}^n$, $r\in(0,\beta]$, $|t|\leq\frac{r}{L(z^0)}$ the definition of class $Q_{\mathbf{b}}(\mathbb{D}^n)$ implies

$$\lambda_1^{\mathbf{b}}(r)L(z^0)\leq L(z^0+t\mathbf{b})\leq\lambda_2^{\mathbf{b}}(r)L(z^0). \tag{43}$$

Now inequalities (42) and (43) for $z=z^0+t\mathbf{b}$, $t\in K$, yield

$$\begin{gathered}\frac{1}{(p+1)!}\left|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}\right)^{p+1}\leq\\ \leq\frac{Cp!}{(p+1)!}\max\left\{\frac{1}{k!}\left|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}\right)^{k}\left(\frac{L(z^0+t\mathbf{b})}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}\right)^{p+1-k}:1\leq k\leq p\right\}\leq\\ \leq\frac{C}{p+1}\max_{1\leq k\leq p}\left\{\frac{1}{k!}\left|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}\right)^{k}\left(\frac{1}{C}\right)^{p+1-k}\right\}\leq\\ \leq\max\left\{\frac{1}{k!}\left|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}\right)^{k}:\ 1\leq k\leq p\right\}.\end{gathered} \tag{44}$$

To complete the proof of the theorem we will apply the following proposition from [27, p.44, Theorem 2.7].

Theorem 15. *[[27, p.44, Theorem 2.7], [16]] Let $D_0=\{t\in\mathbb{C}:\ |t-t_0|<R\}$, $0<R<+\infty$, and f be an analytic function in D_0. If for all $z\in D_0$*

$$\left(\frac{R}{2}\right)^{p+1}\frac{|f^{(p+1)}(t)|}{(p+1)!}\leq\max\left\{\left(\frac{R}{2}\right)^{k}\frac{|f^{(k)}(z)|}{k!}:\ 1\leq k\leq p\right\}, \tag{45}$$

then f is p-valent in $\{t\in\mathbb{C}:\ |t-t_0|\leq\frac{R}{25\sqrt{p+1}}\}$, i. e. the function $f(t)$ attains each value at most p times.

Inequality (44) implies estimate (45) with $R=\frac{2}{C\lambda_2^{\mathbf{b}}(1)L(z^0)}$ for $t_0=0$. By Theorem 15 the function $F(z^0+t\mathbf{b})$ is p-valent in the disc $\{t\in\mathbb{C}:\ |t|\leq\frac{\rho}{L(z^0)}\}$, $\rho=\frac{2}{25C\lambda_2^{\mathbf{b}}(1)\sqrt{p+1}}$.

Let t_j be an arbitrary point in K_0 and $K_j^* = \{t \in \mathbb{C} : |t - t_i| \le \frac{\rho}{L(z^0+t_j\mathbf{b})}\}$. Since by the definition of the class $Q_{\mathbf{b}}(\mathbb{D}^n)$ one has $L(z^0 + t_j\mathbf{b}) \le \lambda_2^{\mathbf{b}}(1)L(z^0)$, i.e. we have that $K_j = \left\{t \in \mathbb{C} : |t - t_j| \le \frac{\rho}{\lambda_2^{\mathbf{b}}(1)L(z^0)}\right\} \subset K_j^*$. We can repeat the above considerations for the set $\left\{t \in \mathbb{C} : |t - t_j| \le \frac{1}{L(z^0+t_j\mathbf{b})}\right\}$. Respectively, we obtain that the function $F(z^0 + t\mathbf{b})$ is p-valent in K_j^*. Since $K_j \subset K_j^*$, the function $F(z^0 + t\mathbf{b})$ is p-valent in K_j.

Finally, we remark that every closed disc of radius R_* can be covered a finite number m_* of closed discs of radius $\rho_* < R_*$ with centers in this disc. Moreover, $m_* < B_*(R_*/\rho_*)^2$, where $B_* > 0$ is a constant. Hence, we can cover the set K_0 by a finite number m of discs K_j, where $m \le 625B^*(p + 1)C^2(\lambda_2^{\mathbf{b}}(1))^2/4$. Since the function $F(z^0 + t\mathbf{b})$ in K_j is p-valent, it is mp-valent in K_0.

In view of arbitrarity of z^0, the theorem is proved. □

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