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ON SOLUTIONS OF CERTAIN COMPATIBLE SYSTEMS OF QUADRATIC TRINOMIAL PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS

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This paper has involved the use of a variety of variations of the Fermat-type equation $f^n(z) + g^n(z) = 1$, where $n(\geq 2) \in \mathbb{N}$. Many researchers have demonstrated a keen interest to investigate the Fermat-type equations for entire and meromorphic solutions of several complex variables over the past two decades. Researchers utilize the Nevanlinna theory as the key tool for their investigations. Throughout the paper, we call the pair (f, g) as a finite order entire solution for the Fermat-type compatible system $\begin{cases} f^{m_1} + g^{n_1} = 1; \\ f^{m_2} + g^{n_2} = 1, \end{cases}$ if f, g are finite order entire functions satisfying the system, where $m_1, m_2, n_1, n_2 \in \mathbb{N} \setminus \{1\}$. Taking into the account the idea of the quadratic trinomial equations, a new system of quadratic trinomial equations has been constructed as follows: $\begin{cases} f^{m_1} + 2\alpha fg + g^{n_1} = 1; \\ f^{m_2} + 2\alpha fg + g^{n_2} = 1, \end{cases}$ where $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$. In this paper, we consider some earlier systems of certain Fermat-type partial differential-difference equations on $\mathbb{C}^2$, especially, those of Xu *et al.* (Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type, J. Math. Anal. Appl. 483(2), 2020) and then construct some systems of certain quadratic trinomial partial differential-difference equations with arbitrary coefficients. Our objective is to investigate the forms of the finite order transcendental entire functions of several complex variables satisfying the systems of certain quadratic trinomial partial differential-difference equations on $\mathbb{C}^n$. These results will extend the further study of this direction.

1. Introduction. By a meromorphic function f on $\mathbb{C}^n$ $(n \in \mathbb{N})$, we mean that f can be written as a quotient of two holomorphic functions without common zero sets in $\mathbb{C}^n$. Notationally, we write $f := \frac{g}{h}$, where g and h are holomorphic functions without common zero sets on $\mathbb{C}^n$ such that $h \not\equiv 0$ and $g \not\equiv 0$.

Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $a \in \mathbb{C} \cup \{\infty\}$, $k \in \mathbb{N}$ and $r > 0$. We consider some notations from [12, 29, 32]. Let $\overline{B}_n(r) := \{z \in \mathbb{C}^n : |z| \leq r\}$, where $|z|^2 := \sum_{j=1}^{n} |z_j|^2$. The exterior derivative splits $d := \partial + \overline{\partial}$ and twists to $d^c := \frac{i}{4\pi}(\overline{\partial} - \partial)$. The standard Kaehler metric on $\mathbb{C}^n$ is given by $v_n(z) := dd^c|z|^2$. Define $\omega_n(z) := dd^c \log |z|^2 \geq 0$ and $\sigma_n(z) := d^c \log |z|^2 \wedge \omega_n^{n-1}(z)$ on $\mathbb{C}^n \setminus \{0\}$. Thus $\sigma_n(z)$ defines a positive measure on $\partial B_n := \{z \in \mathbb{C}^n : |z| = r\}$ with total measure 1. The zero-multiplicity of a holomorphic function h at a point $z \in \mathbb{C}^n$ is defined to be the order of vanishing of h at z and denoted by $\mathcal{D}_h^0(z)$. A divisor of f on $\mathbb{C}^n$

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is an integer valued function which is locally the difference between the zero-multiplicity functions of g and h and it is denoted by $\mathcal{D}_f := \mathcal{D}_g^0 - \mathcal{D}_h^0$ (see [6, p. 381]). Let $a \in \mathbb{C} \cup \{\infty\}$ be such that $f \not\equiv a$. Then the a-divisor ν_f^a of f is the divisor associated with the holomorphic functions $g - ah$ and h (see [12, p. 346]). In [32], Ye has defined the counting function and the valence function with respect to a respectively as follows: $n(r, a, f) := r^{2-2n} \int_{S(r)} \nu_f^a v_n^{n-1}$ and $N(r, a, f) := \int_0^r \frac{n(r,a,f)}{t} dt$. We write

$$N(r, a, f) = \begin{cases} N(r, \frac{1}{f-a}), & \text{when } a \neq \infty; \\ N(r, f), & \text{when } a = \infty. \end{cases}$$

The proximity function [12, 32] of f is defined as follows:

$$m(r, f) := \int_{\partial B_n(r)} \log^+ |f(z)| \sigma_n(z), \text{ when } a = \infty,$$

$$m\Big(r, \frac{1}{f-a}\Big) := \int_{\partial B_n(r)} \log^+ \frac{1}{|f(z) - a|} \sigma_n(z), \text{ when } a \neq \infty.$$

By denoting $S(r) := \overline{B}_n(r) \cap$ supp ν_f^a, where supp $\nu_f^a = \overline{\{z \in \mathbb{C}^n : \nu_f^a(z) \neq 0\}}$ (see [12, p. 346]). The notation $N_k(r, \frac{1}{f-a})$ is known as truncated valence function. In particular, $N_1(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a})$ is the truncated valence function of simple a-divisors of f in $S(r)$. In $N_k(r, \frac{1}{f-a})$, the a-divisors of f in $S(r)$ of multiplicity m are counted m-times if $m < k$ and k-times if $m \geq k$. The Nevanlinna characteristic function is defined by $T(r, f) = N(r, f) + m(r, f)$, which is increasing for r. The order of a meromorphic function f is denoted by $\rho(f)$ and is defined by

$$\rho(f) = \overline{\lim_{r \to \infty}} \frac{\log^+ T(r, f)}{\log r}, \text{ where } \log^+ x = \max\{\log x, 0\}.$$

Given a meromorphic function f, recall that a meromorphic function α is said to be a small function of f, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set E of finite linear measure ($\int_E dr < +\infty$) (see [11, 29, 32]).

Given a meromorphic function $f(z)$ on $\mathbb{C}^n$, $f(z + c)$ is called a shift of f and $\Delta(f) = f(z + c) - f(z)$ is called a difference operator of f, where $c \in \mathbb{C}^n \setminus \{(0, 0, \ldots, 0)\}$.

A significant number of researchers have demonstrated a keen interest in investigating the Fermat-type equations for entire [8, 17, 18, 21] and meromorphic solutions [20, 31] over the past two decades. This has involved the use of a variety of variations of the equation $f^n(z) + g^n(z) = 1$, where $n \in \mathbb{N}$. Yang and Li [31] were the first to undertake the study of transcendental meromorphic solutions of Fermat-type differential equations on $\mathbb{C}$. Liu [20] was the first who investigated on meromorphic solutions of Fermat-type difference equation as well as differential-difference equations on $\mathbb{C}$. For other leading and recent developments in these directions, we also refer to the reader to [7, 22, 23, 25] and the references therein.

A difference polynomial (resp. a partial differential-difference polynomial) in f is a finite sum of difference products of f and its shifts (resp. of products of f, partial derivatives of f and of their shifts) with all the coefficients of these monomials being small functions of f. Below we select a single branch for the square root of a complex number by the condition $\sqrt{1} = 1$.

In 2013, Saleeby [27] considered the quadratic trinomial equations

$$f^2 + 2\alpha f g + g^2 = 1, \quad \alpha \in \mathbb{C} \setminus \{\pm 1\} \tag{1}$$

and the associated partial differential equations

$$\left(\frac{\partial u(z_1,z_2)}{\partial z_1}\right)^2+2\alpha\frac{\partial u(z_1,z_2)}{\partial z_1}\frac{\partial u(z_1,z_2)}{\partial z_2}+\left(\frac{\partial u(z_1,z_2)}{\partial z_2}\right)^2=1,\quad (z_1,z_2)\in\mathbb{C}^2 \tag{2}$$

and obtained an explicit form of all entire and meromoprhic solutions of the equation using their representation by arbitrary entire or meromorphic function, respectively. Moreover, he proved that the entire and meromorphic solutions of (2) are the first degree polynomials in the variables z_1 and z_2. In 2016, Liu and Yang [24] have proved the non-existence of transcendental meromorphic solutions of some trinomial quadratic differential-functional equation and justified that the order of entire solutions of some associated difference equation equals one. In 2020, Xu *et al.* [30] considered the Fermat-type systems of partial differential-difference equations

$$\begin{cases}\left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2+f_2(z_1+c_1,z_2+c_2)^2=1;\\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2+f_1(z_1+c_1,z_2+c_2)^2=1,\end{cases} \tag{3}$$

$$\begin{cases}\left(\frac{\partial f_1(z_1,z_2)}{\partial z_1}\right)^2+\left(f_2(z_1+c_1,z_2+c_2)-f_1(z_1,z_2)\right)^2=1;\\ \left(\frac{\partial f_2(z_1,z_2)}{\partial z_1}\right)^2+\left(f_1(z_1+c_1,z_2+c_2)-f_2(z_1,z_2)\right)^2=1,\end{cases} \tag{4}$$

and obtained an explicit representations of transcendental entire solutions with finite order for system (3) and (4), separately. In 2021, Li *et al.* [19] extended the results of Xu *et al.* [30] by replacing the first partial derivative in variables z_1 and z_2 by their sum, i.e. by the derivative in the direction $(1,1)$ and obtained similar results to Xu's results in [30].

Inspired by the results of Saleeby [27], any researcher can be curious about the following question.

Problem 1. *Is it possible to study further by extending the systems of partial differential-difference equations (3), (4), and Li's systems from [19] to a new system of quadratic trinomial partial differential-difference equations $\mathbb{C}^n$ with arbitrary coefficients?*

Our main objective in this paper is to extend the investigations from the systems of certain Fermat-type partial differential-difference equations on $\mathbb{C}^2$ to the systems of certain quadratic trinomial partial differential-difference equations on $\mathbb{C}^n$ with arbitrary coefficients. Note that our investigations are based on the multidimensional Nevanllinna theory. Given this, our study is limited the class of functions having finite order. There are known two other approaches in the complex analysis which are also used to study analytic solutions of system of partial differential equations. But they allow to consider functions of infinite order. The first approach is the multidimensional Wiman-Valiron theory which examines the properties of the maximal term and the central index of the power series [13–16]. This theory is applicable for any entire solution of differential equations. But even in the case of analytic in the unit disc functions, the question of a complete analogue of the Wiman-Valiron theory is still not fully studied. The second approach is based on the notion of bounded l-index [5]. It allows to study as entire, so analytic in some bounded domain solutions of directional differential equations [1–3], and system of partial differential equations [4]. The method overlaps all analytic functions having bounded multiplicities of zero points.

2. The Main Results. For $I=(i_1,i_2,\ldots,i_n)\in\mathbb{Z}_+^n$ we put $\|I\|=\sum_{k=1}^n i_k$. Then any polynomial $\mathcal{Q}(z)$ on $\mathbb{C}^n$ of degree d can be expressed as $\mathcal{Q}(z)=\sum_{\|I\|=0}^d \alpha_I z_1^{i_1}\cdots z_n^{i_n}$, where

$\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, $\alpha_I \in \mathbb{C}$ such that α_I are not all zero at a time for $\|I\| = d$. Suppose that $\mathcal{Q}(z+c) - \mathcal{Q}(z) \equiv$ constant(say $B \in \mathbb{C}$), for any $c \in \mathbb{C}^n \setminus \{(0,0,\ldots,0)\}$. Let $\mathcal{Q}(z) = \sum_{j=1}^{n} a_j z_j + \Phi(z) + A$, where $A \in \mathbb{C}$ and $\deg(\Phi(z)) \geq 2$. Now, $\mathcal{Q}(z+c) - \mathcal{Q}(z) \equiv B$ implies that $\sum_{j=1}^{n} a_j c_j + \Phi(z+c) - \Phi(z) \equiv B$. Thus, we have $\Phi(z+c) \equiv \Phi(z)$ and $\sum_{j=1}^{n} a_j c_j = B$. Since $\Phi(z)$ is periodic, so we can express $\Phi(z)$ as

$$\Phi(z) = \sum_{\lambda} G_\lambda(z), \quad \text{where} \quad G_\lambda(z) = \prod_{\alpha} G_\alpha(z), \tag{5}$$

where λ belongs to the finite index set I_1 of the family $\{G_\lambda(z) : \lambda \in I_1\}$ and α belongs to the finite index set I_2 of the family $\{G_\alpha(z) : \alpha \in I_2\}$ with

$$\begin{aligned} G_\alpha(z) = &\sum_{\substack{j_1,j_2=1,\\ j_1<j_2}}^{n} \Phi_{2,\alpha,j_1,j_2}(\eta_{j_1} z_{j_1} + \eta_{j_2} z_{j_2}) + \sum_{\substack{j_1,j_2,j_3=1,\\ j_1<j_2<j_3}}^{n} \Phi_{3,\alpha,j_1,j_2,j_3}(\zeta_{j_1} z_{j_1} + \zeta_{j_2} z_{j_2} + \zeta_{j_3} z_{j_3}) + \ldots \\ &\ldots + \sum_{\substack{j_1,j_2,\ldots,j_n=1,\\ j_1<j_2<\ldots<j_n}}^{n} \Phi_{n,\alpha,j_1,j_2,\ldots,j_n}(t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \cdots + t_{j_n} z_{j_n}), \end{aligned}$$

where $\eta_i, \zeta_i, t_i \in \mathbb{C}$ $(1 \leq i \leq n)$, $\deg \Phi(z) = \deg \mathcal{Q}(z)$ and $\Phi_{m,\alpha,j_1,j_2,\ldots,j_m}(t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \ldots + t_{j_m} z_{j_m})$ is a univariate polynomial in $t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \ldots + t_{j_m} z_{j_m}$. Here $\eta_i, \zeta_i, t_i \in \mathbb{C}$ $(1 \leq i \leq n)$ are chosen from the conditions $\eta_{j_1} c_{j_1} + \eta_{j_2} c_{j_2} = 0$, $\zeta_{j_1} c_{j_1} + \zeta_{j_2} c_{j_2} + \zeta_{j_3} c_{j_3} = 0$, $t_{j_1} c_{j_1} + t_{j_2} c_{j_2} + \ldots + t_{j_m} c_{j_m} = 0$ and c_i is given below in system (7) or (8).

It is important to note that, if $\mathcal{Q}(z+c) - \mathcal{Q}(z) \equiv$ constant, for any $c \in \mathbb{C}^n \setminus \{(0,0,\ldots,0)\}$, then we can express $\mathcal{Q}(z)$ as $\mathcal{Q}(z) = \sum_{j=1}^{n} a_j z_j + \Phi(z) + A$, where $A \in \mathbb{C}$,

$$\Phi(z) = \sum_{m=2}^{n} \left(\sum_{\substack{j_1,j_2,\ldots,j_m=1,\\ j_1<j_2<\ldots<j_m}}^{n} \Phi_{m,j_1,j_2,\ldots,j_m}(t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \ldots + t_{j_m} z_{j_m}) \right) \tag{6}$$

and $\Phi_{m,j_1,j_2,\ldots,j_m}(t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \ldots + t_{j_m} z_{j_m})$ is a univariate polynomial in such a variable $t_{j_1} z_{j_1} + t_{j_2} z_{j_2} + \ldots + t_{j_m} z_{j_m}$. Here $t_i \in \mathbb{C}$ $(1 \leq i \leq n)$ is chosen from the condition $t_{j_1} c_{j_1} + t_{j_2} c_{j_2} + \ldots + t_{j_m} c_{j_m} = 0$ and c_i is given below in system (7) or (8).

We will consider the following systems of quadratic trinomial partial differential-difference equations on several complex variables:

$$\begin{cases} \left(a_1 \frac{\partial f_1(z)}{\partial z_1}\right)^2 + 2\alpha a_1 \frac{\partial f_1(z)}{\partial z_1} F_1(z) + F_1(z)^2 = 1; \\ \left(a_1 \frac{\partial f_2(z)}{\partial z_1}\right)^2 + 2\alpha a_1 \frac{\partial f_2(z)}{\partial z_1} F_2(z) + F_2(z)^2 = 1, \end{cases} \tag{7}$$

$$\begin{cases} F_3(z)^2 + 2\alpha F_3(z)\left(a_{n+1} f_1(z) + a_{n+2} f_2(z+c)\right) + \left(a_{n+1} f_1(z) + a_{n+2} f_2(z+c)\right)^2 = 1; \\ F_4(z)^2 + 2\alpha F_4(z)\left(a_{n+1} f_2(z) + a_{n+2} f_1(z+c)\right) + \left(a_{n+1} f_2(z) + a_{n+2} f_1(z+c)\right)^2 = 1, \end{cases} \tag{8}$$

where $a_j \in \mathbb{C}\setminus\{0\}$ for $1 \leq j \leq n+2$, $\alpha \in \mathbb{C}\setminus\{0, \pm 1\}$ and

$$\begin{aligned} &F_1(z) = a_2 f_1(z) + a_3 f_2(z+c) + a_4 \tfrac{\partial^2 f_1(z)}{\partial z_1^2}, \; F_2(z) = a_2 f_2(z) + a_3 f_1(z+c) + a_4 \tfrac{\partial^2 f_2(z)}{\partial z_1^2}, \\ &F_3(z) = a_1 \tfrac{\partial f_1(z)}{\partial z_1} + a_2 \tfrac{\partial f_1(z)}{\partial z_2} + \cdots + a_n \tfrac{\partial f_1(z)}{\partial z_n}, \; F_4(z) = a_1 \tfrac{\partial f_2(z)}{\partial z_1} + a_2 \tfrac{\partial f_2(z)}{\partial z_2} + \cdots + a_n \tfrac{\partial f_2(z)}{\partial z_n}. \end{aligned}$$

Throughout the paper, we denote

$$\begin{aligned} &A_1 = \tfrac{1}{2\sqrt{1+\alpha}} + \tfrac{1}{2i\sqrt{1-\alpha}}, \; A_2 = \tfrac{1}{2\sqrt{1+\alpha}} - \tfrac{1}{2i\sqrt{1-\alpha}}, \; y = \left(a_1 z_2 - a_2 z_1, \ldots, a_1 z_n - a_n z_1\right), \\ &s = \left(a_1 c_2 - a_2 c_1, \ldots, a_1 c_n - a_n c_1\right), \; y_1 = (z_2, z_3, \ldots, z_n), \; s_1 = (c_2, c_3, \ldots, c_n), \\ &\Gamma_1(k) = (a_{k+1} c_1 A_1 + a_1 A_2)/(\sqrt{2} a_1) \quad \text{and} \quad \Gamma_2(k) = (a_1 A_1 + a_{k+1} c_1 A_2)/(\sqrt{2} a_1). \end{aligned} \tag{9}$$

Thus, we have $A_1A_2=\frac{1}{2(1-\alpha^2)}$, $A_1^2+A_2^2=\frac{\alpha}{\alpha^2-1}$ and $A_1^2-A_2^2=\frac{1}{i\sqrt{1-\alpha^2}}$.

In all our statements below we assume that $c=(c_1,c_2,\ldots,c_n)\in\mathbb{C}^n\setminus\{(0,0,\ldots,0)\}$, $a_j\in\mathbb{C}\setminus\{0\}$ for $1\le j\le n+2$. For the finite order transcendental entire solutions of the system (7), we obtain the following results.

Theorem 1. *If $a_2=\pm a_3$, then the functions $f_1(z)=\frac{z_1}{\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}}+g_1(y_1)$, $f_2(z)=\frac{z_1}{\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}}+g_2(y_1)$ are finite order transcendental entire solutions of (7), where $g_1(y_1)$, $g_2(y_1)$ are finite order transcendental entire functions of periods $2s_1$.*

For simpler notation of the following results in Theorems 2-4 we introduce such a condition $(\mathfrak{A})$:

$(\mathfrak{A})$ The constants $b_j,K_i,t_j,\mu,\nu\in\mathbb{C}$ $(1\le j\le n, 1\le i\le 4)$ such that $K_1K_2=1=K_3K_4$, $\Phi_1(z)$ is a polynomial defined in (6) with $\Phi_1(z)\equiv 0$, if $\Phi_1(z)$ contain the variable z_1, and $g_k(y_1)$ $(3\le k\le 8)$ are finite order entire functions satisfying

$$\begin{cases} a_2g_3(y_1)+a_3g_4(y_1+s_1)\equiv\Gamma_1(1)K_1e^{\sum_{j=2}^n b_jz_j+\Phi_1(t)+\mu}+\Gamma_2(1)K_2e^{-\sum_{j=2}^n b_jz_j-\Phi_1(t)-\mu};\\ a_2g_4(y_1)+a_3g_3(y_1+s_1)\equiv\Gamma_1(1)K_3e^{-\sum_{j=2}^n b_jz_j-\Phi_1(t)+\nu}+\Gamma_2(1)K_4e^{\sum_{j=2}^n b_jz_j+\Phi_1(t)-\nu};\\ a_2g_5(y_1)+a_3g_6(y_1+s_1)\equiv 0,\quad a_2g_6(y_1)+a_3g_5(y_1+s_1)\equiv 0,\end{cases}$$

where $\Gamma_1(1)$, $\Gamma_2(1)$ are given in (9).

Theorem 2. *If $a_2=\pm a_3$, then $f_1(z)=\frac{z_1}{\sqrt{2}a_1}\Big(A_1K_1e^{\sum\limits_{j=2}^n b_jz_j+\Phi_1(z)+\mu}+A_2K_2e^{-\sum\limits_{j=2}^n b_jz_j-\Phi_1(z)-\mu}\Big)$ $+g_3(y_1)$, $f_2(z)=\frac{z_1}{\sqrt{2}a_1}\Big(A_1K_3e^{-\sum\limits_{j=2}^n b_jz_j-\Phi_1(z)+\nu}+A_2K_4e^{\sum\limits_{j=2}^n b_jz_j+\Phi_1(z)-\nu}\Big)+g_4(y_1)$ are finite order transcendental entire solutions of (7), for which condition $(\mathfrak{A})$ holds and*

$$\begin{cases} e^{2\sum_{j=2}^n b_jc_j}=1,\quad e^{-\sum_{j=2}^n b_jc_j+\mu+\nu}=-\frac{a_3K_4A_2}{a_2K_1A_1}\\ e^{\sum_{j=2}^n b_jc_j-\mu-\nu}=-\frac{a_3K_3A_1}{a_2K_2A_2},\quad e^{\sum_{j=2}^n b_jc_j+\mu+\nu}=-\frac{a_3K_2A_2}{a_2K_3A_1},\quad e^{-\sum_{j=2}^n b_jc_j-\mu-\nu}=-\frac{a_3K_1A_1}{a_2K_4A_2}.\end{cases}$$

Theorem 3. *If $a_1^2a_2+a_3^2a_4=0$, then $f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)-\mu}+$ $g_5(y_1)$, $f_2(z)=-\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)+\nu}+\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)-\nu}+g_6(y_1)$ are finite order transcendental entire solutions of (7), for which condition $(\mathfrak{A})$ holds, $b_1=\pm a_3/a_1$ and*

$$\begin{cases} e^{2\sum_{j=1}^n b_jc_j}=1,\quad e^{-\sum_{j=1}^n b_jc_j+\mu+\nu}=\frac{a_3K_4}{a_1b_1K_1}\\ e^{\sum_{j=1}^n b_jc_j-\mu-\nu}=-\frac{a_3K_3}{a_1b_1K_2},\quad e^{\sum_{j=1}^n b_jc_j+\mu+\nu}=-\frac{a_3K_2}{a_1b_1K_3},\quad e^{-\sum_{j=1}^n b_jc_j-\mu-\nu}=\frac{a_3K_1}{a_1b_1K_4}.\end{cases}$$

Theorem 4. *The functions $f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)-\mu}+g_5(y_1)$, $f_2(z)=\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)+\nu}-\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)-\nu}+g_6(y_1)$ are finite order transcendental entire solutions for (7), for which condition $(\mathfrak{A})$ holds,*

$$\begin{cases} \frac{a_1K_2}{a_3K_4}\left(-\frac{A_1}{A_2}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)e^{\sum\limits_{j=1}^n b_jc_j-\mu+\nu}=1,\ \frac{a_1K_4}{a_3K_2}\left(-\frac{A_1}{A_2}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)e^{\sum\limits_{j=1}^n b_jc_j+\mu-\nu}=1;\\ \frac{a_1K_1}{a_3K_3}\left(\frac{A_2}{A_1}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)e^{-\sum\limits_{j=1}^n b_jc_j+\mu-\nu}=1,\ \frac{a_1K_3}{a_3K_1}\left(\frac{A_2}{A_1}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)e^{-\sum\limits_{j=1}^n b_jc_j-\mu+\nu}=1.\end{cases}$$

and $(-A_1b_1/A_2-a_4b_1^2/a_1-a_2/a_1)(A_2b_1/A_1-a_4b_1^2/a_1-a_2/a_1)=(a_3/a_1)^2$.

For the finite order transcendental entire solutions of the system (8), we obtain the following result.

Theorem 5. *If* $a_{n+1} = \pm a_{n+2}$, *then* $f_1(z) = \frac{z_1}{\sqrt{a_1^2 - 2\alpha a_1 a_{n+1} c_1 + a_{n+1}^2 c_1^2}} + h_1(y)$ *and* $f_2(z) = \frac{z_1}{\sqrt{a_1^2 - 2\alpha a_1 a_{n+1} c_1 + a_{n+1}^2 c_1^2}} + h_2(y)$ *are finite order transcendental entire solutions of (8), where* $h_j(y)$ $(j = 1, 2)$ *are finite order transcendental entire functions with periods* $2s$ *satisfying* $\sum_{k=1}^{n} a_k \frac{\partial h_j(y)}{\partial z_k} \equiv 0$.

For simpler notation of the following results in Theorems 6–7 we introduce such a condition $(\mathfrak{B})$:

$(\mathfrak{B})$ The constants $b_j, \mu, \nu, K_i \in \mathbb{C}$ $(1 \le j \le n,\ 1 \le i \le 4)$ are such that $K_1K_2 = 1 = K_3K_4$ and $h_k(y)$ $(3 \le k \le 6)$ are finite order entire functions satisfying $\sum_{j=1}^{n} a_j \frac{\partial h_k(y)}{\partial z_j} \equiv 0$ and

$$\begin{cases} a_{n+1}h_3(y) + a_{n+2}h_4(y+s) \equiv \Gamma_1(n)K_1 e^{\sum\limits_{j=2}^{n} b_j z_j + \mu} + \Gamma_2(n)K_2 e^{-\sum\limits_{j=2}^{n} b_j z_j - \mu}; \\ a_{n+1}h_4(y) + a_{n+2}h_3(y+s) \equiv \Gamma_1(n)K_3 e^{-\sum\limits_{j=2}^{n} b_j z_j + \nu} + \Gamma_2(n)K_4 e^{\sum\limits_{j=2}^{n} b_j z_j - \nu}; \\ a_{n+1}h_5(y) + a_{n+2}h_6(y+s) \equiv 0 \quad \text{and} \quad a_{n+1}h_6(y) + a_{n+2}h_5(y+s) \equiv 0, \end{cases}$$

where $\Gamma_1(n)$, $\Gamma_2(n)$ are given in (9).

Theorem 6. *If* $a_{n+1} = \pm a_{n+2}$, *then* $f_1(z) = \frac{z_1}{\sqrt{2}a_1}\left(A_1K_1 e^{\sum\limits_{j=1}^{n} b_j z_j + \mu} + A_2K_2 e^{-\sum\limits_{j=1}^{n} b_j z_j - \mu}\right) + h_3(y)$, $f_2(z) = \frac{1}{\sqrt{2}a_1}\left(A_1K_3 e^{-\sum\limits_{j=1}^{n} b_j z_j + \nu} + A_2K_4 e^{\sum\limits_{j=1}^{n} b_j z_j - \nu}\right) z_1 + h_4(y)$, *are finite order transcendental entire solutions of (8), for which* $\sum_{j=1}^{n} a_j b_j = 0$, $(\mathfrak{B})$ *holds, and*

$$\begin{cases} e^{2\sum\limits_{j=2}^{n} b_j c_j} = 1, \quad e^{-\sum\limits_{j=2}^{n} b_j c_j + \mu + \nu} = -\frac{a_{n+2}K_4A_2}{a_{n+1}K_1A_1}; \\ e^{\sum\limits_{j=2}^{n} b_j c_j - \mu - \nu} = -\frac{a_{n+2}K_3A_1}{a_{n+1}K_2A_2}, \quad e^{\sum\limits_{j=2}^{n} b_j c_j + \mu + \nu} = -\frac{a_{n+2}K_2A_2}{a_{n+1}K_3A_1}, \quad e^{-\sum\limits_{j=2}^{n} b_j c_j - \mu - \nu} = -\frac{a_{n+2}K_1A_1}{a_{n+1}K_4A_2}. \end{cases}$$

Theorem 7. *If* $a_{n+1} \ne \pm a_{n+2}$, *then* $f_1(z) = \frac{1}{\sqrt{2}\sum\limits_{j=1}^{n} a_j b_j}\left(A_1K_1 e^{\sum\limits_{j=1}^{n} b_j z_j + \mu} - A_2K_2 e^{-\sum\limits_{j=1}^{n} b_j z_j - \mu}\right) + h_5(y)$, $f_2(z) = \frac{1}{\sqrt{2}\sum\limits_{j=1}^{n} a_j b_j}\left(A_1K_3 e^{\sum\limits_{j=1}^{n} b_j z_j + \nu} - A_2K_4 e^{-\sum\limits_{j=1}^{n} b_j z_j - \nu}\right) + h_6(y)$ *are finite order transcendental entire solutions of (8), for which* $(\mathfrak{B})$ *holds,* $\left(\sum\limits_{j=1}^{n} a_j b_j + A_2 a_{n+1}/A_1\right)\left(\sum\limits_{j=1}^{n} a_j b_j - A_1 a_{n+1}/A_2\right) = -a_{n+2}^2$ *and*

$$\begin{cases} \frac{-A_1K_2}{a_{n+2}A_2K_4}\left(\sum_{j=1}^{n} a_j b_j + \frac{A_2}{A_1} a_{n+1}\right) e^{\sum_{j=1}^{n} b_j c_j - \mu + \nu} = 1, \\ \frac{-A_1K_4}{a_{n+2}A_2K_2}\left(\sum_{j=1}^{n} a_j b_j + \frac{A_2}{A_1} a_{n+1}\right) e^{\sum_{j=1}^{n} b_j c_j + \mu - \nu} = 1, \\ \frac{A_2K_1}{a_{n+2}A_1K_3}\left(\sum_{j=1}^{n} a_j b_j - \frac{A_1}{A_2} a_{n+1}\right) e^{-\sum_{j=1}^{n} b_j c_j + \mu - \nu} = 1, \\ \frac{A_2K_3}{a_{n+2}A_1K_1}\left(\sum_{j=1}^{n} a_j b_j - \frac{A_1}{A_2} a_{n+1}\right) e^{-\sum_{j=1}^{n} b_j c_j - \mu + \nu} = 1. \end{cases}$$

The key tools in the proof of the main results are Nevanlinna's theory of several complex variables, the difference analogue of the lemma on the logarithmic derivative in several complex variables [12] and the Lagrange's auxiliary equations [28, Chapter 2] for quasi-linear partial differential equations.

3. Some Lemmas. The following are relevant lemmas of this paper and will be used to prove the main results.

Lemma 1 ([11], Lemma 1.5, p. 239). *Let $f_j \not\equiv 0$ $(j = 1, 2, 3)$ be meromorphic functions on $\mathbb{C}^n$ such that f_1 is not constant and $f_1 + f_2 + f_3 \equiv 1$ with $\sum_{j=1}^{3} \left\{N_2(r, 0; f_j) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1))$ holds as $r \to \infty$ out side of a possible exceptional set of finite linear measure, where $\lambda < 1$ is a positive number. Then either $f_2 \equiv 1$ or $f_3 \equiv 1$.*

Let $f(z)$ be an entire function on $\mathbb{C}^n$ $(n > 1)$ such that $f(0) \neq 0$ and $\rho(n(r, 0, f)) < \infty$. Let q be the smallest integer such that the integral $\int_0^\infty \frac{n(r,0,f)}{r^{q+2}} dr$ converges. Then there exists an entire function $\phi(z)$ satisfying the following conditions:

(i) The function $f(z)\phi^{-1}(z)$ is an entire function on $\mathbb{C}^n$ and does not vanish.

(ii) The expansion of the function $\ln \phi(z)$ in the neighborhood of the origin has the form: $\ln \phi(z) = \sum_{\|k\|=q+1}^{\infty} a_k z^k$.

(iii) For any $R > 0$, $\ln M_\phi(R) \leq C_{n,q} R^q \left\{\int_0^R \frac{n(t,0,f)}{t^{q+1}} dt + R \int_R^\infty \frac{n(t,0,f)}{t^{q+2}} ds\right\}$. where $C_{n,q}$ is a constant and $M_\phi(R) = \max_{|z|\leq R} |\phi(z)|$ This function $\phi(z)$ is called *the canonical function* (see [26, Theorem 4.3.2, p. 245]).

Lemma 2 ([26], Theorem 4.3.4, p. 247). *Let $f(z)$ be an entire function on $\mathbb{C}^n$ such that $f(0) \neq 0$ and $\rho(N(r, 0, f)) < \infty$. Then there exists an entire function $g(z)$ and a canonical function $\phi(z)$ such that $f(z) = \phi(z)e^{g(z)}$.*

Lemma 3 ([9], Lemma 2.1, p. 282). *If g is a transcendental entire function on $\mathbb{C}^n$ and if f is a meromorphic function of positive order on $\mathbb{C}$, then $f \circ g$ is of infinite order.*

Lemma 4 ([10], Proposition 3.2, p. 240). *Let P be a non-constant entire function in $\mathbb{C}^n$. Then* $\rho\left(e^P\right) = \begin{cases} \deg(P), & \text{if } P \text{ is a polynomial;} \\ +\infty, & \text{otherwise.} \end{cases}$.

Lemma 5 ([11], Theorem 2.1, p. 242). *Suppose that $a_0(z), a_1(z), \ldots, a_m(z)$ $(m \geq 1)$ are meromorphic functions on $\mathbb{C}^n$ and $g_0(z), g_1(z), \ldots, g_m(z)$ are entire functions on $\mathbb{C}^n$ such that $g_j(z) - g_k(z)$ are not constants for $0 \leq j < k \leq n$. If $\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$ and $T(r, a_j) = o(T(r))$, $j = 0, 1, \ldots, n$ hold as $r \to \infty$ out side of a possible exceptional set of finite linear measure, where $T(r) = \min_{0\leq j<k\leq n} T(r, e^{g_j - g_k})$, then $a_j(z) \equiv 0$ $(j = 0, 1, 2, \ldots, n)$.*

Lemma 6 ([6], Lemma 3.2, p. 385). *Let f be a non-constant meromorphic function on $\mathbb{C}^n$. Then for any $I \in \mathbb{Z}_+^n$, $T(r, \partial^I f) = O(T(r, f))$ for all r except possibly a set of finite Lebesgue measure, where $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ denotes a multiple index with $\|I\| = i_1 + i_2 + \cdots + i_n$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\partial^I f = \frac{\partial^{\|I\|} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}$.*

4. Proofs of the main theorems.

Proof of Theorem 1. Part 1. The first part is common for all Theorems 1–4. Let (f_1, f_2) be a pair of finite order transcendental entire functions satisfies the system (7). Let $a_1\frac{\partial f_1(z)}{\partial z_1} = \frac{1}{\sqrt{2}}(U_1(z)+V_1(z))$ and $a_2f_1(z)+a_3f_2(z+c)+a_4\frac{\partial^2 f_1(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}(U_1(z)-V_1(z))$, where $U_1(z)$ and $V_1(z)$ are finite order entire functions on $\mathbb{C}^n$. The first equation of (7) becomes $(1+\alpha)U_1^2 + (1-\alpha)V_1^2 = 1$, i.e.

$$\left(\sqrt{1+\alpha}U_1 + i\sqrt{1-\alpha}V_1\right)\left(\sqrt{1+\alpha}U_1 - i\sqrt{1-\alpha}V_1\right) = 1.$$

Here $\sqrt{1+\alpha}U_1 \pm i\sqrt{1-\alpha}V_1$ are finite order entire functions and have no zeros on $\mathbb{C}^n$. In view of the Lemma 2, we have $\sqrt{1+\alpha}U_1+i\sqrt{1-\alpha}V_1 = K_1e^{P(z)}$ and $\sqrt{1+\alpha}U_1-i\sqrt{1-\alpha}V_1 = K_2e^{-P(z)}$, where $K_1, K_2 \in \mathbb{C}\setminus\{0\}$ such that $K_1K_2 = 1$ and $P(z)$ is an entire function in $\mathbb{C}^n$. Thus, we have

$$\sqrt{1+\alpha}\; U_1 = \frac{K_1e^{P(z)} + K_2e^{-P(z)}}{2} \;\text{ and }\; \sqrt{1-\alpha}\; V_1 = \frac{K_1e^{P(z)} - K_2e^{-P(z)}}{2i}. \tag{10}$$

Since $\rho(f_i) < +\infty$ $(i = 1, 2)$, by using Lemmas 3, 4 and 6, we get from (10) that $P(z)$ is a polynomial on $\mathbb{C}^n$. Therefore, we have

$$\begin{cases} a_1\frac{\partial f_1(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left(A_1K_1e^{P(z)} + A_2K_2e^{-P(z)}\right); \\ a_2f_1(z) + a_3f_2(z+c) + a_4\frac{\partial^2 f_1(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)} + A_1K_2e^{-P(z)}\right), \end{cases} \tag{11}$$

where A_1 and A_2 are given in (9). Again, let $a_1\frac{\partial f_2(z)}{\partial z_1} = \frac{1}{\sqrt{2}}(U_2(z) + V_2(z))$ and $a_2f_2(z) + a_3f_1(z+c)+a_4\frac{\partial^2 f_2(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}(U_2(z)-V_2(z))$, where $U_2(z), V_2(z)$ are finite order entire functions on $\mathbb{C}^n$. Using similar arguments as above, we get

$$\begin{cases} a_1\frac{\partial f_2(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left(A_1K_3e^{Q(z)} + A_2K_4e^{-Q(z)}\right); \\ a_2f_2(z) + a_3f_1(z+c) + a_4\frac{\partial^2 f_2(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(A_2K_3e^{Q(z)} + A_1K_4e^{-Q(z)}\right), \end{cases} \tag{12}$$

where $K_3, K_4 \in \mathbb{C}\setminus\{0\}$ such that $K_3K_4 = 1$ and $Q(z)$ is a polynomial on $\mathbb{C}^n$. The different cases arise separately in proofs of all Theorems 1-4.

Part 2. Now we begin to prove properly Theorem 1. Let $P(z), Q(z)$ be simultaneously constants. From (11) and (12), we have

$$\begin{cases} a_1\frac{\partial f_1(z)}{\partial z_1} = \varphi_1, \quad a_2f_1(z) + a_3f_2(z+c) + a_4\frac{\partial^2 f_1(z)}{\partial z_1^2} = \varphi_2, \\ a_1\frac{\partial f_2(z)}{\partial z_1} = \varphi_3, \quad a_2f_2(z) + a_3f_1(z+c) + a_4\frac{\partial^2 f_2(z)}{\partial z_1^2} = \varphi_4, \end{cases}$$

where $\varphi_j \in \mathbb{C}$ for $1 \le j \le 4$ with $\varphi_k^2 + 2\alpha\varphi_k\varphi_{k+1} + \varphi_{k+1}^2 = 1$ $(k = 1, 3)$. Hence, we have $f_1(z) = (\varphi_1/a_1)z_1 + g_1(y_1)$ and $f_2(z) = (\varphi_3/a_1)z_1 + g_2(y_1)$, where $g_j(y_1)$ $(j = 1, 2)$ are finite order transcendental entire functions of $z_2, z_3, \ldots, z_n$. Thus, we deduce that

$$\begin{aligned} &((a_2\varphi_1 + a_3\varphi_3)/a_1)z_1 + (a_2g_1(y_1) + a_3g_2(y_1+s_1)) + a_3c_1\varphi_3/a_1 \equiv \varphi_2 \\ \text{and}\quad &((a_2\varphi_3 + a_3\varphi_1)/a_1)z_1 + (a_2g_2(y_1) + a_3g_1(y_1+s_1)) + a_3c_1\varphi_1/a_1 \equiv \varphi_4. \end{aligned}$$

Since $g_j(y_1)$ $(j = 1, 2)$ are finite order transcendental entire functions, so we have

$$\begin{aligned} &a_2\varphi_1 + a_3\varphi_3 = 0, a_2\varphi_3 + a_3\varphi_1 = 0, a_2g_1(y_1) + a_3g_2(y_1+s_1) = 0 \\ &a_2g_2(y_1) + a_3g_1(y_1+s_1) = 0, a_3c_1\varphi_3 = a_1\varphi_2 \quad\text{and}\quad a_3c_1\varphi_1 = a_1\varphi_4. \end{aligned}$$

For non-zero solution of system $a_2\varphi_1+a_3\varphi_3=0$, $a_2\varphi_3+a_3\varphi_1=0$, we must have $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}=0$, *i.e.*, $a_2=\pm a_3$, which implies that $\varphi_1=\pm\varphi_3$. It is easy to see that $\varphi_1/\varphi_2=-a_1/(a_2c_1)$. From $\varphi_1^2+2\alpha\varphi_1\varphi_2+\varphi_2^2=1$, we deduce that $\varphi_2=\pm a_2c_1/\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}$, $\varphi_1=\mp a_1/\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}$ and $\varphi_3=a_1/\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}$. Therefore,

$$f_1(z)=\frac{z_1}{\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}}+g_1(y_1),\quad f_2(z)=\frac{z_1}{\sqrt{a_1^2-2\alpha a_1a_2c_1+a_2^2c_1^2}}+g_2(y_1),$$

where $g_j(y_1)$ $(j=1,2)$ are finite order transcendental entire functions with periods $2s_1$. $\square$

Proof of Theorem 2. Let either $P(z)$ or $Q(z)$ be a constant. Assume that $P(z)$ is a constant and $Q(z)$ is a non-constant polynomial. From (11), we have $a_1\frac{\partial f_1(z)}{\partial z_1}=\varphi_1$, $a_2f_1(z)+a_3f_2(z+c)+a_4\frac{\partial^2 f_1(z)}{\partial z_1^2}=\varphi_2$, where $\varphi_1,\varphi_2\in\mathbb{C}$ with $\varphi_1^2+2\alpha\varphi_1\varphi_2+\varphi_2^2=1$. Thus, we have $a_2f_1(z)+a_3f_2(z+c)=\varphi_2$, which implies that $\frac{\partial f_2(z+c)}{\partial z_1}=-\frac{a_2}{a_3}\frac{\partial f_1(z)}{\partial z_1}=-\frac{a_2}{a_3}\frac{\varphi_1}{a_1}$, which contradicts the fact that $\frac{\partial f_2(z)}{\partial z_1}$ is a transcendental entire function.

Let $P(z),Q(z)$ be both non-constant polynomials. Differentiating partially with respect to z_1 on both sides of the first equation of (11), we get

$$\frac{\partial^2 f_1(z)}{\partial z_1^2}=\frac{A_1K_1e^{P(z)}-A_2K_2e^{-P(z)}}{\sqrt{2}a_1}\frac{\partial P(z)}{\partial z_1}.$$

Using the last equation, we derive from the second equation of (11) that

$$a_2f_1(z)+a_3f_2(z+c)=K_1e^{P(z)}\left(\frac{A_2}{\sqrt{2}}-\frac{a_4A_1}{\sqrt{2}a_1}\frac{\partial P(z)}{\partial z_1}\right)+K_2e^{-P(z)}\left(\frac{A_1}{\sqrt{2}}+\frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial P(z)}{\partial z_1}\right).$$

Differentiating partially the last expression with respect to z_1, we get

$$\begin{aligned}a_2\frac{\partial f_1(z)}{\partial z_1}+a_3\frac{\partial f_2(z+c)}{\partial z_1}&=K_1e^{P(z)}\left(\frac{A_2}{\sqrt{2}}\frac{\partial P(z)}{\partial z_1}-\frac{a_4A_1}{\sqrt{2}a_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2-\frac{a_4A_1}{\sqrt{2}a_1}\frac{\partial^2 P(z)}{\partial z_1^2}\right)+\\&+K_2e^{-P(z)}\left(-\frac{A_1}{\sqrt{2}}\frac{\partial P(z)}{\partial z_1}-\frac{a_4A_2}{\sqrt{2}a_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2+\frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial^2 P(z)}{\partial z_1^2}\right).\end{aligned}\tag{13}$$

Using the first equations of (11) and (12), we get from (13) that

$$\Psi_1(z)e^{P(z)+Q(z+c)}+\Omega_1(z)e^{-P(z)+Q(z+c)}-\frac{A_1K_3}{A_2K_4}e^{2Q(z+c)}\equiv 1,\tag{14}$$

where $\Psi_1(z)=\frac{a_1K_1}{a_3K_4}(\frac{\partial P(z)}{\partial z_1}-\frac{a_4A_1}{a_1A_2}(\frac{\partial P(z)}{\partial z_1})^2-\frac{a_4A_1}{a_1A_2}\frac{\partial^2 P(z)}{\partial z_1^2}-\frac{a_2A_1}{a_1A_2})$, $\Omega_1(z)=\frac{a_1K_2}{a_3K_4}(-\frac{A_1}{A_2}\frac{\partial P(z)}{\partial z_1}-\frac{a_4}{a_1}(\frac{\partial P(z)}{\partial z_1})^2+\frac{a_4}{a_1}\frac{\partial^2 P(z)}{\partial z_1^2}-\frac{a_2}{a_1})$. Using similar arguments as above, we deduce from the first equation of (11) and (12) that

$$\Psi_2(z)e^{Q(z)+P(z+c)}+\Omega_2(z)e^{-Q(z)+P(z+c)}-\frac{A_1K_1}{A_2K_2}e^{2P(z+c)}\equiv 1,\tag{15}$$

where $\Psi_2(z)=\frac{a_1K_3}{a_3K_2}(\frac{\partial Q(z)}{\partial z_1}-\frac{a_4A_1}{a_1A_2}(\frac{\partial Q(z)}{\partial z_1})^2-\frac{a_4A_1}{a_1A_2}\frac{\partial^2 Q(z)}{\partial z_1^2}-\frac{a_2A_1}{a_1A_2})$, $\Omega_2(z)=\frac{a_1K_4}{a_3K_2}(-\frac{A_1}{A_2}\frac{\partial Q(z)}{\partial z_1}-\frac{a_4}{a_1}(\frac{\partial Q(z)}{\partial z_1})^2+\frac{a_4}{a_1}\frac{\partial^2 Q(z)}{\partial z_1^2}-\frac{a_2}{a_1})$. From (14), it is clear that both $\Psi_1(z)$ and $\Omega_1(z)$ are not simultaneously identically zero, otherwise we arrive at a contradiction. Let $\Psi_1(z)\equiv 0$ and $\Omega_1(z)\not\equiv 0$. From (14), we have

$$\Omega_1(z)e^{Q(z+c)}-\frac{A_1K_3}{A_2K_4}e^{2Q(z+c)+P(z)}-e^{P(z)}\equiv 0. \tag{16}$$

From (16), it is easy to see that $Q(z+c)-P(z)$ is a non-constant polynomial. We claim that $Q(z+c)+P(z)$ and $2Q(z+c)+P(z)$ are non-constant polynomials. If possible, let $Q(z+c)+P(z)\equiv k_1$ which implies $Q(z+c)\equiv k_1-P(z)$ and $2Q(z+c)+P(z)\equiv k_2$ which implies $P(z)\equiv k_2-2Q(z+c)$, where $k_1,k_2\in\mathbb{C}$. For above two situations, we deduce from (16) that

$$\begin{cases}\Omega_1(z)e^{k_1}-\frac{A_1K_3}{A_2K_4}e^{2k_1}-e^{2P(z)}\equiv 0;\\ \Omega_1(z)e^{2Q(z+c)-k_2}-\frac{A_1K_3}{A_2K_4}e^{Q(z+c)}-e^{-Q(z+c)}\equiv 0.\end{cases} \tag{17}$$

In both circumstances, we get a contradiction from (17) by using Lemma 5. Hence, $Q(z+c)+P(z)$ and $2Q(z+c)+P(z)$ are non-constant polynomials. In view of Lemma 5, we get a contradiction from (16). Using similar arguments, we again get a contradiction from (14) and (15), when $\Psi_1(z)\not\equiv 0$, $\Omega_1(z)\equiv 0$; $\Psi_2(z)\equiv 0$, $\Omega_2(z)\not\equiv 0$ and $\Psi_2(z)\not\equiv 0$, $\Omega_2(z)\equiv 0$. Now, it easy to see that

$$\begin{aligned}N\big(r,\Psi_1(z)e^{P(z)+Q(z+c)}\big)&=N\big(r,\Omega_1(z)e^{-P(z)+Q(z+c)}\big)=N\big(r,-A_1K_3e^{2Q(z+c)}/(A_2K_4)\big)=\\ &=N\big(r,0;\Psi_1(z)e^{P(z)+Q(z+c)}\big)=N\big(r,0;\Omega_1(z)e^{-P(z)+Q(z+c)}\big)=\\ &=N\big(r,0;-A_1K_3e^{2Q(z+c)}/(A_2K_4)\big)=S\big(r,-A_1K_3e^{2Q(z+c)}/(A_2K_4)\big).\end{aligned}$$

By Lemma 1, we get from (14) that either $\Psi_1(z)e^{P(z)+Q(z+c)}\equiv 1$ or $\Omega_1(z)e^{-P(z)+Q(z+c)}\equiv 1$ where $\Psi_1(z)$ and $\Omega_1(z)$ are given after (14). Similarly, by using Lemma 1, we deduce from (15) that either $\Psi_2(z)e^{Q(z)+P(z+c)}\equiv 1$ or $\Omega_2(z)e^{-Q(z)+P(z+c)}\equiv 1$, where $\Psi_2(z)$ and $\Omega_2(z)$ are given after (15). Now we will discuss the following cases.

Let

$$\Psi_1(z)e^{P(z)+Q(z+c)}\equiv 1,\quad \Psi_2(z)e^{Q(z)+P(z+c)}\equiv 1. \tag{18}$$

Using (18), we get from (14) and (15) respectively

$$\tfrac{A_2K_4}{A_1K_3}\Omega_1(z)e^{-P(z)-Q(z+c)}\equiv 1,\quad \tfrac{A_2K_2}{A_1K_1}\Omega_2(z)e^{-Q(z)-P(z+c)}\equiv 1. \tag{19}$$

From (18), it is clear that $P(z)+Q(z+c)$ and $Q(z)+P(z+c)$ are both constants, say ξ_1 and ξ_2 respectively, where $\xi_1,\xi_2\in\mathbb{C}$. Now $P(z)-P(z+2c)=(P(z)+Q(z+c))-(Q(z+c)+P(z+2c))\equiv\xi_1-\xi_2$ and $Q(z)-Q(z+2c)\equiv\xi_2-\xi_1$. It is easy to see that $P(z)=\sum_{j=1}^n b_jz_j+\Phi_1(z)+\mu$ and $Q(z)=\sum_{j=1}^n d_jz_j+\Phi_2(z)+\nu$, where $b_i,d_i,\mu,\nu\in\mathbb{C}$ $(1\le i\le n)$ and $\Phi_k(z)$ $(k=1,2)$ is a polynomial defined in (6). From (18), we have

$$\begin{aligned}&\left(b_1+\tfrac{\partial\Phi_1(z)}{\partial z_1}\right)-\tfrac{a_4A_1}{a_1A_2}\left(b_1+\tfrac{\partial\Phi_1(z)}{\partial z_1}\right)^2-\tfrac{a_4A_1}{a_1A_2}\tfrac{\partial^2\Phi_1(z)}{\partial z_1^2}-\tfrac{a_2A_1}{a_1A_2}\equiv\tfrac{a_3K_4}{a_1K_1}e^{-\xi_1},\\ &\left(d_1+\tfrac{\partial\Phi_2(z)}{\partial z_1}\right)-\tfrac{a_4}{a_1}\left(d_1+\tfrac{\partial\Phi_2(z)}{\partial z_1}\right)^2+\tfrac{a_4}{a_1}\tfrac{\partial^2\Phi_1(z)}{\partial z_1^2}-\tfrac{a_2}{a_1}\equiv\tfrac{a_3K_2}{a_1K_3}e^{-\xi_2}.\end{aligned} \tag{20}$$

If $\Phi_k(z)$ $(k=1,2)$ contain the variable z_1, then by comparing the degrees on both sides of (20), we get that $\deg(\Phi_k(z))\le 1$ for $k=1,2$. For simplicity, we still denote $P(z)=\sum_{j=1}^n b_jz_j+\mu$ and $Q(z)=\sum_{j=1}^n d_jz_j+\nu$, where $b_j,d_j,\mu,\nu\in\mathbb{C}$ $(1\le j\le n)$. This implies that $\Phi_k(z) \equiv 0$ for $k = 1, 2$. Since $P(z) + Q(z + c)$ is a constant, so we must have $b_j + d_j = 0$ for $1 \leq j \leq n$. Therefore $P(z) = \sum_{j=1}^{n} b_j z_j + \mu$ and $Q(z) = -\sum_{j=1}^{n} b_j z_j + \nu$. From (18) and (19), we deduce that

$$\begin{cases} \frac{a_1 K_1 A_1}{a_3 K_4 A_2}\left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) e^{-\sum\limits_{j=1}^{n} b_j c_j + \mu + \nu} = 1; \\ \frac{a_1 K_3 A_1}{a_3 K_2 A_2}\left(-\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) e^{\sum\limits_{j=1}^{n} b_j c_j + \mu + \nu} = 1; \\ \frac{a_1 K_2 A_2}{a_3 K_3 A_1}\left(-\frac{A_1}{A_2}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) e^{\sum\limits_{j=1}^{n} b_j c_j - \mu - \nu} = 1; \\ \frac{a_1 K_4 A_2}{a_3 K_1 A_1}\left(\frac{A_1}{A_2}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) e^{-\sum\limits_{j=1}^{n} b_j c_j - \mu - \nu} = 1. \end{cases} \tag{21}$$

From (21), we have

$$\left(\frac{a_1}{a_3}\right)^2 \left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right)\left(\frac{A_1}{A_2}b_1 + \frac{a_4}{a_1}b_1^2 + \frac{a_2}{a_1}\right) =$$
$$= \left(\frac{a_1}{a_3}\right)^2 \left(\frac{A_2}{A_1}b_1 + \frac{a_4}{a_1}b_1^2 + \frac{a_2}{a_1}\right)\left(\frac{A_1}{A_2}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right),$$

i.e. $\frac{(A_1^2 - A_2^2)}{A_1 A_2}\left(\frac{a_4}{a_1}b_1^2 + \frac{a_2}{a_1}\right) b_1 = 0$. Since $(A_1^2 - A_2^2)/(A_1 A_2) = -2i\sqrt{1-\alpha^2} \neq 0$, so, either $b_1 = 0$ or $a_4 b_1^2 + a_2 = 0$. It is clear that both b_1 and $a_4 b_1^2 + a_2$ are not simultaneously zero, otherwise we get $a_2 = 0$, which is a contradiction.

Now two different cases are possible $b_1 = 0$ and $a_4 b_1^2 + a_2 = 0$. The second case is considered in the proof of Theorem 4.

If $b_1 = 0$, then we deduce from (21) that

$$\begin{cases} e^{2\sum\limits_{j=2}^{n} b_j c_j} = \left(\frac{a_3}{a_2}\right)^2 = e^{-2\sum\limits_{j=2}^{n} b_j c_j}, \quad e^{-\sum\limits_{j=2}^{n} b_j c_j + \mu + \nu} = -\frac{a_3 K_4 A_2}{a_2 K_1 A_1}, e^{\sum\limits_{j=2}^{n} b_j c_j - \mu - \nu} = -\frac{a_3 K_3 A_1}{a_2 K_2 A_2}; \\ e^{\sum\limits_{j=2}^{n} b_j c_j + \mu + \nu} = -\frac{a_3 K_2 A_2}{a_2 K_3 A_1}, \quad e^{-\sum\limits_{j=2}^{n} b_j c_j - \mu - \nu} = -\frac{a_3 K_1 A_1}{a_2 K_4 A_2}. \end{cases} \tag{22}$$

From (11) and (12), we deduce that

$$\begin{cases} f_1(z) = \frac{1}{\sqrt{2}a_1}\left(A_1 K_1 e^{\sum\limits_{j=2}^{n} b_j z_j + \mu} + A_2 K_2 e^{-\sum\limits_{j=2}^{n} b_j z_j - \mu}\right) z_1 + g_3(y_1); \\ f_2(z) = \frac{1}{\sqrt{2}a_1}\left(A_1 K_3 e^{-\sum\limits_{j=2}^{n} b_j z_j + \nu} + A_2 K_4 e^{\sum\limits_{j=2}^{n} b_j z_j - \nu}\right) z_1 + g_4(y_1), \end{cases} \tag{23}$$

where $g_j(y_1) (j = 3, 4)$ are finite order entire functions. From (22), it is clear that $a_2 = \pm a_3$. Using (22) and (23), we get from the second equation of (11) that

$$a_2 g_3(y_1) + a_3 g_4(y_1 + s_1) = \Gamma_1(1) K_1 e^{\sum\limits_{j=2}^{n} b_j z_j + \mu} + \Gamma_2(1) K_2 e^{-\sum\limits_{j=2}^{n} b_j z_j - \mu},$$

where $\Gamma_k(1)$ $(k = 1, 2)$ are given in (9). Similarly, we deduce from the second equation of (12) that $a_2 g_4(y_1) + a_3 g_3(y_1 + s_1) = \Gamma_1(1) K_3 e^{-\sum_{j=2}^{n} b_j z_j + \nu} + \Gamma_2(1) K_4 e^{\sum_{j=2}^{n} b_j z_j - \nu}$.

If $\Phi_k(z)$ $(k=1,2)$ is independent of z_1, then we have $P(z)=\sum_{j=1}^{n} b_j z_j+\Phi_1(z)+\mu$ and $Q(z)=\sum_{j=1}^{n} d_j z_j+\Phi_2(z)+\nu$, where $b_j, d_j, \mu, \nu \in \mathbb{C}$ $(1 \leq j \leq n$ and $2 \leq i \leq n)$ and $\Phi_k(z)$ $(k=1,2)$ is a polynomial defined in (6). Since $P(z)+Q(z+c)$ is a constant, so we must have $b_j+d_j=0$ for $1 \leq j \leq n$ and $\Phi_1(z)+\Phi_2(z) \equiv 0$. Therefore $P(z)=\sum_{j=1}^{n} b_j z_j+\Phi_1(z)+\mu$ and $Q(z)=-\sum_{j=1}^{n} b_j z_j-\Phi_1(z)+\nu$. From (18) and (19), we again have (21) and either $b_1=0$ or $a_4 b_1^2+a_2=0$. The second case is considered in the proof of Theorem 4.

If $b_1=0$, then we have (22). Using arguments similar to those presented above, we deduce that

$$\begin{cases} f_1(z)=\frac{z_1}{\sqrt{2}a_1}\left(A_1K_1e^{\sum\limits_{j=2}^{n} b_jz_j+\Phi_1(z)+\mu}+A_2K_2e^{-\sum\limits_{j=2}^{n} b_jz_j-\Phi_1(z)-\mu}\right)+h_1(y_1); \\ f_2(z)=\frac{z_1}{\sqrt{2}a_1}\left(A_1K_3e^{-\sum\limits_{j=2}^{n} b_jz_j-\Phi_1(z)+\nu}+A_2K_4e^{\sum\limits_{j=2}^{n} b_jz_j+\Phi_1(z)-\nu}\right)+h_2(y_1), \end{cases}$$

where $a_2=\pm a_3$ and $h_j(y_1)(j=1,2)$ are finite order entire functions satisfying

$$a_2h_1(y_1)+a_3h_2(y_1+s_1)=\Gamma_1(1)K_1e^{\sum\limits_{j=2}^{n} b_jz_j+\Phi_1(z)+\mu}+\Gamma_2(1)K_2e^{-\sum\limits_{j=2}^{n} b_jz_j-\Phi_1(z)-\mu},$$

$$a_2h_2(y_1)+a_3h_1(y_1+s_1)=\Gamma_1(1)K_3e^{-\sum\limits_{j=2}^{n} b_jz_j-\Phi_1(z)+\nu}+\Gamma_2(1)K_4e^{\sum\limits_{j=2}^{n} b_jz_j+\Phi_1(z)-\nu},$$

where $\Gamma_k(1)$ $(k=1,2)$ are given in (9). □

Proof of Theorem 3. Let $\Phi_k(z)$ $(k=1,2)$ contain the variable z_1. Taking into account the proof of Theorem 2 we assume that $a_4b_1^2+a_2=0$. Then from (21) we have

$$\begin{cases} e^{2\sum\limits_{j=1}^{n} b_jc_j}=\left(\frac{a_3}{a_1b_1}\right)^2=e^{-2\sum\limits_{j=1}^{n} b_jc_j}; \\ e^{-\sum\limits_{j=1}^{n} b_jc_j+\mu+\nu}=\frac{a_3K_4}{a_1b_1K_1}, e^{\sum\limits_{j=1}^{n} b_jc_j-\mu-\nu}=-\frac{a_3K_3}{a_1b_1K_2}; \\ e^{\sum\limits_{j=1}^{n} b_jc_j+\mu+\nu}=-\frac{a_3K_2}{a_1b_1K_3}, e^{-\sum\limits_{j=1}^{n} b_jc_j-\mu-\nu}=\frac{a_3K_1}{a_1b_1K_4}. \end{cases} \tag{24}$$

From (24), it is easy to see that $a_3=\pm a_1b_1$. The Lagrange's auxiliary equations [28, Chapter 2] of the first equation of (11) are

$$\frac{dz_1}{1}=\frac{dz_2}{0}=\cdots=\frac{dz_n}{0}=\frac{\sqrt{2}a_1df_1}{A_1K_1e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}+A_2K_2e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu}}.$$

Note that $\alpha_j=z_j$ for $2 \leq j \leq n$ and $df_1=\frac{A_1K_1}{\sqrt{2}a_1}e^{\sum_{j=1}^{n} b_jz_j+\mu}dz_1+\frac{A_2K_2}{\sqrt{2}a_1}e^{-\sum_{j=1}^{n} b_jz_j-\mu}dz_1$, i.e.

$$df_1=\frac{A_1K_1}{\sqrt{2}a_1}e^{b_1z_1+\sum\limits_{j=2}^{n} b_je_j+\mu}dz_1+\frac{A_2K_2}{\sqrt{2}a_1}e^{-b_1z_1-\sum\limits_{j=2}^{n} b_je_j-\mu}dz_1,$$

$$f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu}+\alpha_1.$$

Note that after integration with respect to z_1, replacing α_2 by z_2,..., α_n by z_n, where $\alpha_j \in \mathbb{C}$ for $1 \leq j \leq n$. Hence, the solution is $\chi(\alpha_1, \alpha_2, \ldots, \alpha_n)=0$. For simplicity, we suppose

$$f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}+g_5(y_1),$$

where $g_5(y_1)$ is a finite order entire function of $z_2, z_3, \ldots, z_n$. Similarly, we deduce from the first equation in (12) that

$$f_2(z)=-\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_jz_j+\nu}+\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_jz_j-\nu}+g_6(y_1),$$

where $g_6(y_1)$ is a finite order entire function. Using (24) and representations for f_1, f_2 given above, we deduce from the second equation of (11) that

$$a_2\Bigg(\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}+g_5(y_1)\Bigg)+a_3\Bigg(-\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_j(z_j+c_j)+\nu}+$$
$$+\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_j(z_j+c_j)-\nu}+g_6(y_1+s_1)\Bigg)+a_4\left(\frac{A_1K_1b_1}{\sqrt{2}a_1}e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}-\frac{A_2K_2b_1}{\sqrt{2}a_1}e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}\right)=$$
$$=\frac{A_2K_1e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}+A_1K_2e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}}{\sqrt{2}},$$

$$\text{i.e., }\frac{K_1e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}}{\sqrt{2}}\left(\frac{a_2A_1}{a_1b_1}+A_2+\frac{a_4b_1A_1}{a_1}\right)-\frac{K_2e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}}{\sqrt{2}}\left(\frac{a_2A_2}{a_1b_1}-A_1+\frac{a_4b_1A_2}{a_1}\right)+$$
$$+a_2g_5(y_1)+a_3g_6(y_1+s_1)\equiv\frac{A_2K_1e^{\sum\limits_{j=1}^{n}b_jz_j+\mu}+A_1K_2e^{-\sum\limits_{j=1}^{n}b_jz_j-\mu}}{\sqrt{2}}.$$

Hence, $a_2g_5(y_1)+a_3g_6(y_1+s_1)\equiv 0$. From (24) and representations for f_1, f_2 given above, using arguments similar as above, we deduce from the second equation in (12) that

$$a_2g_6(y_1)+a_3g_5(y_1+s_1)\equiv 0.$$

Let $\Phi_k(z)$ $(k=1,2)$ is independent of z_1. In view of proof of Theorem 2 one has $a_4b_1^2+a_2=0$. Then (24) is true. Using arguments similar to presented above, we deduce from (11) and (12) that

$$\begin{cases} f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_jz_j+\Phi_1(z)+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_jz_j-\Phi_1(z)-\mu}+h_3(y_1);\\ f_2(z)=-\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n}b_jz_j-\Phi_1(z)+\nu}+\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n}b_jz_j+\Phi_1(z)-\nu}+h_4(y_1), \end{cases}$$

where $a_3=\pm a_1b_1$ and $h_j(y_1)(j=3,4)$ are finite order entire functions satisfying

$$a_2h_3(y_1)+a_3h_4(y_1+s_1)\equiv 0,\quad a_2h_4(y_1)+a_3h_3(y_1+s_1)\equiv 0.$$

□

Proof of Theorem 4. Let

$$\Psi_1(z)e^{P(z)+Q(z+c)}\equiv 1,\quad \Omega_2(z)e^{-Q(z)+P(z+c)}\equiv 1. \tag{25}$$

Since $P(z), Q(z)$ are non-constant polynomials, so it is clear from (25) that $P(z)+Q(z+c)\equiv \xi_1$ and $-Q(z)+P(z+c)\equiv\xi_2$, where $\xi_1,\xi_2\in\mathbb{C}$. Therefore, we have $P(z)+P(z+2c)\equiv\xi_1+\xi_2$, which is not possible, since $P(z)$ is a non-constant polynomial.

Let

$$\Omega_1(z)e^{-P(z)+Q(z+c)} \equiv 1, \quad \Psi_2(z)e^{Q(z)+P(z+c)} \equiv 1. \tag{26}$$

From (26), it is clear that $-P(z)+Q(z+c) \equiv \xi_3$ and $Q(z)+P(z+c) \equiv \xi_4$, where $\xi_3, \xi_4 \in \mathbb{C}$. Using arguments similar to those presented in the proof of Theorem 7, we get a contradiction.

Let

$$\Omega_1(z)e^{-P(z)+Q(z+c)} \equiv 1, \quad \Omega_2(z)e^{-Q(z)+P(z+c)} \equiv 1. \tag{27}$$

Using (27), we get from (14) and (15) respectively

$$\frac{A_2K_4}{A_1K_3}\Psi_1(z)e^{P(z)-Q(z+c)} \equiv 1, \quad \frac{A_2K_2}{A_1K_1}\Psi_2(z)e^{Q(z)-P(z+c)} \equiv 1. \tag{28}$$

From (27), it is clear that $P(z) - Q(z+c)$ and $Q(z) - P(z+c)$ are both constants, say ξ_3 and ξ_4 respectively, where $\xi_3, \xi_4 \in \mathbb{C}$. Now $P(z) - P(z+2c) = (P(z) - Q(z+c)) + (Q(z+c) - P(z+2c)) \equiv \xi_3 + \xi_4$ and $Q(z) - Q(z+2c) \equiv \xi_3 + \xi_4$. Thus $P(z) = \sum_{j=1}^{n} b_j z_j + \Phi_1(z) + \mu$ and $Q(z) = \sum_{j=1}^{n} d_j z_j + \Phi_2(z) + \nu$ where $b_i, d_i, \mu, \nu \in \mathbb{C}$ $(1 \le i \le n)$ and $\Phi_k(z)$ $(k = 1, 2)$ is a polynomial defined in (6). From (27), we have

$$\begin{aligned} -\tfrac{A_1}{A_2}\left(b_1 + \tfrac{\partial\Phi_1(z)}{\partial z_1}\right) - \tfrac{a_4}{a_1}\left(b_1 + \tfrac{\partial\Phi_1(z)}{\partial z_1}\right)^2 + \tfrac{a_4}{a_1}\tfrac{\partial^2\Phi_1(z)}{\partial z_1^2} - \tfrac{a_2}{a_1} &\equiv \tfrac{a_3K_4}{a_1K_2}e^{\xi_3},\\ -\tfrac{A_1}{A_2}\left(d_1 + \tfrac{\partial\Phi_2(z)}{\partial z_1}\right) - \tfrac{a_4}{a_1}\left(d_1 + \tfrac{\partial\Phi_2(z)}{\partial z_1}\right)^2 + \tfrac{a_4}{a_1}\tfrac{\partial^2\Phi_2(z)}{\partial z_1^2} - \tfrac{a_2}{a_1} &\equiv \tfrac{a_3K_2}{a_1K_4}e^{\xi_4}. \end{aligned} \tag{29}$$

If $\Phi_k(z)$ $(k = 1, 2)$ contain the variable z_1, then by comparing the degrees on both sides of (29), we get that $\deg(\Phi_k(z)) \le 1$ for $k = 1, 2$. For simplicity, we still denote $P(z) = \sum_{j=1}^{n} b_j z_j + \mu$ and $Q(z) = \sum_{j=1}^{n} d_j z_j + \nu$, where $b_j, d_j \in \mathbb{C}$ $(1 \le j \le n+1)$. This implies that $\Phi_k(z) \equiv 0$ for $k = 1, 2$. Since $P(z) - Q(z+c)$ is a constant, so we must have $b_j = d_j$ for $1 \le j \le n$. Therefore $P(z) = \sum_{j=1}^{n} b_j z_j + \mu$ and $Q(z) = \sum_{j=1}^{n} b_j z_j + \nu$, where $b_j, \mu, \nu \in \mathbb{C}$ for $1 \le j \le n$. From (27) and (28), we obtain

$$\begin{cases} \tfrac{a_1K_2}{a_3K_4}\left(-\tfrac{A_1}{A_2}b_1 - \tfrac{a_4}{a_1}b_1^2 - \tfrac{a_2}{a_1}\right)e^{\sum\limits_{j=1}^{n} b_jc_j - \mu + \nu} = 1;\\ \tfrac{a_1K_4}{a_3K_2}\left(-\tfrac{A_1}{A_2}b_1 - \tfrac{a_4}{a_1}b_1^2 - \tfrac{a_2}{a_1}\right)e^{\sum\limits_{j=1}^{n} b_jc_j + \mu - \nu} = 1;\\ \tfrac{a_1K_1}{a_3K_3}\left(\tfrac{A_2}{A_1}b_1 - \tfrac{a_4}{a_1}b_1^2 - \tfrac{a_2}{a_1}\right)e^{-\sum\limits_{j=1}^{n} b_jc_j + \mu - \nu} = 1;\\ \tfrac{a_1K_3}{a_3K_1}\left(\tfrac{A_2}{A_1}b_1 - \tfrac{a_4}{a_1}b_1^2 - \tfrac{a_2}{a_1}\right)e^{-\sum\limits_{j=1}^{n} b_jc_j - \mu + \nu} = 1. \end{cases} \tag{30}$$

From (30), we have

$$\left(-\frac{A_1}{A_2}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right)\left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) = \left(\frac{a_3}{a_1}\right)^2. \tag{31}$$

By similar arguments as in the proof of Theorem 2, we obtain from (11) and (12) that

$$\begin{cases} f_1(z) = \tfrac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n} b_jz_j + \mu} - \tfrac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n} b_jz_j - \mu} + g_7(y_1);\\ f_2(z) = \tfrac{A_1K_3}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^{n} b_jz_j + \nu} - \tfrac{A_2K_4}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^{n} b_jz_j - \nu} + g_8(y_1), \end{cases}$$

where $g_j(y_1)(j=7,8)$ are finite order entire functions satisfying $a_2g_7(y_1)+a_3g_8(y_1+s_1)\equiv 0$ and $a_2g_8(y_1)+a_3g_7(y_1+s_1)\equiv 0$.

If $\Phi_k(z)$ $(k=1,2)$ is independent of z_1, then, we have $P(z)=\sum_{j=1}^n b_jz_j+\Phi_1(z)+\mu$ and $Q(z)=\sum_{j=1}^n d_jz_j+\Phi_2(z)+\nu$, where $b_j,d_j,\mu,\nu\in\mathbb{C}$ $(1\leq j\leq n)$ and $\Phi_k(z)$ $(k=1,2)$ is a polynomial defined in (6). Since $P(z)-Q(z+c)$ is a constant, so we must have $b_j=d_j$ for $1\leq j\leq n$ and $\Phi_1(z)\equiv\Phi_2(z)$. Therefore $P(z)=\sum_{j=1}^n b_jz_j+\Phi_1(z)+\mu$ and $Q(z)=\sum_{j=1}^n b_jz_j+\Phi_1(z)+\nu$, where $b_j,\mu,\nu\in\mathbb{C}$ for $1\leq j\leq n$. By similar arguments as above in the case $\Phi_k(z)$ $(k=1,2)$ contain the variable z_1, we again obtain (30), (31) and

$$\begin{cases} f_1(z)=\frac{A_1K_1}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)+\mu}-\frac{A_2K_2}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)-\mu}+h_5(y_1);\\ f_2(z)=\frac{A_1K_3}{\sqrt{2}a_1b_1}e^{\sum\limits_{j=1}^n b_jz_j+\Phi_1(z)+\nu}-\frac{A_2K_4}{\sqrt{2}a_1b_1}e^{-\sum\limits_{j=1}^n b_jz_j-\Phi_1(z)-\nu}+h_6(y_1), \end{cases}$$

where $h_j(y_1)(j=5,6)$ are finite order entire functions satisfying $a_2h_5(y_1)+a_3h_6(y_1+s_1)\equiv 0$ and $a_2h_6(y_1)+a_3h_5(y_1+s_1)\equiv 0$. □

Proof of Theorem 5. Let (f_1,f_2) be a pair of finite order transcendental entire functions satisfies the system (8). Using arguments similar to those presented in Theorem 1, we get

$$\begin{cases} \sum\limits_{j=1}^n a_j\frac{\partial f_1(z)}{\partial z_j}=\frac{1}{\sqrt{2}}\left(A_1K_1e^{P(z)}+A_2K_2e^{-P(z)}\right);\\ a_{n+1}f_1(z)+a_{n+2}f_2(z+c)=\frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)}+A_1K_2e^{-P(z)}\right);\\ \sum\limits_{j=1}^n a_j\frac{\partial f_2(z)}{\partial z_j}=\frac{1}{\sqrt{2}}\left(A_1K_3e^{Q(z)}+A_2K_4e^{-Q(z)}\right);\\ a_{n+1}f_2(z)+a_{n+2}f_1(z+c)=\frac{1}{\sqrt{2}}\left(A_2K_3e^{Q(z)}+A_1K_4e^{-Q(z)}\right), \end{cases} \tag{32}$$

where $K_1,K_2,K_3,K_4\in\mathbb{C}\setminus\{0\}$ such that $K_1K_2=1=K_3K_4$, $P(z),Q(z)$ are polynomials on $\mathbb{C}^n$ and A_1,A_2 are given in (9). The following cases arise.

Let $P(z),Q(z)$ be simultaneously constants. Then from (32), we have

$$\sum_{j=1}^n a_j\frac{\partial f_1(z)}{\partial z_j}=\gamma_1,\quad a_{n+1}f_1(z)+a_{n+2}f_2(z+c)=\gamma_2, \tag{33}$$

$$\sum_{j=1}^n a_j\frac{\partial f_2(z)}{\partial z_j}=\gamma_3\quad a_{n+1}f_2(z)+a_{n+2}f_1(z+c)=\gamma_4, \tag{34}$$

where $\gamma_j\in\mathbb{C}$ for $1\leq j\leq 4$ such that $\gamma_1^2+2\alpha\gamma_1\gamma_2+\gamma_2^2=1$ and $\gamma_3^2+2\alpha\gamma_3\gamma_4+\gamma_4^2=1$. The Lagrange's auxiliary equations of the first equation of (33) are

$$\frac{dz_1}{a_1}=\frac{dz_2}{a_2}=\frac{dz_3}{a_3}=\cdots=\frac{dz_n}{a_n}=\frac{df_1(z)}{\gamma_1}.$$

Note that $z_j=(\alpha_j+a_jz_1)/a_1$ for $2\leq j\leq n$ and $df_1(z)=(\gamma_1/a_1)dz_1$ implies that $f_1(z)=(\gamma_1/a_1)z_1+\alpha_1$, where $\alpha_j\in\mathbb{C}$ for $1\leq j\leq n$. Hence the solution is $\chi(\alpha_1,\alpha_2,\ldots,\alpha_n)=0$. For simplicity, we suppose $f_1(z)=(\gamma_1/a_1)z_1+h_1(y)$, where $h_1(y)$ is a finite order transcendental entire function of $a_1z_2-a_2z_1,\ldots,a_1z_n-a_nz_1$. In view of this, we deduce from the first equation of (33) that $\sum_{j=1}^n a_j\frac{\partial h_1(y)}{\partial z_j}\equiv 0$.

Using arguments similar as above, we deduce from the first equation of (34) that

$$f_2(z)=(\gamma_3/a_1)z_1+h_2(y), \tag{35}$$

where $h_2(y)$ is a finite order transcendental entire function satisfying $\sum_{j=1}^{n} a_j \frac{\partial h_2(y)}{\partial z_j} \equiv 0$. Using (35) and the representation $f_1(z) = (\gamma_1/a_1)z_1 + h_1(y)$ we get from the second equation of (33) that

$$(a_{n+1}\gamma_1 + a_{n+2}\gamma_3)z_1/a_1 + a_{n+1}h_1(y) + a_{n+2}h_2(y+s) + (a_{n+2}c_1\gamma_3)/a_1 \equiv \gamma_2. \tag{36}$$

Comparing both sides of (36), we get $a_{n+1}\gamma_1 + a_{n+2}\gamma_3 = 0$, $a_{n+1}h_1(y) + a_{n+2}h_2(y+s) \equiv 0$ and $a_{n+2}c_1\gamma_3/a_1 = \gamma_2$. Similarly, by using (35) and the representation $f_1(z) = (\gamma_1/a_1)z_1 + h_1(y)$, we get from the second equation of (34) that $a_{n+1}\gamma_3 + a_{n+2}\gamma_1 = 0$, $a_{n+1}h_2(y) + a_{n+2}h_1(y+s) \equiv 0$ and $a_{n+2}c_1\gamma_1/a_1 = \gamma_4$. Similarly, as in proof of Theorem 1, we deduce that

$$a_{n+1} = \pm a_{n+2},\ f_1(z) = \frac{z_1}{\sqrt{a_1^2 - 2\alpha a_1 a_{n+1} c_1 + a_{n+1}^2 c_1^2}} + h_1(y)$$

and $f_2(z) = \frac{z_1}{\sqrt{a_1^2 - 2\alpha a_1 a_{n+1} c_1 + a_{n+1}^2 c_1^2}} + h_2(y)$, where $h_j(y)$ $(j = 1, 2)$ are finite order transcendental entire functions with periods $2s$ satisfying $\sum_{k=1}^{n} a_k \frac{\partial h_j(y)}{\partial z_k} \equiv 0$. □

Proof of Theorem 6. Let either $P(z)$ or $Q(z)$ be a constant. Repeating arguments from proof of Theorem 2 in the same case, we get a contradiction.

Let $P(z), Q(z)$ be both non-constant polynomials. Now differentiating partially with respect to $z_j (1 \leq j \leq n)$ on both sides of the second equation in (32) and summarizing them in j we get

$$a_{n+1} \sum_{j=1}^{n} a_j \frac{\partial f_1(z)}{\partial z_j} + a_{n+2} \sum_{j=1}^{n} a_j \frac{\partial f_2(z+c)}{\partial z_j} = \frac{A_2 K_1 e^{P(z)} - A_1 K_2 e^{-P(z)}}{\sqrt{2}} \sum_{j=1}^{n} a_j \frac{\partial P(z)}{\partial z_j}.$$

Applying (32) to the last equation we deduce that

$$\Psi_3(z) e^{P(z)+Q(z+c)} + \Omega_3(z) e^{-P(z)+Q(z+c)} - \frac{A_1 K_3}{A_2 K_4} e^{2Q(z+c)} \equiv 1, \tag{37}$$

where $\Psi_3(z) = \frac{K_1}{a_{n+2}K_4}\Big(\sum\limits_{j=1}^{n} a_j \frac{\partial P(z)}{\partial z_j} - \frac{A_1}{A_2} a_{n+1}\Big)$ and $\Omega_3(z) = -\frac{A_1 K_2}{a_{n+2} A_2 K_4}\Big(\sum\limits_{j=1}^{n} a_j \frac{\partial P(z)}{\partial z_j} + \frac{A_2}{A_1} a_{n+1}\Big)$. By using arguments similar as above, we deduce from (32) that

$$\Psi_4(z) e^{Q(z)+P(z+c)} + \Omega_4(z) e^{-Q(z)+P(z+c)} - \frac{A_1 K_1}{A_2 K_2} e^{2P(z+c)} \equiv 1, \tag{38}$$

where $\Psi_4(z) = \frac{K_3}{a_{n+2}K_2}\big(\sum_{j=1}^{n} a_j \frac{\partial Q(z)}{\partial z_j} - \frac{A_1}{A_2} a_{n+1}\big)$, $\Omega_4(z) = -\frac{A_1 K_4}{a_{n+2} K_2 A_2}\big(\sum_{j=1}^{n} a_j \frac{\partial Q(z)}{\partial z_j} + \frac{A_2}{A_1} a_{n+1}\big)$. Using arguments similar to those presented above in proof of Theorem 2 and in view of Lemma 1, we obtain from (37) and (38) respectively

$$\Psi_3(z) e^{P(z)+Q(z+c)} \equiv 1 \quad \text{or} \quad \Omega_3(z) e^{-P(z)+Q(z+c)} \equiv 1$$

and either

$$\Psi_4(z) e^{Q(z)+P(z+c)} \equiv 1 \quad \text{or} \quad \Omega_4(z) e^{-Q(z)+P(z+c)} \equiv 1.$$

Let

$$\Psi_3(z) e^{P(z)+Q(z+c)} \equiv 1, \quad \Psi_4(z) e^{Q(z)+P(z+c)} \equiv 1. \tag{39}$$

Using (39), we get from (37) and (38) respectively that

$$\frac{A_2 K_4}{A_1 K_3} \Omega_1(z) e^{-P(z)-Q(z+c)} \equiv 1, \quad \frac{A_2 K_2}{A_1 K_1} \Omega_2(z) e^{-Q(z)-P(z+c)} \equiv 1. \tag{40}$$

From (39), it is clear that $P(z) + Q(z+c)$ and $Q(z) + P(z+c)$ are both constants, say ξ_1 and ξ_2 respectively, where $\xi_1, \xi_2 \in \mathbb{C}$. By using arguments similar to those presented in the proof of Theorem 2, we have $P(z)=\sum_{j=1}^{n} b_j z_j+\Phi_1(z)+\mu$ and $Q(z)=\sum_{j=1}^{n} d_j z_j+\Phi_2(z)+\nu$, where $b_i, d_i, \mu, \nu \in \mathbb{C}$ $(1 \leq i \leq n)$ and $\Phi_k(z)$ $(k=1,2)$ is a polynomial defined in (6). From (39), we have

$$\sum_{j=1}^{n} a_j\Big(b_j+\frac{\partial \Phi_1(z)}{\partial z_j}\Big)-\frac{A_1}{A_2}a_{n+1} \equiv \frac{a_{n+2}K_4}{K_1}e^{-\xi_1},\ \sum_{j=1}^{n} a_j\Big(d_j+\frac{\partial \Phi_2(z)}{\partial z_j}\Big)-\frac{A_1}{A_2}a_{n+1} \equiv \frac{a_{n+2}K_2}{K_3}e^{-\xi_2}.$$

Since $a_j \neq 0$ for all $j=1,2,\ldots,n$, by comparing the degrees on both sides of the last equations, we get that $\deg(\Phi_k(z)) \leq 1$ for $k=1,2$. For simplicity, we still denote $P(z)=\sum_{j=1}^{n} b_j z_j+\mu$ and $Q(z)=\sum_{j=1}^{n} d_j z_j+\nu$, where $b_j, d_j, \mu, \nu \in \mathbb{C}$ $(1 \leq j \leq n)$. This implies that $\Phi_k(z) \equiv 0$ for $k=1,2$. Since $P(z)+Q(z+c)$ is a constant, so we must have $b_j+d_j=0$ for $1 \leq j \leq n$. Therefore $P(z)=\sum_{j=1}^{n} b_j z_j+\mu$ and $Q(z)=-\sum_{j=1}^{n} b_j z_j+\nu$, where $b_j, \mu, \nu \in \mathbb{C}$ for $1 \leq j \leq n$. From (39) and (40), we have

$$\begin{cases} \frac{K_1}{a_{n+2}K_4}\left(\sum\limits_{j=1}^{n} a_j b_j-\frac{A_1}{A_2}a_{n+1}\right)e^{-\sum\limits_{j=1}^{n} b_j c_j+\mu+\nu}=1; \\ \frac{K_3}{a_{n+2}K_2}\left(-\sum\limits_{j=1}^{n} a_j b_j-\frac{A_1}{A_2}a_{n+1}\right)e^{\sum\limits_{j=1}^{n} b_j c_j+\mu+\nu}=1; \\ -\frac{K_2}{a_{n+2}K_3}\left(\sum\limits_{j=1}^{n} a_j b_j+\frac{A_2}{A_1}a_{n+1}\right)e^{\sum\limits_{j=1}^{n} b_j c_j-\mu-\nu}=1; \\ -\frac{K_4}{a_{n+2}K_1}\left(-\sum\limits_{j=1}^{n} a_j b_j+\frac{A_2}{A_1}a_{n+1}\right)e^{-\sum\limits_{j=1}^{n} b_j c_j-\mu-\nu}=1. \end{cases}$$

From the last system we deduce that

$$\left(\sum_{j=1}^{n} a_j b_j-\frac{A_1}{A_2}a_{n+1}\right)\left(\sum_{j=1}^{n} a_j b_j+\frac{A_2}{A_1}a_{n+1}\right)=\left(\sum_{j=1}^{n} a_j b_j+\frac{A_1}{A_2}a_{n+1}\right)\left(\sum_{j=1}^{n} a_j b_j-\frac{A_2}{A_1}a_{n+1}\right),$$

i.e., $\frac{a_{n+1}(A_1^2-A_2^2)}{A_1A_2}(a_1b_1+a_2b_2+\cdots+a_nb_n)=0$. Hence, $a_1b_1+a_2b_2+\cdots+a_nb_n=0$. Also, that system implies

$$\begin{cases} e^{2\sum\limits_{j=1}^{n} b_j c_j}=\left(\frac{a_{n+2}}{a_{n+1}}\right)^2=e^{-2\sum\limits_{j=1}^{n} b_j c_j}; \\ e^{-\sum\limits_{j=1}^{n} b_j c_j+\mu+\nu}=-\frac{a_{n+2}K_4A_2}{a_{n+1}K_1A_1}, e^{\sum\limits_{j=1}^{n} b_j c_j+\mu+\nu}=-\frac{a_{n+2}K_2A_2}{a_{n+1}K_3A_1}; \\ e^{\sum\limits_{j=1}^{n} b_j c_j-\mu-\nu}=-\frac{a_{n+2}K_3A_1}{a_{n+1}K_2A_2}, e^{-\sum\limits_{j=1}^{n} b_j c_j-\mu-\nu}=-\frac{a_{n+2}K_1A_1}{a_{n+1}K_4A_2}. \end{cases}$$

Hence, it is clear that $a_{n+1}=\pm a_{n+2}$. The Lagrange's auxiliary equations [28, Chapter 2] of the first equation of (32) are

$$\frac{dz_1}{a_1}=\frac{dz_2}{a_2}=\frac{dz_3}{a_3}=\cdots=\frac{dz_n}{a_n}=\frac{\sqrt{2}df_1(z)}{A_1K_1e^{\sum_{j=1}^{n} b_j z_j+\mu}+A_2K_2e^{-\sum_{j=1}^{n} b_j z_j-\mu}}.$$

Note that $z_j=(\alpha_j+a_jz_1)/a_1$ for $2 \leq j \leq n$, $\sum_{k=1}^{n} b_k z_k=b_1z_1+\sum_{k=2}^{n} b_k\left(\frac{\alpha_k+a_kz_1}{a_1}\right)=$

$\left(b_1+\sum_{k=2}^{n} a_k b_k/a_1\right) z_1+\sum_{k=2}^{n} \alpha_k b_k/a_1=\sum_{k=2}^{n} \alpha_k b_k/a_1$ and

$$df_1=\frac{A_1K_1}{\sqrt{2}a_1}e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}dz_1+\frac{A_2K_2}{\sqrt{2}a_1}e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu}dz_1=$$
$$=\frac{A_1K_1}{\sqrt{2}a_1}e^{\sum\limits_{j=2}^{n} \alpha_jb_j/a_1+\mu}dz_1+\frac{A_2K_2}{\sqrt{2}a_1}e^{-\sum\limits_{j=2}^{n} \alpha_jb_j/a_1-\mu}dz_1,$$

that is,

$$f_1(z)=\frac{A_1K_1z_1}{\sqrt{2}a_1}e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}+\frac{A_2K_2z_1}{\sqrt{2}a_1}e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu}+\alpha_1,$$

where $\alpha_j\in\mathbb{C}$ for $1\le j\le n$. Hence, the solution is $\chi(\alpha_1,\alpha_2,\ldots,\alpha_n)=0$. For simplicity, we suppose

$$f_1(z)=\frac{z_1}{\sqrt{2}a_1}\left(A_1K_1e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}+A_2K_2e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu}\right)+h_3\left(y\right), \tag{41}$$

where $h_3\left(y\right)$ is a finite order entire function satisfying $\sum_{j=1}^{n} a_j\frac{\partial h_3(y)}{\partial z_j}\equiv 0$. Similarly, from the third equation of (32), we obtain

$$f_2(z)=\frac{1}{\sqrt{2}a_1}\left(A_1K_3e^{-\sum\limits_{j=1}^{n} b_jz_j+\nu}+A_2K_4e^{\sum\limits_{j=1}^{n} b_jz_j-\nu}\right)z_1+h_4(y), \tag{42}$$

where $h_4(y)$ is a finite order entire function satisfying $\sum_{j=1}^{n} a_j\frac{\partial h_4(y)}{\partial z_j}\equiv 0$. Using (41) and (42), we deduce from the second and fourth equations of (32) that

$$a_{n+1}h_3(y)+a_{n+2}h_4(y+s)\equiv\Gamma_1(n)K_1e^{\sum\limits_{j=1}^{n} b_jz_j+\mu}+\Gamma_2(n)K_2e^{-\sum\limits_{j=1}^{n} b_jz_j-\mu},$$
$$a_{n+1}h_4(y)+a_{n+2}h_3(y+s)\equiv\Gamma_1(n)K_3e^{-\sum\limits_{j=1}^{n} b_jz_j+\nu}+\Gamma_2(n)K_4e^{\sum\limits_{j=1}^{n} b_jz_j-\nu},$$

where $\Gamma_1(n)$ and $\Gamma_2(n)$ are given in (9). □

Proof of Theorem 7. Let $\Psi_1(z)e^{P(z)+Q(z+c)}\equiv 1$ and $\Omega_2(z)e^{-Q(z)+P(z+c)}\equiv 1$. Repeating arguments similar to the proof of Theorem 2, we get a contradiction.

Suppose that $\Omega_1(z)e^{-P(z)+Q(z+c)}\equiv 1$ and $\Psi_2(z)e^{Q(z)+P(z+c)}\equiv 1$. Using arguments similar to those presented in the proof of Theorem 3, we get a contradiction.

Let

$$\Omega_1(z)e^{-P(z)+Q(z+c)}\equiv 1,\quad \Omega_2(z)e^{-Q(z)+P(z+c)}\equiv 1. \tag{43}$$

Using (43), we get from (37) and (38) respectively that

$$\frac{A_2K_4}{A_1K_3}M_1(z)e^{P(z)-Q(z+c)}\equiv 1,\quad \frac{A_2K_2}{A_1K_1}M_2(z)e^{Q(z)-P(z+c)}\equiv 1. \tag{44}$$

Using arguments similar to those presented in Theorem 6, we deduce that $P(z)=\sum\limits_{j=1}^{n} b_jz_j+\mu$ and $Q(z)=\sum\limits_{j=1}^{n} b_jz_j+\nu$, where $b_j,\mu,\nu\in\mathbb{C}$ for $1\le j\le n$. From (43) and

(44), we have

$$\begin{cases} -\frac{A_1K_2}{a_{n+2}A_2K_4}\left(\sum_{j=1}^n a_jb_j+\frac{A_2}{A_1}a_{n+1}\right)e^{\sum_{j=1}^n b_jc_j-\mu+\nu}=1;\\ -\frac{A_1K_4}{a_{n+2}A_2K_2}\left(\sum_{j=1}^n a_jb_j+\frac{A_2}{A_1}a_{n+1}\right)e^{\sum_{j=1}^n b_jc_j+\mu-\nu}=1;\\ \frac{A_2K_1}{a_{n+2}A_1K_3}\left(\sum_{j=1}^n a_jb_j-\frac{A_1}{A_2}a_{n+1}\right)e^{-\sum_{j=1}^n b_jc_j+\mu-\nu}=1;\\ \frac{A_2K_3}{a_{n+2}A_1K_1}\left(\sum_{j=1}^n a_jb_j-\frac{A_1}{A_2}a_{n+1}\right)e^{-\sum_{j=1}^n b_jc_j-\mu+\nu}=1. \end{cases}$$

Hence, it is easy to see that

$$\left(\sum_{j=1}^n a_jb_j+\frac{A_2}{A_1}a_{n+1}\right)\left(\sum_{j=1}^n a_jb_j-\frac{A_1}{A_2}a_{n+1}\right)=-a_{n+2}^2. \tag{45}$$

If $a_{n+1}=\pm a_{n+2}$, then from (45), we get $\sum_{j=1}^n a_jb_j=0$ and hence we obtain the same conclusions as in the proof of Theorem 6. Therefore, we consider that $a_{n+1}\neq\pm a_{n+2}$. By using similar arguments as in the proof of Theorem 6, from the first and third equations of (32), we get

$$\begin{cases} f_1(z)=\frac{1}{\sqrt{2}\sum_{j=1}^n a_jb_j}\left(A_1K_1e^{\sum_{j=1}^n b_jz_j+\mu}-A_2K_2e^{-\sum_{j=1}^n b_jz_j-\mu}\right)+h_5(y);\\ f_2(z)=\frac{1}{\sqrt{2}\sum_{j=1}^n a_jb_j}\left(A_1K_3e^{\sum_{j=1}^n b_jz_j+\nu}-A_2K_4e^{-\sum_{j=1}^n b_jz_j-\nu}\right)+h_6(y), \end{cases} \tag{46}$$

where $h_j(y)$ $(j=5,6)$ are finite order entire functions satisfying $\sum_{k=1}^n a_k\frac{\partial h_j(y)}{\partial z_k}\equiv 0$. Using (45) and (46), we deduce from the second and fourth equations of (32) that $a_{n+1}h_5(y)+a_{n+2}h_6(y+s)\equiv 0$ and $a_{n+1}h_6(y)+a_{n+2}h_5(y+s)\equiv 0$. □

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