УДК 517.547.2

M. M. Sheremeta

PROPERTIES OF LAPLACE-STIELTJES-TYPE INTEGRALS

M. M. Sheremeta. *Properties of Laplace-Stieltjes-type integrals*, Mat. Stud. **60** (2023), 115–131.

The properties of Laplace-Stieltjes-type integrals $I(r) = \int_0^\infty a(x)f(xr)dF(x)$ are studied, where F is a non-negative non-decreasing unbounded continuous on the right function on $[0, +\infty)$, $f(z) = \sum_{k=0}^\infty f_k z^k$ is an entire transcendental function with $f_k \geq 0$ for all $k \geq 0$, and a function $a(x) \geq 0$ on $[0, +\infty)$ is such that the Lebesgue-Stieltjes integral $\int_0^K a(x)f(xr)dF(x)$ exists for every $r \geq 0$ and $K \in [0, +\infty)$.

For the maximum of the integrand $\mu(r) = \sup\{a(x)f(xr)\colon x \geq 0\}$ it is proved that if

$$\varliminf_{x\to+\infty} \frac{f^{-1}(1/a(x))}{x} = R_\mu$$

then $\mu(r) < +\infty$ for $r < R_\mu$ and $\mu(r) = +\infty$ for $r > R_\mu$. The relationship between R_μ and the radius R_c of convergence of the integral $I(r)$ was found. The concept of the central point $\nu(r)$ of the maximum of the integrand is introduced and the formula for finding $\ln \mu(r)$ over $\nu(r)$ is proved. Under certain conditions on the function F, estimates of $I(r)$ in terms of $\mu(r)$ are obtained, and in the case when $R_\mu = +\infty$, in terms of generalized orders, a relation is established between the growth $\mu(r)$ and $I(r)$ and the decrease of the function $a(x)$.

I dedicate to the memory of my first graduate student B. V. Vynnyts'kyi (1953–2020).

Introduction. Let

$$f(z) = \sum_{k=0}^\infty f_k z^k \tag{1}$$

be an entire transcendental function, $M_f(r) = \max\{|f(z)|\colon |z| = r\}$ and (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z) = \sum_{n=1}^\infty a_n f(\lambda_n z) \tag{2}$$

in the system $\{f(\lambda_n z)\}$ regularly convergent in $\mathbb{C}$, i.e.

$$\sum_{n=1}^\infty |a_n| M_f(r\lambda_n) < +\infty$$

for all $r \in [0, +\infty)$. It is clear that many functional series arising in various sections of the analysis can be written as series by a system of functions of the form $f(\lambda_n z)$ (see for example

2020 *Mathematics Subject Classification:* 26A42, 30B50.
Keywords: Laplace-Stieltjes integral; maximum of the integrand; central point; generalized order.
doi:10.30970/ms.60.2.115-131

[1–14]). We will specify also on the monography of B .V. Vynnyts'kyi [15], where references are to many other works. In particular, for example, in articles [4–8] B. V. Vynnyts'kyi investigated under the most general conditions on a function f itself and on the sequence (λ_n), both the basicity of this system of functions and the properties of series on this system. Property of systems of Bessel and Mittag-Leffler type functions are investigated, for example, in [1–3, 9–11]. At the end, modern e-search systems will allow the reader to easily find both other articles about the series on this general system of functions, and on the specific systems of functions, such as Mittag-Leffler functions, Bessel functions, and many others.

If $f_k \geq 0$ and $a_n \geq 0$ for all k and n then $M_f(r) = f(r)$ and series (2) regularly converges in $\mathbb{C}$ if and only if $\sum_{n=1}^{\infty} a_n f(\lambda_n r) < +\infty$, i.e. $\int_0^{\infty} a(x)f(xr)dn(x) < +\infty$ for all $r \in [0, +\infty)$, where $n(x) = \sum_{\lambda_n \leq x} 1$ and $a(\lambda_n) = a_n$.

Let V be the class of non-negative non-decreasing unbounded continuous from the right functions F on $[0, +\infty)$ and $f_k \geq 0$ for all $k \geq 0$. Assume that a function $a(x) \geq 0$ on $[0, +\infty)$ is such that the Lebesgue-Stieltjes integral $\int_0^K a(x)f(xr)dF(x)$ exists for every $r \geq 0$ and $K \in [0, +\infty)$. The integral

$$I(r) = \int_0^{\infty} a(x)f(xr)dF(x), \quad r \geq 0, \tag{3}$$

is called Laplace-Stieltjes-type integral and is a direct generalization of the Laplace-Stieltjes integral $\int_0^{\infty} a(x)e^{xr}dF(x)$, the study of which many works are devoted to (the bibliography can be found in [16]).

As in [17] (see also [16, p. 76]) by L we denote the class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$. If $\alpha \in L$, $\beta \in L$ and integral (3) exists for all $r \geq 0$ then the value

$$\varrho_{\alpha,\beta}[I] := \varlimsup_{r\to+\infty} \frac{\alpha(\ln I(r))}{\beta(r)}$$

is called the generalized (α, β)-order of I. If instead of $I(r)$ we put $M_f(r)$ then we get the definition of $\varrho_{\alpha,\beta}[f]$.

Suppose that $\alpha \in L_{si}$, $\beta \in L^0$, $\dfrac{\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ and

$$\ln x = o\Big(\beta^{-1}\Big(c\alpha\Big(\frac{1}{\ln x}\ln\frac{1}{a(x)}\Big)\Big)\Big) \quad (x \to +\infty)$$

for each $c \in (0, +\infty$. If $\ln F(x) = O(\Gamma_f(x))$ as $x \to +\infty$, where $\Gamma_f(x) = \dfrac{d\ln f(x)}{d\ln x}$, then [18] $\varrho_{\alpha,\beta}[I] = \varrho_{\alpha,\beta}[f]$.

In this paper, to find $\varrho_{\alpha,\beta}[I]$, another formula will be found that describes the relationship between the increase of I and the decrease of a. To do this, we study the convergence of the integral $I(r)$, the behavior of the maximum of the integrand $\mu(r)$ and its central point $\nu(r)$, and obtain estimates of $I(r)$ in terms of $\mu(r)$.

1. Convergence of Laplace-Stieltjes type integrals. It is clear that integral (3) either converges for all $r > 0$ or diverges for every $r > 0$ or there exists a number R_c such that integral (3) either converges for $r < R_c$ and diverges for $r > R_c$. In the latter case the number R_c is called radius of the convergence of integral (3). If integral (3) converges for all $r > 0$ then we put $R_c = +\infty$, and if it diverges for every $r > 0$ then we put $R_c = 0$ (assume that $I(0)$ exists).

For $r \geq 0$ let $\mu(r) = \mu(r, I) = \sup\{a(x)f(xr)\colon x \geq 0\}$ be the maximum of the integrand. Then either $\mu(r) < +\infty$ for all $r > 0$ or $\mu(r) = +\infty$ for all $r > 0$ or there exists a number R_μ such that $\mu(r) < +\infty$ for all $r < R_\mu$ and or $\mu(r) = +\infty$ for all $r > R_\mu$. By analogy the number R_μ is called radius of the existence of maximum of the integrand.

Clearly, if $a(x) = 0$ for all $x \geq x_0$ then $R_\mu = +\infty$. Further we exclude this case from the consideration, i. e. we suppose that $a(x) \not\equiv 0$ on each interval $[x_0, +\infty)$.

Since $f_k \geq 0$ for all $k \geq 0$ and the function $\ln M_f(r)$ is logarithmically convex, we have $M_f(r) = f(r)$ and

$$\Gamma_f(r) := \frac{d\ln M_f(r)}{d\ln r} = \frac{d\ln f(r)}{d\ln r} \nearrow +\infty, \quad r \to +\infty.$$

Lemma 1. *If* $0 < r_1 \leq r_2 < +\infty$ *then*

$$\exp\Big\{-\Gamma_f(r_2)\ln\frac{r_2}{r_1}\Big\} \leq \frac{f(r_1)}{f(r_2)} \leq \exp\Big\{-\Gamma_f(r_1)\ln\frac{r_2}{r_1}\Big\}.$$

Indeed,

$$-\Gamma_f(r_2)\ln\frac{r_2}{r_1} \leq \ln f(r_1) - \ln f(r_2) = -\int_{r_1}^{r_2}\Gamma_f(x)d\ln x \leq -\Gamma_f(r_1)\ln\frac{r_2}{r_1}.$$

We remark also that $\frac{d\ln f^{-1}(x)}{d\ln x} \searrow 0$ as $x \to +\infty$ and, thus, the function f^{-1} is slowly increasing, i. e. $f^{-1}(cx) = (1 + o(1))f^{-1}(x)$ as $x \to +\infty$ for every $c \in (0, +\infty)$.

Let us start finding formulas for calculating R_μ and R_c.

Proposition 1. *The following equality is true*

$$R_\mu = \theta := \varliminf_{x\to+\infty}\frac{1}{x}f^{-1}\left(\frac{1}{a(x)}\right). \tag{4}$$

Proof. If $\theta > 0$ and $0 < r < \theta$ then $\frac{1}{x}f^{-1}\left(\frac{1}{a(x)}\right) > \theta - \frac{\theta-r}{2} = \frac{\theta+r}{2}$, that is $a(x) \leq \frac{1}{f((\theta+r)x/2)}$ for all $x \geq x_0 = x_0(r)$ and, therefore,

$$\begin{aligned}\ln\mu(r) = \max\{\sup\{\ln a(x) + \ln f(rx)\colon x \leq x_0\},\ \sup\{\ln a(x) + \ln f(xr)\colon x \geq x_0\}\} \leq \\ \leq \max\{\ln f(rx_0) + \sup\{\ln a(x)\colon x \leq x_0\},\ \sup\{-\ln f((\theta+r)x/2) + \ln f(xr)\colon x \geq x_0\}\} \leq \\ \leq \max\{\ln f(rx_0) + \sup\{\ln a(x)\colon x \leq x_0\},\ 0\} < +\infty,\end{aligned}$$

i.e. $R_\mu \geq \theta$. For $\theta = 0$ this inequality is obvious.

On the other hand, if $\theta < +\infty$ and $r > \theta$ then $\frac{1}{x_k}f^{-1}(\frac{1}{a(x_k)}) < \theta + \frac{r-\theta}{2} = \frac{\theta+r}{2}$, that is $a(x_k) \geq \frac{1}{f((\theta+r)x_k/2)}$ for some sequence $(x_k) \uparrow +\infty$. Therefore, by Lemma 1

$$\begin{aligned}\ln\mu(r) = \sup\{\ln a(x) + \ln f(rx)\colon x \geq 0\} \geq \sup\{\ln a(x_k) + \ln f(rx_k)\colon k \geq 1\} \geq \\ \geq \sup\{\ln f(rx_k) - \ln f((r+\theta)x_k/2)\colon k \geq 1\} \geq \sup\left\{\Gamma\left(\frac{(r+\theta)x_k}{2}\right)\ln\frac{2r}{r+\theta}\colon k \geq 1\right\} = +\infty,\end{aligned}$$

i.e. $R_\mu \leq \theta$. For $\theta = +\infty$ this inequality is obvious. □

Let at first $R_\mu = +\infty$. Then $\frac{1}{x}f^{-1}(\frac{1}{a(x)}) \ge eK$ for every $K > 1$ and all $x \ge x_0$, whence $a(x) \le \frac{1}{f(eKx)}$. Therefore, for $1 \le r < K$ and $h > 0$ by Lemma 1

$$\int_{x_0}^{\infty} a(x)f(xr)dF(x) \le \int_{x_0}^{\infty} \frac{f(xr)}{f(eKx)}dF(x) \le \int_{x_0}^{\infty} \exp\left\{-\Gamma_f(rx)\ln\frac{eK}{r}\right\}dF(x) \le$$
$$\le \int_{x_0}^{\infty} \exp\{-\Gamma_f(x) + (1+h)\ln F(x)\}\frac{dF(x)}{F(x)^{1+h}}.$$

Suppose that $\overline{\lim\limits_{x\to+\infty}}\big(\ln F(x)/\Gamma_f(x)\big) < 1$. Then there exists $h > 0$ such that $\ln F(x) \le \Gamma_f(x)/(1+h)$ for all $x \ge x_1 \ge x_0$, i.e. $-\Gamma_f(x) + (1+h)\ln F(x) \le 0$ and, thus,

$$\int_{x_1}^{\infty} a(x)f(xr)dF(x) \le \int_{x_1}^{\infty} \frac{dF(x)}{F(x)^{1+h}}. \tag{5}$$

It is known [16, p. 10] (see also [18]) that $\int_{x_1}^{\infty} \frac{dF(x)}{F(x)^{1+h}} < +\infty$. Therefore, $\int_{x_1}^{\infty} a(x)f(xr)dF(x) < +\infty$ for every $1 \le r < K$ and in view of the arbitrariness of K we get $R_c = +\infty$. Thus, the following statement is true.

Proposition 2. *If* $\overline{\lim\limits_{x\to+\infty}}\big(\ln F(x)/\Gamma_f(x)\big) < 1$ *and* $R_\mu = +\infty$ *then* $R_c = +\infty$.

Now, let $0 < R_\mu < +\infty$. Then $\frac{1}{x}f^{-1}(1/a(x)) \ge R_\mu - \delta$ for every $\delta \in (0, R_\mu)$ and all $x \ge x_0$, whence $a(x) \le 1/f((R_\mu - \delta)x)$. Therefore, for $r < R_\mu - \delta$ by Lemma 1

$$\int_{x_0}^{\infty} a(x)f(xr)dF(x) \le \int_{x_0}^{\infty} \frac{f(xr)}{f((R_\mu-\delta)x)}dF(x) \le \int_{x_0}^{\infty} \exp\left\{-\Gamma_f(rx)\ln\frac{R_\mu-\delta}{r}\right\}dF(x).$$

Using this inequality we prove the following statement.

Proposition 3. *If* $\ln F(x) = o(\Gamma_f(x))$ *as* $x \to +\infty$ *and either* $R_\mu > 1$ *or* $R_\mu \le 1$ *and* $\Gamma_f \in L^0$ *then* $R_c \ge R_\mu$.

Proof. If $R_\mu > 1$ then choosing $\delta \in (0, R_\mu)$ such that $R_\mu - \delta > 1$ for $r \in [1, R_\mu - \delta)$ we obtain

$$(1+h)\ln F(x) - \Gamma_f(rx)\ln\frac{R_\mu-\delta}{r} \le (1+h)\ln F(x) - \Gamma_f(x)\ln\frac{R_\mu-\delta}{r} =$$
$$= -(1+o(1))\Gamma_f(x)\ln\frac{R_\mu-\delta}{r} \le 0, \quad x \to +\infty,$$

and, as above, we obtain $\int_{x_1}^{\infty} a(x)f(xr)dF(x) < +\infty$ for each $r < R_\mu - \delta$ and, thus, $R_c \ge R_\mu - \delta$. In view of the arbitrariness of δ we get $R_c \ge R_\mu$.

Now let $R_\mu \le 1$ and $\Gamma_f \in L^0$. It is known [19] that if $\beta \in L^0$ and $B(\delta) = \overline{\lim\limits_{x\to+\infty}} \frac{\beta((1+\delta)x)}{\beta(x)}$, $\delta > 0$, then in order that $\beta \in L^0$, it is necessary and sufficient that $B(\delta) \searrow 1$ as $\delta \searrow 0$. Therefore, $\overline{\lim\limits_{x\to+\infty}} \Gamma_f(x)/\Gamma_f(rx) = B_1(r) \searrow 1$ as $r \nearrow 1$ and, thus, $\Gamma_f(rx) \ge \Gamma_f(x)/B_2(r)$, $B_2(r) \in (1, +\infty)$, i.e.

$$(1+h)\ln F(x) - \Gamma_f(rx)\ln\frac{R_\mu-\delta}{r} \le -(1+o(1))\Gamma_f(x)\frac{1}{B_2(r)}\ln\frac{R_\mu-\delta}{r} \le 0, \quad x \to +\infty,$$

whence as above it follows that $R_c \ge R_\mu$. □

In general, the reverse inequality $R_c \le R_\mu$ is not true. This is indicated, for example, by the following assertion proved in [16, p. 16].

Proposition 4. *For every $-\infty \le \gamma < \beta \le +\infty$ there exists a continuous function $a\colon [0,+\infty) \to (0,+\infty)$ such that for the integral $I(r) = \int_0^\infty a(x)e^{rx}dx$ the inequality $R_c = \beta > \gamma = R_\mu$ holds.*

Therefore, as in [16, p. 21] we say that a positive function a on $[0,+\infty)$ has regular variation with respect to $F \in V$ if there exist $b \ge 0$, $c \ge 0$ and $h > 0$ such that for all $x \ge b$

$$\int\limits_{x-b}^{x+c} a(t)dF(t) \ge ha(x). \tag{6}$$

Proposition 5. *If function a has regular variation with respect to $F \in V$ and $\Gamma_f(r) = O(r)$ as $r \to +\infty$ then the inequality $R_c \le R_\mu$ holds.*

Proof. For all $x \ge b$ and $r \ge 0$

$$I(r) = \int\limits_0^\infty a(t)f(tr)dF(t) \ge \int\limits_{x-b}^{x+c} a(t)f(tr)dF(t) = f(xr)\int\limits_{x-b}^{x+c} a(t)\frac{f(tr)}{f(xr)}dF(t) \ge$$

$$\ge f(xr)\min\left\{\frac{f(tr)}{f(xr)}\colon x-b \le t \le x+c\right\}\int\limits_{x-b}^{x+c} a(t)dF(t) \ge qha(x)f(xr),$$

where $q = \min\{f(tr)/f(xr)\colon x-b \le t \le x+c\} = f((x-b)r)/f(xr)$. Since $\Gamma_f(r) \le Kr$ for some $K = \text{const} > 0$ and all $r > 0$, by Lemma 1 we have

$$q = \exp\{-(\ln f(xr) - \ln f((x-b)r))\} \ge \exp\left\{-\Gamma_f(xr)\ln\frac{x}{x-b}\right\} \ge$$

$$\ge \exp\left\{-Krx\ln\left(1+\frac{b}{x-b}\right)\right\} \ge \exp\left\{-Kr\frac{bx}{x-b}\right\},$$

i.e. $q \ge \exp\{-2bKr\}$ for $x \ge 2b$ and, thus, $a(x)f(xr) \le I(r)\exp\{2bKr\}/h$ for all $x \ge 2b$, whence it follows that $\mu(r, I)$ exists if $r < R_c$ and, thus, $R_\mu \ge R_c$. □

From Proposition 2, 3 and 5 the following theorem follows.

Theorem 1. *Let $\Gamma_f \in L^0$ and $\Gamma_f(r) = O(r)$ as $r \to +\infty$. If $F \in V$, $\ln F(x) = o(\Gamma_f(x))$ as $x \to +\infty$ and the function a has regular variation with respect to $F \in V$ then $R_c = R_\mu$.*

Note that the conditions of Proposition 5 can be omitted if there exists the limit

$$R_\mu = \lim_{x\to+\infty}\frac{1}{x}f^{-1}\left(1/a(x)\right).$$

Indeed, if $R_\mu < +\infty$ and $r > R_\mu$ then $a(x) \ge 1/f(rx)$ for all $x \ge x_0 = x_0(r)$. Therefore, $\int_{x_0}^\infty a(x)f(xr)dF(x) \ge \int_{x_0}^\infty dF(x) = +\infty$, whence we obtain the inequality $R_c \le R_\mu$, which is obvious provided $R_\mu = +\infty$.

Now suppose that $\ln F(x) = o\left(\ln\left(1/a(x)\right)\right)$ as $x \to +\infty$ and $R_\mu > 0$.

Choose $0 < q < \delta < R_\mu$, and let $r < q$. Then for $\varepsilon \in (0,1)$ and all $x \ge x_0 = x_0(\varepsilon,\delta)$ in view of (4)

$$\int\limits_{x_0}^\infty a(x)f(xr)dF(x) = \int\limits_{x_0}^\infty \exp\left\{-\ln\frac{1}{a(x)} + (1+h)\ln F(x) + \ln f(xr)\right\}\frac{dF(x)}{F(x)^{1+h}} \le$$

$$\leq \int_{x_0}^{\infty} \exp\left\{-(1-\varepsilon)\ln\frac{1}{a(x)} + \ln f(xr)\right\} \frac{dF(x)}{F(x)^{1+h}} \leq$$
$$\leq \int_{x_0}^{\infty} \exp\left\{-(1-\varepsilon)\ln f(\delta x) + \ln f(qx)\right\} \frac{dF(x)}{F(x)^{1+h}}.$$

Also suppose that $\Gamma_f(x)/\ln f(x) = \frac{d}{d\ln x}\ln\ln f(x) > c_0 > 0$ for $x \geq x_0$. Then for such x we have

$$\ln\ln f(\delta x) - \ln\ln f(qx) = \int_{qx}^{\delta x} \frac{\ln\ln f(t)}{d\ln t} d\ln t \geq c_0 \ln\frac{\delta}{q}.$$

Therefore, if $\varepsilon \in (0,1)$ is such that $c_0 \ln\frac{\delta}{q} \geq \ln\frac{1}{1-\varepsilon}$ then $\ln\ln f(\delta x) - \ln\ln f(qx) \geq$ $\leq \ln(1/(1-\varepsilon))$ and, thus, $(1-\varepsilon)\ln f(\delta x) \geq \ln f(qx)$, i. e. $\int_{x_1}^{\infty} a(x) f(xr) dF(x) < +\infty$ and, thus, $R_c \geq R_\mu$. Therefore, the following statement is true.

Proposition 6. *If* $\ln F(x) = o(\ln(1/a(x))$ *and* $\ln f(x) = O(\Gamma_f(x))$ *as* $x \to +\infty$ *then* $R_c \geq R_\mu$.

The following statement complements Proposition 6.

Proposition 7. *If* $\overline{\lim\limits_{x\to+\infty}} \ln F(x)/\ln(1/a(x)) < 1$ *and* $\lim\limits_{x\to+\infty} \Gamma_f(x)/\ln f(x) = +\infty$ *then* $R_c \geq R_\mu$.

Proof. It is clear that $\ln F(x) \leq \tau \ln(1/a(x))$ for some $\tau \in (0, 1)$ and all $x \geq x_0 = x_0(\tau)$. We can choose $h > 0$ such that $(1+h)\tau \leq p$ for some $p \in (0, 1)$. Then $-\ln(1/a(x)) + (1+h)\ln F(x) \leq -(1-p)\ln(1/a(x))$ for $x \geq x_0$. On the other hand, since $\frac{d\ln\ln f(x)}{d\ln x} > K$ for every $K > 0$ and all $x \geq x_0^* = x_0^*(K) \geq x_0$, we get as above $\ln\ln f(\delta x) - \ln\ln f(qx) \geq K\ln\frac{\delta}{q}$. Therefore, if we choose $K \geq \frac{\ln(1/(1-p))}{\ln(\delta/q)}$ then as above we obtain

$$\int_{x_0^*}^{\infty} a(x) f(xr) dF(x) \leq \int_{x_0^*}^{\infty} \exp\{-(1-p)\ln f(\delta x) + \ln f(xr)\} \frac{dF(x)}{F(x)^{1+h}} \leq \int_{x_0^*}^{\infty} \frac{dF(x)}{F(x)^{1+h}} < +\infty,$$

i.e. $R_c \geq R_\mu$. □

From Proposition 6, 7 and 5 the following theorem follows.

Theorem 2. *Let* $F \in V$, *the function* a *has regular variation with respect to* $F \in V$ *and* $\Gamma_f(r) = O(r)$ *as* $r \to +\infty$. *If either* $\ln F(x) = o(\ln(1/a(x))$ *and* $\ln f(x) = O(\Gamma_f(x))$ *as* $x \to +\infty$ *or* $\overline{\lim\limits_{x\to+\infty}} \ln F(x)/\ln(1/a(x)) < 1$ *and* $\lim\limits_{x\to+\infty} \Gamma_f(x)/\ln f(x) = +\infty$ *then* $R_c = R_\mu$.

2. Maximum of the integrand and central point. As above, let R_μ be the radius of the existence of the maximum of the integrand $\mu(r) = \sup\{a(x)f(xr)\colon x \geq 0\}$ in integral (3).

Proposition 8. *The function* $\ln \mu(r)$ *is logarithmically convex (convex concerning of the logarithm) on* $(0, R_\mu)$.

Proof. By Hadamard's theorem on three circles the function $\ln M_f(r)$ is logarithmically convex on $(0, +\infty)$. Therefore, since $M_f(r) = f(r)$, we have

$$\ln f(r) \le \frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1} \ln f(r_1) + \frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1} \ln f(r_2)$$

for all $0 < r_1 \le r \le r_2 < +\infty$, i.e. $\ln f(r_1^{1-\lambda} r_2^{\lambda}) \le (1-\lambda)\ln f(r_1) + \lambda \ln f(r_2)$ for all $\lambda \in [0,1]$ and $0 < r_1 \le r_2 < +\infty$. Clearly, if $\ln f(r)$ is logarithmically convex then $\ln f(xr)$ is logarithmically convex for every $x \ge 0$. Therefore,

$$\ln f(xr_1^{1-\lambda} r_2^{\lambda}) = \ln f((xr_1)^{1-\lambda}(xr_2)^{\lambda}) \le (1-\lambda)\ln f(xr_1) + \lambda \ln f(xr_2)$$

for all $x > 0$, $\lambda \in [0,1]$ and $0 < r_1 \le r_2 < +\infty$ and, thus, for all $\lambda \in [0,1]$ and $0 < r_1 \le r_2 < R_\mu$ we get

$$\ln \mu(r_1^{1-\lambda} r_2^{\lambda}) = \sup\{\ln a(x) + \ln f(xr_1^{1-\lambda} r_2^{\lambda})\colon x \ge 0\} \le$$
$$\le \sup\{(1-\lambda)\ln a(x) + (1-\lambda)\ln f(xr_1) + \lambda \ln a(x) + \lambda \ln f(xr_2)\colon x \ge 0\} \le$$
$$\le (1-\lambda)\sup\{\ln a(x) + \ln f(xr_1)\colon x \ge 0\} + \lambda \sup\{\ln a(x) + \ln f(xr_2)\colon x \ge 0\} =$$
$$= (1-\lambda)\ln \mu(r_1) + \lambda \ln \mu(r_2),$$

i.e. the function $\ln \mu(r)$ is logarithmically convex on $(0,\, R_\mu)$. □

For $r \in [0,\, R_\mu)$ and $\varepsilon > 0$ we put

$$\nu(r,\varepsilon) = \sup\{x \ge 0\colon\ \ln a(x) + \ln f(xr) \ge \ln \mu(r) - \varepsilon\}.$$

Clearly, for fixed $r \in [0,\, R_\mu)$ the function $\nu(r,\cdot)$ is non-decreasing on $(0,\, +\infty)$ and, thus, there exists a quantity

$$\nu(r) = \nu(r,\, I) := \lim_{\varepsilon \downarrow 0} \nu(r,\varepsilon),$$

which we will call the *central point* of the maximum of the integrand.

Proposition 9. *The function $\nu(r)$ is non-decreasing on $(0,\, +\infty)$ and $\frac{d}{d\ln r}\ln \mu(r) = \Gamma_f(r\nu(r))$ for all $r \in [0,\, R_\mu)$ with the exception of an at most countable set.*

Proof. At first we prove that for arbitrary $r \in [0,\, R_\mu)$ and $\varepsilon > 0$ the set

$$E(r,\varepsilon) = \{x \ge 0\colon |x - \nu(r)| < \varepsilon,\, \ln a(x) + \ln f(xr) \ge \ln \mu(r) - \varepsilon\}$$

is non-empty. Indeed, we fix $r \in [0,\, R_\mu)$ and $\varepsilon > 0$. Then there exits $\delta \in (0,\varepsilon)$ such that

$$|\nu(r) - \sup\{x \ge 0\colon\ \ln a(x) + \ln f(xr) \ge \ln \mu(r) - \delta\}| < \varepsilon/2 \tag{7}$$

and there exists $x_0 \ge 0$ such that

$$\ln a(x_0) + \ln f(x_0 r) \ge \ln \mu(r) - \delta \tag{8}$$

and

$$|x_0 - \sup\{x \ge 0\colon\ \ln a(x) + \ln f(xr) \ge \ln \mu(r) - \delta\}| < \varepsilon/2. \tag{9}$$

From (7), (8) and (9) we get $|x_0 - \nu(r)| < \varepsilon$ and $\ln a(x_0) + \ln f(x_0 r) \ge \ln \mu(r) - \varepsilon$, i.e. $x_0 \in E(r,\varepsilon)$ and, thus, $E(r,\varepsilon)$ is a non-empty set.

Since $E(r,\varepsilon)$ is not empty for arbitrary $r \in [0,\, R_\mu)$ and $\varepsilon > 0$, by the axiom of choice for each $\varepsilon > 0$ there exists a function $\nu_\varepsilon(r) \ge 0$ such that for all $r \in [0,\, R_\mu)$

$$\ln \mu(r) \ge \ln a(\nu_\varepsilon(r)) + \ln f(r\nu_\varepsilon(r)) \ge \ln \mu(r) - \varepsilon \tag{10}$$

and

$$|\nu_\varepsilon(r) - \nu(r)| < \varepsilon. \tag{11}$$

Suppose that $r_1, r_2 \in [0,\ R_\mu)$. By definition

$$\ln \mu(r_1) \ge \ln a(\nu_\varepsilon(r_2)) + \ln f(r_1 \nu_\varepsilon(r_2)), \tag{12}$$

and from (10) we have

$$\ln \mu(r_2) \le \ln a(\nu_\varepsilon(r_2)) + \ln f(r_2 \nu_\varepsilon(r_2)) + \varepsilon. \tag{13}$$

Combining (12) and (13) we obtain $\ln \mu(r_2) - \ln \mu(r_1) \le \ln f(r_2\nu_\varepsilon(r_2)) - \ln f(r_1\nu_\varepsilon(r_2)) + \varepsilon$, whence passing to the limit as $\varepsilon \to 0$ and taking account (11) we get

$$\ln \mu(r_2) - \ln \mu(r_1) \le \ln f(r_2\nu(r_2)) - \ln f(r_1\nu(r_2)).$$

Since r_1 and r_2 are arbitrary, we can obtain also the following inequality

$$\ln \mu(r_1) - \ln \mu(r_2) \le \ln f(r_1\nu(r_1)) - \ln f(r_2\nu(r_1)).$$

Suppose that $r_1 < r_2$. Then

$$\ln f(r_2\nu(r_1)) - \ln f(r_1\nu(r_1)) \le \ln \mu(r_2) - \ln \mu(r_1) \le \ln f(r_2\nu(r_2)) - \ln f(r_1\nu(r_2)). \tag{14}$$

We put $\Phi(x) = \ln f(e^x)$, $x_j = \ln r_j$ and $p_j = \ln \nu(r_j)$ for $j = 1,\ 2$. Then (14) implies

$$\Phi(p_1 + x_2) - \Phi(p_1 + x_1) \le \Phi(p_2 + x_2) - \Phi(p_2 + x_1)$$

and, since the function Φ is convex, we obtain the inequality $p_1 \le p_2$, i. e. $\nu(r_1) \le \nu(r_2)$, the function $\nu(r)$ is non-decreasing and, thus, continuous with the exception of an at most countable set of points.

Using Lemma 1 from (14) we get

$$\Gamma_f(r_1\nu(r_1)) \le \frac{\ln f(r_2\nu(r_1)) - \ln f(r_1\nu(r_1))}{\ln r_2 - \ln r_1} \le \frac{\ln \mu(r_2) - \ln \mu(r_1)}{\ln r_2 - \ln r_1} \le$$

$$\le \frac{\ln f(r_2\nu(r_2)) - \ln f(r_1\nu(r_2))}{\ln r_2 - \ln r_1} \le \Gamma_f(r_2\nu(r_2))$$

Passing to the limit as $r_1 \to r_2$ (and afterwards $r_2 \to r_1$), we obtain the equality

$$\frac{d \ln \mu(r)}{d \ln r} = \Gamma_f(r\nu(r)).$$

□

From Proposition 9 the following statement follows.

Corollary 1. *For all* $0 \le r_0 \le r < R_\mu$

$$\ln \mu(r) - \ln \mu(r_0) = \int_{r_0}^{r} \frac{\Gamma_f(x\nu(x))}{x} dx. \tag{15}$$

Let us point out other properties of the maximum of the integrand.

Proposition 10. *If* $R_\mu = +\infty$ *then* $\ln \mu(r)/\ln r \to +\infty$ *as* $r \to +\infty$.

Proof. If $\ln\mu(r_k)/\ln r_k \le K < +\infty$ for some sequence (r_k) increasing to $+\infty$ then $\ln a(x) + \ln f(xr_k) \le K\ln r_k$ and, thus, $a(x) \le r_k^K/f(xr_k) = x^{-K}(xr_k)^K/f(xr_k)$, i.e. for every $x > 0$
$$a(x)x^K \le \varlimsup_{k\to\infty}\frac{(xr_k)^K}{f(xr_k)} \le \varlimsup_{r\to+\infty}\frac{r^K}{f(r)} = 0,$$
because the function f is transcendental, this is impossible. Thus, $\ln\mu(r)/\ln r \to +\infty$ as $r \to +\infty$. □

Remark that in the case $R_\mu < +\infty$ the situation is different; the function $\mu(r)$ may be bounded. The following statement is true.

Proposition 11. *Let* $0 < R_\mu < +\infty$. *In order that* $\mu(r) \nearrow +\infty$ *as* $r \to R_\mu$ *it is necessary and sufficient that* $\sup\{a(x)f(xR_\mu)\colon x \ge 0\} = +\infty$.

Proof. If $\sup\{a(x)f(xR_\mu)\colon x \ge 0\} \le K < +\infty$ then $\mu(r) = \sup\{a(x)f(xr)\colon x \ge 0\} \le$ $\le \sup\{a(x)f(xR_\mu)\colon x \ge 0\} \le K < +\infty$.

On the contrary, if $\mu(r) \le K$ for all $r \in [0, R_\mu)$ then $a(x)f(xr) \le K$ and $\frac{1}{x}f^{-1}(\frac{K}{a(x)}) \ge r$ for each $x \ge 0$ and all $r \in [0, R_\mu)$. Passing to the limit as $r \to R_\mu$, we obtain $\frac{1}{x}f^{-1}(\frac{K}{a(x)}) \ge R_\mu$ for each $x \ge 0$ and, thus, $a(x)f(xR_\mu) \le K$ for each $x \ge 0$. □

Proposition 12. *If the function* $a(x)$ *is semi-continuous from above then*
$$\mu(r) = \max\{a(x)f(xr)\colon x \ge 0\} \text{ and } \nu(r) = \max\{x \ge 0\colon a(x)f(xr) = \mu(r)\}$$
for each $r \in [0, R_\mu)$.

Proof. Since the function $a(x)$ is semi-continuous from above, the function $\ln a(x) + \ln f(xr)$ is semi-continuous from above for every $r \in [0, R_\mu)$. From (4) it follows that for every $\varepsilon \in (0, R_\mu)$ all $x \ge x_0(\varepsilon)$ and $r < R_\mu - \varepsilon$
$$\ln a(x) + \ln f(xr) \le -\ln f(x(R_\mu - \varepsilon)) + \ln f(xr) \le -\Gamma_f(xr)(\ln(R_\mu - \varepsilon) - \ln r) \to -\infty$$
as $x \to +\infty$, i. e. $\ln a(x) + \ln f(xr) \to -\infty$ as $x \to +\infty$ for every $r \in [0, R_\mu)$. Therefore, $\max\{\ln a(x) + \ln f(xr)\colon x \ge 0\} = \ln\mu(r)$ exists.

Now suppose that $\ln a(x_k) + \ln f(x_kr)\colon x \ge 0\} = \ln\mu(r)$ and $x_k \uparrow x^*$. Since the function $\ln a(x)$ is semi-continuous from above, for every $\varepsilon > 0$ and all $k \ge k_0(\varepsilon)$ we have $\ln a(x_k) \le \ln a(x^*) + \varepsilon$. Therefore,
$$\ln\mu(r) \ge \ln a(x^*) + \ln f(x^*r) \ge \ln a(x_k) + \ln f(x_kr) - \varepsilon + \ln f(x^*r) - \ln f(x_kr) \ge$$
$$\ge \ln a(x_k) + \ln f(x_kr) - \varepsilon = \ln\mu(r) - \varepsilon,$$
whence in view of the arbitrariness of ε we have $\ln\mu(r) = \ln a(x^*) + \ln f(x^*r)$ and the existence of $\max\{x \ge 0\colon a(x)f(xr) = \mu(r)\} = \nu(r)$. □

3. Estimates of Laplace-Stieltjes type integrals. At first we suppose that $R_\mu = +\infty$ and put
$$\tau := \varlimsup_{x\to+\infty}\frac{\ln F(x)}{\Gamma_f(x)}, \quad \omega := \varlimsup_{x\to+\infty}\frac{\ln F(x)}{\ln(1/a(x))}.$$
We denote by $K(\varepsilon)$ constants depending on ε.

Theorem 3. *Let $F \in V$ and $R_\mu = +\infty$.*
1. If $\tau < +\infty$ then $I(r) \le K(\varepsilon)\mu(re^{\tau+\varepsilon})$ for every $\varepsilon > 0$;
2. If $\tau < +\infty$ and the function f'/f is non-decreasing then $I(r) \le K(\varepsilon)\mu(r+\tau+\varepsilon)$ for every $\varepsilon > 0$;
3. If $\omega < 1$ and $\varlimsup\limits_{x\to+\infty} \Gamma_f(x)/\ln f(x) = p < +\infty$ then $I(r) \le K(\varepsilon)\mu\left(r/(1-\omega-\varepsilon)^{p+\varepsilon}\right)$ for every $\varepsilon \in (0, 1-\omega)$.

Proof. If $\tau < +\infty$ then $\ln F(x) \le (\tau + \varepsilon/2)\Gamma_f(x)$ for every $\varepsilon > 0$ and all $x \ge x_0 = x_0(\varepsilon)$. Therefore, using Lemma 1 we get for $q = e^{\tau+\varepsilon}$, $h = \varepsilon/(2\tau+\varepsilon)$ and $r \ge 1$

$$I(r) = \int_0^\infty a(x)f(qxr)\frac{f(xr)}{f(qxr)}dF(x) \le$$

$$\le \mu(qr)\Big(\int_0^{x_0} \frac{f(xr)}{f(qxr)}dF(x) + \int_{x_0}^\infty \exp\{-(\ln f(qxr) - \ln f(xr))\}dF(x)\Big) \le$$

$$\le \mu(qr)\Big(\int_0^{x_0} dF(x) + \int_{x_0}^\infty \exp\{-\Gamma_f(xr)\ln q\}dF(x)\Big) \le$$

$$\le \mu(qr)\Big(K_1(\varepsilon) + \int_0^\infty \exp\{-\Gamma_f(xr)\ln q + (1+h)\ln F(x)\}\frac{dF(x)}{F(x)^{1+h}}\Big) \le$$

$$\le \mu(qr)\Big(K_1(\varepsilon) + \int_0^\infty \exp\{-\Gamma_f(x)\ln q + (\tau+\varepsilon)\Gamma_f(x)\}\frac{dF(x)}{F(x)^{1+h}}\Big) \le$$

$$\le \mu(qr)\Big(K_1(\varepsilon) + \int_0^\infty \frac{dF(x)}{F(x)^{1+h}}\Big) \le (K_1(\varepsilon) + K_2(\varepsilon))\mu(qr) = K(\varepsilon)\mu(re^{\tau+\varepsilon}).$$

If $\tau < +\infty$ and the function f'/f is non-decreasing then for $q = \tau + \varepsilon$, $h = \frac{\varepsilon}{2\tau+\varepsilon}$ and $r \ge 1$ we get

$$I(r) = \int_0^\infty a(x)f(x(r+q))\frac{f(xr)}{f(x(r+q))}dF(x) \le$$

$$\le \mu(r+q)\Big(K_1(\varepsilon) + \int_{x_0}^\infty \exp\Big\{-\int_{xr}^{x(r+q)} \frac{f'(t)}{f(t)}dt\Big\}dF(x)\Big) \le$$

$$\le \mu(r+q)\Big(K_1(\varepsilon) + \int_0^\infty \exp\Big\{-qx\frac{f'(xr)}{f(xr)}\Big\}dF(x)\Big) \le$$

$$\le \mu(r+q)\Big(K_1(\varepsilon) + \int_0^\infty \exp\Big\{-qx\frac{f'(x)}{f(x)}\Big\}dF(x)\Big) =$$

$$= \mu(r+q)\Big(K_1(\varepsilon) + \int_0^\infty \exp\big\{-q\Gamma_f(x) + (1+h)\ln F(x)\big\}\frac{dF(x)}{F(x)^{1+h}}\Big) \le$$

$$\le \mu(r+q)\Big(K_1(\varepsilon)+\int\limits_0^\infty \exp\big\{-(q-(1+h)(\tau+\varepsilon/2))\Gamma_f(x)\big\}\frac{dF(x)}{F(x)^{1+h}}\Big)\le$$
$$\le (K_1(\varepsilon)+K_2(\varepsilon))\mu(r+q)=K(\varepsilon)\mu(r+\tau+\varepsilon).$$

Finally, if $\omega<1$ and $\varlimsup\limits_{x\to+\infty}\ln f(x)/\Gamma_f(x)=p<+\infty$ then $\ln F(x)\le(\omega+\varepsilon/2)\ln(1/a(x))$ and $\ln f(x)\le(p+\varepsilon)\Gamma_f(x)$ for every $\varepsilon\in(0,\,1-\omega)$ and all $x\ge x_0=x_0(\varepsilon)$. Choose $h>0$ such that $(1+h)(\omega+\varepsilon/2)\le\omega+\varepsilon=\eta<1$. Then

$$I(r)=\int\limits_0^{x_0} a(x)f(xr)dF(x)+\int\limits_{x_0}^\infty \exp\{\ln a(x)+\ln f(xr)+(1+h)\ln F(x)\}\frac{dF(x)}{F(x)^{1+h}}\le$$
$$\le\mu(r)\int\limits_0^{x_0} dF(x)+\int\limits_{x_0}^\infty \exp\{\ln a(x)+\ln f(xr)+(1+h)(\omega+\varepsilon/2)\ln(1/a(x))\}\frac{dF(x)}{F(x)^{1+h}}=$$
$$=K_1(\varepsilon)\mu(r)+\int\limits_{x_0}^\infty \exp\{(1-\eta)\ln a(x)+\ln f(xr)\}\frac{dF(x)}{F(x)^{1+h}}.$$

Let $q=1/(1-\eta)^{p+\varepsilon}$. Then $q>1$ and

$$\ln\ln f(qxr)-\ln\ln f(xr)=\int\limits_{xr}^{qxr}\frac{\ln\ln f(t)}{d\ln t}d\ln t=\int\limits_{xr}^{qxr}\frac{\Gamma_f(t)}{\ln f(t)}d\ln t\ge\frac{\ln q}{p+\varepsilon}\ge\ln\frac{1}{1-\eta}$$

i.e. $\ln f(xr)\le(1-\eta)\ln f(qxr)$ and, thus,

$$(1-\eta)\ln a(x)+\ln f(xr)=(1-\eta)(\ln a(x)+\ln f(qxr))-(1-\eta)\ln f(qxr)+\ln f(xr)\le$$
$$\le(1-\eta)\ln\mu(qr).$$

Therefore,

$$I(r)\le K_1(\varepsilon)\mu(r)+K_2(\varepsilon)\mu(qr)^{1-\eta}\le K(\varepsilon)\mu(qr)=K(\varepsilon)\mu\Big(r/(1-\omega-\varepsilon)^{p+\varepsilon}\Big).$$

The proof of Theorem 3 is complete. □

Consider now the case when $0<R_\mu<+\infty$. This case, by replacing r by r/R_μ, reduces to the case $R_\mu=1$. If $R_\mu=1$ then by Proposition 11 $\mu(r)\nearrow+\infty$ as $r\to1$ if and only if $\sup\{a(x)f(x)\colon x\ge0\}=+\infty$. The last condition is satisfied if $\varlimsup\limits_{x\to+\infty}a(x)>0$ and even more so if $a(x)\to+\infty$ as $x\to+\infty$. We put

$$\theta=\varlimsup_{x\to+\infty}\frac{\ln F(x)}{\ln a(x)}.$$

Theorem 4. *Let $F\in V$ and $R_\mu=1$. If $\theta<+\infty$ then $I(r)\le K(\varepsilon)\mu(r^{1-\varepsilon})^{1+\theta+\varepsilon}$ for every $\varepsilon\in(0,\,1)$.*

Proof. Since $\ln F(x)\le(\theta+\varepsilon/2)\ln a(x)$ for every $\varepsilon\in(0,\,1)$ and all $x\ge x_0=x_0(\varepsilon)$, we have for $r<1$ and $h=\frac{\varepsilon}{2\theta+\varepsilon}$

$$I(r)\le\int\limits_0^{x_0} a(x)f(x)dF(x)+\int\limits_{x_0}^\infty \exp\{\ln a(x)+\ln f(xr)+(1+h)\ln F(x)\}\frac{dF(x)}{F(x)^{1+h}}\le$$

$$\le K_1(\varepsilon) + \int\limits_{x_0}^{\infty} \exp\{(1+\theta+\varepsilon)\ln a(x) + \ln f(xr)\}\frac{dF(x)}{F(x)^{1+h}} = K_1(\varepsilon) +$$

$$+\int\limits_{x_0}^{\infty} \exp\{(1+\theta+\varepsilon)(\ln a(x) + \ln f(xr^{1-\varepsilon})) - (1+\theta+\varepsilon)\ln f(xr^{1-\varepsilon}) + \ln f(xr)\}\frac{dF(x)}{F(x)^{1+h}} \le$$

$$\le K_1(\varepsilon) + \mu(r^{1-\varepsilon})^{(1+\theta+\varepsilon)} \int\limits_{x_0}^{\infty} \exp\{-((1+\theta+\varepsilon)\ln f(xr^{1-\varepsilon}) - \ln f(xr))\}\frac{dF(x)}{F(x)^{1+h}} \le$$

$$\le K_1(\varepsilon) + \mu(r^{1-\varepsilon})^{(1+\theta+\varepsilon)} \int\limits_{x_0}^{\infty} \frac{dF(x)}{F(x)^{1+h}} \le K_1(\varepsilon) + K_2(\varepsilon)\mu(r^{1-\varepsilon})^{(1+\theta+\varepsilon)} \le K(\varepsilon)\mu(r^{1-\varepsilon})^{(1+\theta+\varepsilon)},$$

because $f(xr^{1-\varepsilon}) \ge f(xr)$ and, thus, $(1+\theta+\varepsilon)\ln f(xr^{1-\varepsilon}) - \ln f(xr) > 0$. □

In general, the estimate of $I(r)$ from below via $\mu(r)$ is impossible. Indeed, if $f(x) = e^x$, $F(x) = x$, $a(x) = 0$ for $x \ne n$ and $a(x) = b_n$ for $x = n$ then $I(r) = \int_0^\infty a(x)e^{rx}dx = 0$ and $\mu(r) = \sup\{b_n e^{rn} \colon n \ge 0\} > 0$.

However, by imposing certain conditions on a and F one can estimate of $I(r)$ from below by $\mu(r)$.

Proposition 13. *Let $F \in V$, a has the regular variation in regard to F and $\Gamma_f(r) = O(r)$ as $r \to +\infty$. If $R_\mu = 1$ then $\ln \mu(r) \le (1+o(1))\ln I(r)$ as $r \uparrow 1$, and if $R_\mu = +\infty$ then $\ln \mu(r) \le (1+o(1))\ln I(r) + O(r)$ as $r \to \infty$.*

Proof. If the function a has regular variation with respect to $F \in V$ then there exist $b \ge 0$, $c \ge 0$ and $h > 0$ such that (6) holds for all $x \ge b$. Therefore, as in the proof of Proposition 5, we get $I(r) \ge \int_{x-b}^{x+c} a(t)f(tr)dF(t) \ge ha(x)f(xr)\exp\{-2bKr\}$ for all $x \ge 2b$ and $r > 0$, where $K = \text{const} > 0$, i. e.

$$\sup\{a(x)f(xr) \colon x \ge 2b\} \le I(r)\exp\{2bKr\}/h. \tag{16}$$

If $R_\mu = 1$ then for $r < 1$ we have

$$\sup\{a(x)f(xr) \colon x \le 2b\} \le \sup\{a(x)f(x) \colon x \le 2b\} = C > 0,$$

and in view of (16), $\sup\{a(x)f(xr) \colon x \ge 2b\} \le I(r)\exp\{2bK\}/h$, whence the inequality $\ln \mu(r) \le (1+o(1))\ln I(r)$ as $r \uparrow 1$ follows.

Now, if $R_\mu = +\infty$ then for every $c_0 > 0$ and all $r > c_0$

$$\frac{\mu(r)}{f(c_0 r)} = \sup\left\{a(x)\frac{f(xr)}{f(c_0 r)} \colon x \ge 0\right\} \ge \sup\left\{a(x)\frac{f(xr)}{f(c_0 r)} \colon x \ge 2c_0\right\} =$$

$$= \sup\left\{a(x)\exp\left\{\int\limits_{c_0 r}^{xr} \Gamma_f(t)d\ln t\right\} \colon x \ge 2c_0\right\} \ge \sup\left\{a(x)\exp\left\{\Gamma_f(c_0 r)\ln\frac{x}{c_0}\right\} \colon x \ge 2c_0\right\} \ge$$

$$\ge \exp\{\Gamma_f(c_0 r)\ln 2\}\sup\{a(x) \colon x \ge 2c_0\} \to +\infty, \quad r \to +\infty.$$

Therefore, $f(c_0 r) = o(\mu(r))$ as $r \to +\infty$ and

$$\sup\{a(x)f(xr) \colon x \le 2b\} \le f(2br)\sup\{a(x) \colon x \le 2b\} = o(\mu(r)), \quad r \to +\infty.$$

From hence and (16) we obtain the inequality $\ln \mu(r) \le (1+o(1))\ln I(r) + O(r)$ as $r \to \infty$. □

4. Growth of Laplace-Stieltjes type integrals with infinite radius of convergence. If $R_\mu = +\infty$, $\alpha \in L$ and $\beta \in L$ then we put

$$\varrho_{\alpha,\beta}[\mu] := \overline{\lim_{r\to+\infty}} \frac{\alpha(\ln \mu(r))}{\beta(r)}.$$

Proposition 14. *Let $\alpha(e^x) \in L^0$, $\beta(x) \in L^0$ and $\frac{\ln r}{\ln \alpha^{-1}(c\beta(r))} \to 0$ as $r \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $\ln f(r) = O(\Gamma_f(r))$, $\Gamma_f(r) = O(r)$ as $r \to +\infty$ and $\ln f(r) \ge \eta r$ for some $\eta > 0$ and all $r \ge r_0$. Then*

$$\varrho_{\alpha,\beta}[\mu] = \zeta_{\alpha,\beta}[I], \quad \zeta_{\alpha,\beta}[I] := \overline{\lim_{x\to+\infty}} \frac{\alpha(x)}{\beta\left(\frac{1}{x} f^{-1}\left(1/a(x)\right)\right)}. \tag{17}$$

Proof. Suppose that $\varrho_{\alpha,\beta}[\mu] < +\infty$. Then $\ln \mu(r) \le \alpha^{-1}(\varrho\beta(r))$ for every $\varrho > \varrho_{\alpha,\beta}[\mu]$ and all $r \ge r_0$, i. e. $\ln a(x)| \le \alpha^{-1}(\varrho\beta(r)) - \ln f(rx)$ for all $x \ge 0$ and $r \ge r_0$. Choosing $r = \beta^{-1}\left(\alpha(x)/\varrho\right)$ we get $e^x/a(x) \ge f\left(x\beta^{-1}\left(\alpha(x)/\varrho\right)\right)$ for all $x \ge x_0$, i. e.

$$\alpha(x) \le \varrho\beta\left(\frac{1}{x} f^{-1}\left(\frac{e^x}{a(x)}\right)\right), \quad x \ge x_0.$$

From (4) it follows that $\frac{1}{x} f^{-1}\left(1/a(x)\right) \to +\infty$ as $x \to +\infty$, and the condition $\ln f(r) \ge \eta r$ for $r \ge r_0$ implies $f^{-1}\left(1/a(x)\right) \le \frac{1}{\eta}\ln(1/a(x))$ for $x \ge x_0$. Therefore, $-\frac{1}{x}\ln a(x) \to +\infty$ and $\frac{e^x}{a(x)} = \exp\left\{-(1+o(1))\ln a(x)\right\}$ as $x \to +\infty$. Since $\ln f(r) = O(\Gamma_f(r))$ as $r \to +\infty$, we have $\frac{d}{d\ln r}\ln\ln f(r) \ge c > 0$ for all r, i.e. $\frac{d}{d\ln x}\ln f^{-1}(e^x) \le \frac{1}{c} < +\infty$ for all $x \ge 0$. Hence it follows that the function $\gamma(x) = f^{-1}(e^x)$ belongs to L^0 and, thus, $f^{-1}(e^{(1+o(1))x}) = (1+o(1))(f^{-1}(e^x))$ as $x \to +\infty$. Since $\beta(x) \in L^0$, we get

$$\alpha(x) \le \varrho\beta\Big(\frac{1}{x} f^{-1}\left(\exp\left\{-(1+o(1))\ln a(x)\right\}\right)\Big) = (1+o(1))\varrho\beta\Big(\frac{1}{x} f^{-1}\left(1/a(x)\right)\Big), \quad x \to +\infty,$$

whence in view of arbitrariness of ϱ we obtain the inequality $\zeta_{\alpha,\beta}[I] \le \varrho_{\alpha,\beta}[\mu]$, which is obvious if $\varrho_{\alpha,\beta}[\mu] = +\infty$.

Now, to prove the equality $\zeta_{\alpha,\beta}[I] = \varrho_{\alpha,\beta}[\mu]$, suppose by contradiction that $\zeta_{\alpha,\beta}[I] < \varrho_{\alpha,\beta}[\mu]$ and choose $\zeta_{\alpha,\beta}[A] < \zeta < q < \varrho_{\alpha,\beta}[\mu]$. Then $a(x) \le \frac{1}{f(x\beta^{-1}(\alpha(x)/\zeta))}$ for $x \ge x_0(\zeta)$. Since (see the proof of Proposition 13) $f(cr) = o(\mu(r))$ as $r \to +\infty$ for every $c > 0$, we have $\sup\{a(x)f(xr) \colon x \le x_0(\zeta)\} = o(\mu(r))$ as $r \to +\infty$. Therefore,

$$(1+o(1))\mu(r) = \sup\{a(x)f(xr) \colon x \ge x_0(\zeta)\} \le \max\left\{\frac{f(xr)}{f(x\beta^{-1}(\alpha(x)/\zeta))} \colon x \ge x_0(\zeta)\right\}$$

as $r \to +\infty$. Since $\mu(r) \uparrow +\infty$ as $r \to +\infty$ and $\frac{f(xr)}{f(x\beta^{-1}(\alpha(x)/\zeta))} \le 1$ for $r < \beta^{-1}(\alpha(x)/\zeta))$, for the central point $\nu(r)$ of the $\mu(r)$ we obtain $\beta^{-1}(\alpha(\nu(r))/\zeta)) \le r$, i. e. $\nu(r) \le \alpha^{-1}(\zeta\beta(r))$ for all $r \ge r_0$, and in view of condition $\Gamma_f(r) = O(r)$ as $r \to +\infty$ by formula (15)

$$\ln \mu(r) - \ln \mu(r_0) = \int_{r_0}^{r} \frac{\Gamma_f(x\alpha^{-1}(\zeta\beta(x))}{x} dx \le K\alpha^{-1}(\zeta\beta(r))r, \quad K = \text{const} > 0.$$

Since $\alpha(e^x) \in L^0$, $\beta(x) \in L^0$ and $\frac{\ln r}{\ln \alpha^{-1}(c\beta(r))} \to 0$ as $r \to +\infty$ for each $c \in (0, +\infty)$, from hence we obtain

$$\alpha(\ln \mu(r)) \le \alpha(\exp\{\ln \alpha^{-1}(\zeta\beta(r)) + \ln(Kr)\}) =$$

$$= \alpha(\exp\{(1+o(1))\ln\alpha^{-1}(\zeta\beta(r))\}) = (1+o(1))\zeta\beta(r), \quad r\to+\infty,$$

i. e. $\varrho_{\alpha,\beta}[\mu]\le\zeta$, which is contrary to the inequality $\zeta<\varrho_{\alpha,\beta}[\mu]$. □

We remark that the functions $\alpha(x)=\ln^+ x$ and $\beta(x)=x^+$ satisfy the conditions of Proposition 14 and, thus, the following statement is true.

Corollary 2. *If $\Gamma_f(r)=O(r)$, $\ln f(r)=O(\Gamma_f(r))$ and $r=O(\ln f(r))$ as $r\to+\infty$ then*

$$\varlimsup_{r\to+\infty}\frac{\ln\ln\mu(r)}{r}=\varlimsup_{x\to+\infty}\frac{x\ln x}{f^{-1}(1/a(x))}.$$

The functions $\alpha(x)=\beta(x)=\ln^+ x$ not satisfy the conditions of Proposition 14. In this case we put $\varrho_l[\mu]=\varlimsup\limits_{r\to+\infty}\frac{\ln\ln\mu(r)}{\ln r}$ and prove the following statement.

Proposition 15. *If $\ln f(r)=O(\Gamma_f(r))$, $r=o(\ln f(r))$ as $r\to+\infty$ and $\varlimsup\limits_{n\to\infty}\frac{\ln\ln f(r)}{\ln r}\le 1$ then*

$$\varrho_l[\mu]=\zeta_l[I]+1,\quad \zeta_l[I]:=\varlimsup_{x\to+\infty}\frac{\ln x}{\ln\left(\frac{1}{x}f^{-1}\left(1/a(x)\right)\right)}.$$

Proof. Let $1\le\varrho_l[\mu]<+\infty$. Then for every $\varrho>\varrho_l[\mu]$ and all $r\ge r_0(\varrho)$ we have $\ln\mu(r)\le r^{\varrho}$, i. e. $\ln a(x)\le r^{\varrho}-\ln f(rx)$ for all $x\ge 0$ and $r\ge r_0(\varrho)$. Choose $r=x^{1/(\varrho-1)}$. Then $\ln a(x)\le x^{\varrho/(\varrho-1)}-\ln f(x^{\varrho/(\varrho-1)})$ for $x\ge x_0(\varrho)$. The condition $r=o(\ln f(r))$ as $r\to+\infty$ implies $\ln a(x)\le-(1+o(1))\ln f(x^{\varrho/(\varrho-1)})$ as $x\to+\infty$, i. e.

$$x^{\varrho/(\varrho-1)}\le f^{-1}(\exp\{(1+o(1))\ln(1/a(x)\}) = (1+o(1))f^{-1}(1/a(x)),\quad x\to+\infty,$$

because $f^{-1}(e^x)\in L^0$. From hence it follows that $\zeta_l[I]\le\varrho-1$. In view of the arbitrariness of ϱ we get the inequality $\zeta_l[I]+1\le\varrho_l[\mu]$, which is obvious if $\varrho_l[\mu]=+\infty$.

To prove the equality $\zeta_l[I]+1\le\varrho_l[\mu]$, suppose by contradiction that $\zeta_l[I]<\varrho_l[\mu]-1$, i. e. for every $\zeta\in(\zeta_l[I],\varrho_l[\mu]-1)$ we have $a(x)\le 1/f(x^{1+1/\zeta})$ for all $x\ge x_0(\zeta)$. Therefore, as above $(1+o(1))\mu(r)\le\max\left\{\frac{f(xr)}{f(x^{1+1/\zeta})}: x\ge x_0(\zeta)\right\}$ as $r\to+\infty$, whence $r\ge\nu(r)^{1/\zeta}$ for $r\ge r_0$. Therefore, (15) implies

$$\ln\mu(r)-\ln\mu(r_0)\le\int\limits_{r_0}^{r}\frac{\Gamma_f(t^{1+\zeta})}{t}dt=\frac{1}{1+\zeta}\int\limits_{r_0^{1+\zeta}}^{r^{1+\zeta}}\Gamma_f(t)d\ln t=$$

$$=\frac{1}{1+\zeta}\int\limits_{r_0^{1+\zeta}}^{r^{1+\zeta}}\frac{d\ln f(t)}{d\ln t}d\ln t=\frac{1}{1+\zeta}\big(\ln f(r^{1+\zeta})-\ln f(r_0^{1+\zeta})\big),$$

whence $\varrho_l[\mu]\le\varlimsup\limits_{r\to+\infty}\dfrac{\ln\ln f(r^{1+\zeta})}{\ln r}=(1+\zeta)\varlimsup\limits_{r\to+\infty}\dfrac{\ln\ln f(r)}{\ln r}\le 1+\zeta$, which is contrary to the inequality $\zeta<\varrho_l[\mu]-1$. □

Now, using Theorem 3, Propositions 13 and 14, we prove the following theorem.

Theorem 5. *Let $F \in V$, $R_\mu = +\infty$, $\varlimsup\limits_{x\to+\infty} \big(\ln F(x)/\Gamma_f(x)\big) = \tau < +\infty$ and the function a has the regular variation in regard to F. Suppose that $\Gamma_f(r)/r \nearrow \xi \in (0, +\infty)$ as $r_0 \le r \to +\infty$. If $\alpha(e^x) \in L^0$, $\beta \in L^0$ and $\ln r = o(\ln \alpha^{-1}(c\beta(r)))$ as $r \to +\infty$ for each $c \in (0, +\infty)$ then $\varrho_{\alpha,\beta}[I] = \zeta_{\alpha,\beta}[I]$.*

Proof. At first we remark that the condition $\frac{\Gamma_f(r)}{r} \nearrow \xi \in (0, +\infty)$ as $r_0 \le r \to +\infty$ implies $\Gamma_f(r) \asymp r$, $\ln f(r) \ge \eta r$ for $r \ge r_0$ and f'/f is non-decreasing function. Also it is easy to check that the condition $\alpha(e^x) \in L^0$ implies the condition $\alpha \in L_{si}$ that is $\alpha(cx) = (1 + o(1)\alpha(x)$ as $x \to +\infty$ for every $c \in (0, +\infty)$. Therefore, by Theorem 3 we get $I(r) \le K(\varepsilon)\mu(r + \tau + \varepsilon))$ for every $\varepsilon > 0$, whence it follows that $\varrho_{\alpha,\beta}[I] \le \varrho_{\alpha,\beta}[\mu]$, because $\beta \in L^0$.

On the other hand, by Proposition 13

$$\alpha(\ln \mu(r)) \le \alpha(q \ln I(r) + pr) \le \alpha(2 \max\{q \ln I(r), pr\}) = (1 + o(1))\alpha(\max\{q \ln I(r), pr\}) =$$
$$= (1 + o(1)) \max\{\alpha(q \ln I(r)), \alpha(pr)\} \le (1 + o(1))\alpha(\ln I(r)) + (1 + o(1))\alpha(r), \quad r \to +\infty.$$

From the condition $\ln r = o(\ln \alpha^{-1}(c\beta(r)))$ as $r \to +\infty$ it follows that $\alpha(r) = o(\beta(r))$ as $r \to +\infty$. Therefore $\varrho_{\alpha,\beta}[\mu] \le \varrho_{\alpha,\beta}[I]$ and, thus, $\varrho_{\alpha,\beta}[I] = \varrho_{\alpha,\beta}[\mu]$.

Finally, by Proposition 14 $\varrho_{\alpha,\beta}[\mu] = \zeta_{\alpha,\beta}[I]$ and, thus, $\varrho_{\alpha,\beta}[I] = \zeta_{\alpha,\beta}[I]$. □

Using Theorem 3, Propositions 13 and 15, we prove the following theorem.

Theorem 6. *Let $F \in V$, $R_\mu = +\infty$,*

$$\varlimsup_{x\to+\infty} \frac{\ln F(x)}{\Gamma_f(x)} = \tau < +\infty$$

and the function a has the regular variation in regard to F. Suppose that $\Gamma_f(r) = O(r)$, $r = o(\ln f(r))$ as $r \to +\infty$ and $\varlimsup\limits_{n\to\infty} \ln\ln M_f(r)/\ln r \le 1$. Then $\varrho_l[I] \le \zeta_l[I] + 1 \le \varrho_l[I] + 1$.

Proof. From the inequality $I(r) \le K(\varepsilon)\mu(re^{\tau+\varepsilon})$ we obtain the inequality $\varrho_l[I] \le \varrho_l[\mu]$. By Proposition 13 $\ln\ln \mu(r) \le \ln(q \ln I(r) + pr) \le \ln(q \ln I(r)) + \ln(pr) + O(1)$ as $r \to +\infty$, whence $\varrho_l[\mu] \le \varrho_l[I] + 1$. Thus, $\varrho_l[I] \le \varrho_l[\mu] \le \varrho_l[I] + 1$ and, since By Proposition 15 $\varrho_l[\mu] = \zeta_l[I] + 1$, we get $\varrho_l[I] \le \zeta_l[I] + 1 \le \varrho_l[I] + 1$. □

5. Growth of Laplace-Stieltjes type integrals with finite radius of convergence. If $R_\mu = 1$ then to characterize the growth of $I(r)$ as $r \uparrow 1$, we can introduce a generalized order

$$\varrho^*_{\alpha,\beta}[I] := \varlimsup_{r\uparrow 1} \frac{\alpha(\ln I(r))}{\beta(1/(1-r))}.$$

If instead of $I(r)$ we put $\mu(r)$ then we get the definition $\varrho^*_{\alpha,\beta}[\mu]$. Suppose that $\mu(r) \nearrow +\infty$ as $r \uparrow 1$, i.e. by Proposition 11 $\varlimsup\limits_{x\to+\infty} a(x)f(x) = +\infty$.

Proposition 16. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and for every $c \in (0, +\infty)$*

$$\frac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right) = (1 + o(1))\alpha(x) \tag{18}$$

as $x_0 \le x \to +\infty$. Suppose that $\Gamma_f(r) \asymp r$ as $r \to +\infty$. Then

$$\varrho^*_{\alpha,\beta}[\mu] = \zeta^*_{\alpha,\beta}[I] := \varlimsup_{x\to+\infty} \frac{\alpha(x)}{\beta\big(x/\ln(a(x)f(x))\big)}.$$

Proof. At first we remark that the condition $\Gamma_f(r) \asymp r$ as $r \to +\infty$ implies the inequalities $cr \le \ln f(r) \le Cr$ for all r, where $0 < c \le C < +\infty$.

Let $\varrho^*_{\alpha,\beta}[\mu] < +\infty$ then $\ln\mu(r) \le \alpha^{-1}(\varrho\beta(1/(1-r)))$ for every $\varrho > \varrho^*_{\alpha,\beta}[\mu]$ and all $r \in [r_0, 1)$. Therefore, $\ln a(x) \le \alpha^{-1}(\varrho\beta(1/(1-r))) - \ln f(rx)$ for all $r \in [r_0, 1)$ and $x \ge 0$ and, thus,

$$\ln(a(x)f(x)) \le \alpha^{-1}\left(\varrho\beta\left(1/(1-r)\right)\right) + \ln f(x) - \ln f(rx) =$$
$$= \alpha^{-1}\left(\varrho\beta\left(1/(1-r)\right)\right)) + \int_{rx}^{x} \frac{\Gamma_f(x)}{x}dx \le \alpha^{-1}\left(\varrho\beta\left(1/(1-r)\right)\right) + Cx(1-r)$$

Choosing r so that $\frac{1}{1-r} = \beta^{-1}\left(\frac{1}{\varrho}\alpha\left(\frac{x}{\beta^{-1}(\alpha(x)/\varrho)}\right)\right)$ in view of conditions (18) we get

$$\ln(a(x)f(x)) \le \frac{x}{\beta^{-1}(\alpha(x)/\varrho)} + \frac{Cx}{\beta^{-1}\left(\frac{1}{\varrho}\alpha\left(\frac{x}{\beta^{-1}(\alpha(x)/\varrho)}\right)\right)} =$$
$$= \frac{x}{\beta^{-1}(\alpha(x)/\varrho)} + \frac{Cx}{\beta^{-1}\left(\frac{1+o(1)}{\varrho}\alpha(x)\right)} \le \frac{(C+1)x}{\beta^{-1}\left(\frac{1}{q}\alpha(x)\right)}$$

for every $q > \varrho$ and all $x \ge x_0(q)$. Since $\beta \in L_{si}$, from hence it follows that $\zeta^*_{\alpha,\beta}[I] \le q$ and, thus, in view of the arbitrariness of q we get the inequality $\zeta^*_{\alpha,\beta}[I] \le \varrho^*_{\alpha,\beta}[\mu]$, which is obvious if $\varrho^*_{\alpha,\beta}[\mu] = +\infty$.

Now to prove the equality $\zeta^*_{\alpha,\beta}[I] = \varrho^*_{\alpha,\beta}[\mu]$, suppose by contradiction that $\zeta^*_{\alpha,\beta}[I] < \varrho^*_{\alpha,\beta}[\mu]$ and choose $\zeta^*_{\alpha,\beta}[A] < \zeta < \varrho^*_{\alpha,\beta}[\mu]$. Then $\ln(a(x)f(x)) \le \frac{x}{\beta^{-1}(\alpha(x)/\zeta)}$ for $x \ge x_0$ and, thus,

$$\ln\mu(r)) = \sup\{\ln a(x) + \ln f(rx) : x \ge 0\} =$$
$$= \max\{K(\zeta), \sup\{\ln(a(x)f(x)) - \ln f(x) + \ln f(rx) \colon x \ge x_0\}\} \le$$
$$\le K(\zeta) + \max\left\{\frac{x}{\beta^{-1}\left(\alpha(x)/\zeta\right)} - \int_{rx}^{x}\frac{\Gamma_f(x)}{x}dx \colon x \ge 0\right\} \le$$
$$\le K(\zeta) + \max\left\{x\left(\frac{1}{\beta^{-1}\left(\alpha(x)/\zeta\right)} - c(1-r)\right) \colon x \ge 0\right\}.$$

Since $\mu(r) \to +\infty$ as $r \uparrow 1$, from hence we obtain $1/\beta^{-1}(\alpha(\nu(r))/\zeta \ge c(1-r)$, i.e. $\nu(r) \le \alpha^{-1}\left(\zeta\beta\left(1/c(1-r)\right)\right)$, and in view of (15)

$$\ln\mu(r) - \ln\mu(r_0) = \int_{r_0}^{r}\frac{\Gamma_f(x\nu(x))}{x}dx \le C\int_{r_0}^{r}\nu(x))dx \le$$
$$\le C\nu(r)(r-r_0) \le C\alpha^{-1}\Big(\zeta\beta\left(1/(c(1-r))\right)\Big).$$

Since $\alpha \in L_{si}$ and $\beta \in L_{si}$, this inequality implies the inequality $\varrho^*_{\alpha,\beta}[\mu] \le \zeta$, which is contrary to the condition $\zeta < \varrho^*_{\alpha,\beta}[\mu]$. □

From Theorem 4 and Propositions 13 and 16 the following theorem follows.

Theorem 7. *Let $F \in V$, $R_\mu = 1$, $\overline{\lim\limits_{x\to+\infty}}(\ln F(x)/\ln a(x)) = \theta < +\infty$ and the function a has the regular variation with respect to F. Suppose that $\Gamma_f(r) \asymp r$ and the functions $\alpha \in L_{si}$, $\beta \in L_{si}$ satisfy conditions* (18). *Then $\varrho^*_{\alpha,\beta}[I] = \zeta^*_{\alpha,\beta}[I]$.*

REFERENCES

1. Boas R.P., Pollard H., *Complete sets of Bessel and Legendre functions*, Annals of Math., **48** (1947), №2, 366–384.
2. Guadalupe J.J., Pérez M., Ruiz F.J., Varona J.L., *Mean and weak convergence of Fourier–Bessel series*, J. Math. Anal. Appl., **173** (1993), 370–389.
3. Hochstadt H., *The mean convergence of Fourier-Bessel series*, SIAM Rev., **9** (1967), 211–218.
4. Vinnitskii B.V. *Representation of analytic functions by series* $\sum_{n=1}^{+\infty} d_n f(\lambda_n z)$, Ukr. Math. J., **31** (1979), 501–506. https://doi.org/10.1007/BF01092529
5. Vinnitskii B.V., *Representation of functions by series* $\sum_{n=1}^{+\infty} d_n f(\lambda_n z)$, Ukr. Math. J., **31** (1979), 198–204. https://doi.org/10.1007/BF01089018
6. Vinnitskii B.V., *Completeness of the system* $f(\lambda_n z)$, Ukr. Math. J., **36** (1984), 493–495. https://doi.org/10.1007/BF01086778
7. Vinnitskii B.V., *On the description of certain absolutely representating systems*, Ukr. Mat. Zh., **38** (1986), №1, 93–95.
8. Vinnitskii B.V., *Effective expansion of analytic functions in series in generalized systems of exponents*, Ukr. Math. J., **41** (1989), 269–273. https://doi.org/10.1007/BF01060309
9. Vynnyts'kyi B.V., Khats' R.V., *Some approximation properties of the systems of Bessel functions of index* $-3/2$, Mat. Stud., **34** (2010), 152–159. http://matstud.org.ua/texts/2010/34_2/152-159.pdf
10. Vynnyts'kyi B.V., Khats' R.V. *A remark on basis property of systems of Bessel and Mittag-Leffler type functions* J. Contemp. Math. Anal., **50** (2015), №6, 300–305.
11. Vynnyts'kyi B.V., Khats' R.V., *Complete biorthogonal systems of Bessel functions*, Mat. Stud., **48** (2017), №2, 150–155. https://doi.org/10.15330/ms.48.2.150-155
12. Skaskiv O.B., Trusevych O.M., *Relations of Borel type for generalizations of exponential series*, Ukr. Math. J., **53** (2001), 1926–1931. https://doi.org/10.1023/A:1015219417195
13. Skaskiv O.B., *Rate of convergence of positive series*, Ukr. Math. J., **56** (2004), 1975–1988. https://doi.org/10.1007/s11253-005-0162-2
14. Sheremeta M.M., *On the growth of series in system of functions and Laplace-Stieltjes integrals*, Mat. Stud., **55** (2021), №2, 124–131. https://doi.org/10.30970/ms.55.2.124-131
15. B.V. Vinnitsky, Some approximation properties of generalized systems of exponentials, Drogobych, 1991, Dep. in UkrNIINTI 25.02.1991. (in Russian)
16. Sheremeta M.M. Asymptotical behavior of Laplace-Stieltjes integrals, Lviv: VNTL Publishers, 2010.
17. Sheremeta M.M., *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*, Izv. Vyssh. Uchebn. Zaved. Mat., (1967), №2, 100–108. (in Russian)
18. Posiko O.S., Skaskiv O.B., Sheremeta M.M., *Estimate of Laplace-Stieltjes integrals*, Mat. Stud., **21** (2004), №2, 180–186.
19. Sheremeta M.M., *On two classes of positive functions and the belonging to them of main characteristics of entire functions*, Mat. Stud., **19** (2003), №1, 75–82.
20. Sheremeta M.M., *Relative growth of series in system functions and Laplace-Stieltjes type integrals*, Axioms, **10** (2021), 43.
21. Sheremeta M.M., *On the growth of series in systems of functions and Laplace-Stieltjes integrals*, Mat. Stud., **55** (2021), №2, 124–131.

Ivan Franko National University of L'viv
Lviv, Ukraine
m.m.sheremeta@gmail.com

Received 07.02.2023
Revised 06.07.2023