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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION IM

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In the paper, we discuss the distribution of uniqueness and its elements over the extended complex plane from different polynomials of view. We obtain some new results regarding the structure and position of uniqueness. These new results have immense applications like classifying different expressions to be or not to be unique. The principal objective of the paper is to study the uniqueness of meromorphic functions when sharing a small function $a(z)$ IM with restricted finite order and its nonlinear differential polynomials. The lemma on the logarithmic derivative by Halburb and Korhonen (Journal of Mathematical Analysis and Applications, **314** (2006), 477–87) is the starting point of this kind of research. In this direction, the current focus in this field involves exploring unique results for the differential-difference polynomials of meromorphic functions, covering both derivatives and differences. Liu et al. (Applied Mathematics A Journal of Chinese Universities, **27** (2012), 94–104) have notably contributed to this research. Their research establishes that when $n \leq k+2$ for a finite-order transcendental entire function f the differential-difference polynomial

$$[f^n f(z+c)]^{(k)} - \alpha(z)$$

has infinitely many zeros. Here, $\alpha(z)$ is characterized by its smallness relatively to f. Additionally, for two distinct meromorphic functions f and g, both of finite order, if the differential-difference polynomials

$$[f^n f(z+c)]^{(k)} \text{ and } [g^n g(z+c)]^{(k)}$$

share the value 1 in the same set, then $f(z) = c_1 e^{dz}$, $g(z) = c_2 e^{-dz}$. We prove two results, which significantly generalize the results of Dyavanal and Mathai (Ukrainian Math. J., **71** (2019), 1032–1042), and Zhang and Xu (Comput. Math. Appl., **61** (2011), 722-730) and citing a proper example we have shown that the result is true only for a particular case. Finally, we present the compact version of the same result as an improvement.

1. Introduction. By a meromorphic function we shall always mean a meromorphic function in the complex plane. In this paper, we use the standard notations of Nevanlinna value distribution theory [1,2,4]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$ as $r \to +\infty$ possibly outside of a set E of finite linear measure. We say that the meromorphic function $a(z)$ ($\not\equiv \infty$) is a small function of f, if $T(r,a) = S(r,f)$. Let f and g be two nonconstant meromorphic functions. We say that f and g share the value a CM (counting multiplicities), if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

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It is well-known that for two positive integers k, n with $n \geq k+1$ and for transcendental entire functions f, $(f^n)^{(k)} - 1$ has infinitely many zeros [5]. In 2002, Fang [6], considering sharing value problems and for $n > 2k+4$ proved that for two entire functions f and g if $(f^n)^{(k)}$ and $(f^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $f(z) = c_1 e^{cz}$ where $(-1)^k (c_1 c_2)^n$ or $f \equiv tg$ for a constant t satisfying $t^n = 1$. Later, in 2008, Zhang et al. [7] extended the result of Wang and Fang confirming that $(f^n P(f))^{(k)} - 1$ has infinitely many zeros, where $P(f)$ is a polynomial expression in f. We observe that the polynomial expression $f^n P(f) = a_m f^{n+m} + ... + a_{1f^{n+1}} + a_0 f^n$ is not complete. To continue the research, for more general setting, we recall the polynomials as defined in the following definition. Let m be a nonnegative integer and let $a_0(\neq 0), a_1, a_2, ..., a_{m-1}, a_m(\neq 0)$ be complex constants. Define

$$P(w) = a_w w^m + a_{m-1} w^{m-1} + ... + a_1 w + a_0. \tag{1}$$

A differential polynomial $H[f]$ of a nonconstant meromorphic function f is defined as

$$H[f] = \sum_{i=1}^{m} M_i[f] \tag{2}$$

where

$$M_i[f] = b_i \prod_{j=1}^{l} (f^{(j)}(z+c_j))^{n_{ij}}$$

with $n_{i0}, n_{i1}, ..., n_{il}$ as nonnegative integers and $b_i(\not\equiv 0)$ are meromorphic functions satisfying $T(r, b_i) = o(T(r,f))$ as $r \to \infty$. The numbers

$$\overline{\gamma}_p = \max_{1 \leq i \leq m} \sum_{j=1}^{l} n_{ij} \quad \text{and} \quad \underline{\gamma}_p = \min_{1 \leq i \leq m} \sum_{j=1}^{l} n_{ij}$$

are respectively called the *upper degree* and *lower degree* of $H[f]$, respectively. If $\overline{\gamma}_p = \underline{\gamma}_p = \gamma$ (say), then we say that $H[f]$ is a homogeneous differential polynomial of degree γ. Also we define

$$Q = \max_{1 \leq i \leq m} \{n_{i0} + n_{i1} + ... + l n_{il}\},$$

we know that $H[f]$ is called homogeneous if $\overline{\gamma}_p = \underline{\gamma}_p$ and $H[f]$ is called a linear differential polynomial generated by f if $\overline{\gamma}_p = 1$. Otherwise, $P[f]$ is called a non-linear differential polynomial.

Recently, the exploration of difference polynomials has garnered significant attention within the academic literature, largely owing to the groundbreaking development of the analogous lemma for logarithmic derivatives by Halburd and Korhonen [3]. A notable current trend in this field revolves around the investigation of uniqueness results for differential-difference polynomials of meromorphic functions, which encompass both derivatives and differences. In this context, Liu et al. [8] have made a notable contribution. Their work establishes that for $n \leq k+2$ and a finite-order transcendental entire function f, the differential-difference polynomial $[f^n f(z+c)]^{(k)} - \alpha(z)$ possesses infinitely many zeros, where $\alpha(z)$ exhibits smallness concerning f. Furthermore, for two distinct meromorphic functions, both of finite order, if the differential-difference polynomials $[f^n f(z+c)]^{(k)}$ and $[g^n g(z+c)]^{(k)}$ share the value 1 in the same set for $n \geq 5k+12$, then Liu et al. [8] have established that f and g assume the forms $f(z) = c_1 e^{dz}$ and $g(z) = c_2 e^{-dz}$, where

$$(-1)(c_1 c_2)^{n+1} [(n+1)d]^{2k} = 1,$$

or in the case of $t^{n+1} = 1$, it follows that $f = tg$.

The investigation is currently focused on exploring different avenues related to the sharing of values. On one hand, the situation is being examined in cases where value sharing is substituted by small functions. On the other hand, a more general framework involving differential-difference polynomials is being considered. In this context, Dyavanal and Mathai [9] have presented the following result within a broader setting, specifically involving differential-difference polynomials $[f^n P(f) f(z+c)]^{(k)}$ and $[g^n P(g) g(z+c)]^{(k)}$ instead of $f^n f(z+c)$ and $g^n g(z+c)$, respectively.

Theorem 1. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions. Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$ which has no common zeros or poles with $f(z)$ and $g(z)$. Let k (> 0) and m (> 0) be two integers satisfying the inequality $n > 4m + 13k + 19$, let $P(w)$ be defined as in definition (1), and let c be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c, the poles of $f(z)$ are not zeros of $f(z+c)$, and the poles of $g(z)$ are not zeros of $g(z+c)$. If $[f^n P(f) f(z+c)]^{(k)}$ and $[g^n P(g) g(z+c)]^{(k)}$ share $a(z)$ IM and $f(z)$ and $g(z)$ share the value ∞ IM, then one of the following two cases is realized:*

(i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+1, ..., n+m+1-i, ..., n+1)$ and $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$;

(ii) $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n (a_m w_1^m + a_{m-1} w_1^{m-1} + ... + a_0) w_1(z+c) - \\ - w_2^n (a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0) w_2(z+c).$$

Theorem 2. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions. Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let k (> 0) be an integer satisfying the inequality $n > 13k+19$, let $P(w) = a_0$, where $a_0 \neq 0$, and let c be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c, the poles of $f(z)$ are not zeros of $f(z+c)$, and the poles of $g(z)$ are not zeros of $g(z+c)$. If $[f^n P(f) f(z+c)]^{(k)}$ and $[g^n P(g) g(z+c)]^{(k)}$ share $a(z)$ IM and $f(z)$ and $g(z)$ share the value ∞ IM, then one of the following two cases is realized:*

(i) $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

(ii) $a_0^2 [f^n f(z+c)]^{(k)} [g^n g(z+c)]^{(k)} = a^2(z)$.

Recently, the differential polynomials and difference analogue of the Nevanlinna theory has been established (see [10,14,17]). Many researchers started to consider the uniqueness of meromorphic functions sharing small function or sets or differences operators (see [11–13,25]).

It is important that when it comes to difference-differential polynomials within a broader context, no investigation has been conducted to date. Consequently, in order to extend the applicability of Theorems 1-2, several inevitable questions arise. In light of the findings presented by R. S. Dyavanal and M. M. Mathai mentioned earlier, it is natural to pose the following question, which serves as the driving force behind this present paper.

Question 1. *What happens if one replaces the difference-differential polynomial $[f^n P(f) f(z+c)]^{(k)}$ by more general nonlinear differential polynomials of the form $[f^n P(f) H[f]]^{(k)}$ in Theorems 1 and 2?*

The primary objective of this paper is to address the aforementioned question in a positive manner. The paper is structured as follows. Section 2 contains the answers to these questions through the proof of two key results. In Section 3, we establish several critical lemmas that will aid in substantiating the main findings. Section 4 is dedicated to the verification of the principal results. Section 5 is some possible application of the main results. Section 6 contains few more open problems relevant to main results within this paper.

2. Main Results. Corresponding to Question 1, we prove the following results.

Theorem 3. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions. Suppose that $a(z)$ ($\not\equiv 0, \infty$) is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let k, n, m, l $\overline{\gamma}$, Q be positive integers satisfying the inequality $n > 4m + 4k(Q+2) + \overline{\gamma}_p(5k+5l+6) + 8Q + 11$. Let $P(w)$ and $H[f]$ be defined as in definitions (1) and (2), c_j ($j = 1, 2, ..., l$) be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c_j, the poles of $f(z)$ (resp., $g(z)$) are not zeros of $H[f]$ (resp., $H[g]$). If $[f^n P(f) H[f]]^{(k)}$ and $[g^n P(g) H[g]]^{(k)}$ share $a(z)$ IM and $f(z)$ and $g(z)$ share the value ∞ IM, then one of the following two cases is realized:*

(i) $f(z) \equiv t g(z)$ for a constant t such that $t^d = 1$, where

$$d = GCD(n + m + \overline{\gamma}_p, ..., n + m + \overline{\gamma}_p - 1, ..., n + 1)$$

and $a_{m-i} \neq 0$ for $i = 0, 1, ..., m$.

(ii) $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + ... + a_0) H[w_1] - \\ - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0) H[w_2].$$

Theorem 4. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions. Suppose that $a(z)$ ($\not\equiv 0, \infty$) is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let k, n, l $\overline{\gamma}_p$, Q be positive integers satisfying the inequality*

$$n > 4k(Q+2) + \overline{\gamma}_p(5k + 5l + 6) + 8Q + 11.$$

Let $P(w) = a_0$, where $a_0 \neq 0$ is a complex constant and $H[f]$ be defined as in definition (2), c_j ($j = 1, 2, ..., l$) be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions with period c_j, the poles of $f(z)$ (resp., $g(z)$) are not zeros of $H[f]$ (resp., $H[f]$). If $[f^n P(f) H[f]]^{(k)}$ and $[g^n P(g) H[g]]^{(k)}$ share $a(z)$ IM and $f(z)$ and $g(z)$ share the value ∞ IM, then one of the following two cases is realized:

(i) $f(z) \equiv t g(z)$ for a constant t such that $t^{n+\overline{\gamma}_p} = 1$.

(ii) $a_0^2 [f^n H[f]]^{(k)} [g^n H[g]]^{(k)} = a^2(z)$.

Remark 1. Since Theorems 1–2 are the special cases of Theorems 3–4 respectively for $Q = 1$ and $\overline{\gamma}_p = 0$ then Theorems 3–4 improve and extend Theorems 1–2, respectively.

Example 1. Let $P(z) = a_m z^m$ $z = 1$, $f(z) = e^z$, $g(z) = te^z$, where $t^{m+n+1} = 1$, n, $m \in \mathbb{N}$. Let $H[f] = f(z + c)$, where c is a non-zero complex constant. Then it is easy to see that $[f^n P(f) H[f]]'$ and $[g^n P(g) H[g]]'$ share z CM. Clearly, f and g satisfy Theorems 3 and 4.

Example 2. Let $P(z) = a_m z^m$ $z = 1$, $f(z) = \frac{e^z}{e^{2\pi i z/c}-1}$, $g(z) = tf(z)$, where $t^{m+n+1} = 1$, n, $m \in \mathbb{N}$. Let $H[f] = f(z+c)$, where c is a non-zero complex constant. Then it is easy to see that $[f^n P(f)H[f]]'$ and $[g^n P(g)H[g]]'$ share z CM. Clearly, f and g satisfy Theorems 3 and 4.

3. Some Lemmas. In this section, we summarize some lemmas, which will be used to prove our main results. Henceforth, let F and G be two nonconstant meromorphic functions defined by

$$F = \frac{[f^n P(f)H[f]]^{(k)}}{a(z)}, \qquad G = \frac{[g^n P(g)H[g]]^{(k)}}{a(z)}. \tag{3}$$

Henceforth, we shall denote by H and V in the following

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{4}$$

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{5}$$

Lemma 1 ([18]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then, for any $\epsilon > 0$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2 ([19]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then, for any $\epsilon > 0$,*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}).$$

It is evident that $S(r, f(z+c)) = S(r, f)$.

Lemma 3 ([20]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then*
(i) $N\left(r, \frac{1}{f(z+c)}\right) \le N\left(r, \frac{1}{f}\right) + S(r, f)$; *(ii)* $N(r, f(z+c)) \le N(r, f) + S(r, f)$;
(iii) $\overline{N}\left(r, \frac{1}{f(z+c)}\right) \le \overline{N}\left(r, \frac{1}{f}\right) + S(r, f)$; *(iv)* $\overline{N}(r, f(z+c)) \le \overline{N}(r, f) + S(r, f)$;
an exceptional set with finite logarithmic measure.

Lemma 4 ([21]). *Let $f(z)$ be a nonconstant meromorphic function and let p and k be two positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f),$$
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \le k\overline{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 5 ([22]). *Let $f(z)$ and $g(z)$ be a nonconstant meromorphic functions. If $f(z)$ and $g(z)$ share the value 1 CM, then one of the following three cases is realized* *(i)* $T(r,f) \leq N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$, *the same inequality holds for* $T(r,g)$;

(ii) $fg = 1$;

(iii) $f \equiv g$.

Lemma 6 ([21]). *Let $f_1(z)$, $f_2(z)$ be two nonconstant meromorphic functions such that $c_1f_1 + c_2f_2 = c_3$, where c_1, c_2, c_3 are three nonzero constants. Then*

$$T(r,f_1) \leq \overline{N}(r,f_1) + \overline{N}\left(r,\frac{1}{f_1}\right) + \overline{N}\left(r,\frac{1}{f_2}\right) + S(r,f_1).$$

Lemma 7 ([23]). *Let F, G, and H be defined as in* (3) *and* (4). *If F and G share 1 IM and ∞ IM and, moreover, $H \not\equiv 0$, then $F \not\equiv G$,*

$$\begin{aligned} T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \\ +7\overline{N}(r,F) + S(r,F) + S(r,G). \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 8 ([24]). *Let F, G, and V be defined as in* (3) *and* (5). *If F and G share ∞ IM and $V \equiv 0$, then $F \equiv G$.*

Lemma 9 ([24]). *If F and G share 1 IM, then*

$$\overline{N}_L\left(r,\frac{1}{F-1}\right) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,F) + S(r,G).$$

Lemma 10. *Let $f(z)$ is a nonconstant meromorphic function with finite order. Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$, $P(w)$ and $H[f]$ are defined as in definitions* (1) *and* (2). *Then we have,*

$$(n+m-\overline{\gamma}_p)T(r,f) + S(r,f) \leq T(r,F) \leq (n+m+\overline{\gamma}_p)T(r,f) + S(r,f).$$

Proof. Let $F = f^nP(f)H[f]$, we know that

$$\begin{aligned} T(r,F) = T(r,f^nP(f)H[f]) &\leq T(r,f^nP(f)) + T(r,H[f]) + S(r,f) \\ &\leq (n+m+\overline{\gamma}_p)T(r,f) + S(r,f). \end{aligned} \tag{6}$$

A quick calculation reveals that

$$\begin{aligned} (n+m+1)T(r,f) + S(r,f) = T(r,f^nP(f)f) + S(r,f) \leq \\ \leq m(r,f^nP(f)f) + N(r,f^nP(f)f) + S(r,f) \leq \\ \leq m\left(r,F\frac{f}{H[f]}\right) + N\left(r,F\frac{f}{H[f]}\right) + S(r,f) \leq \\ \leq T(r,F) + (1+\overline{\gamma}_p)T(r,f) + S(r,f). \end{aligned} \tag{7}$$

It is follows from (6) and (7) that

$$(n+m-\overline{\gamma}_p)T(r,f) + S(r,f) \leq T(r,F) \leq (n+m+\overline{\gamma}_p)T(r,f) + S(r,f).$$

□

Lemma 11. *If $f(z)$ and $g(z)$ are two nonconstant meromorphic functions with finite order. If c_j $(j = 1, 2, ..., l)$ is a nonzero complex constant, f and g are not periodic functions of period c_j and k, n, m, $\overline{\gamma}_p$, Q are positive integers satisfying the inequality $n > k+2Q+\overline{\gamma}_p(k+l)+2$. Let $P(w)$ and $H[f]$ be defined as in definitions (1) and (2). Let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If $[f^nP(f)H[f]]^{(k)}$ and $[g^nP(g)H[g]]^{(k)}$ share $a(z)$ IM, then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.*

Proof. Let $F_1 = f^nP(f)H[f]$. By the Second Fundamental Theorem for small functions and for all $\varepsilon > 0$, we get

$$T(r,F^{(k)}) \le \overline{N}(r,F_1) + \overline{N}\left(r,\frac{1}{F_1^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F_1^{(k)}-a(z)}\right) + (\varepsilon+O(1))T(r,F_1) \le$$
$$\le \overline{N}(r,f) + \overline{N}(r,H[f]) + \overline{N}\left(r,\frac{1}{F_1^{(k)}}\right) + \overline{N}\left(r,\frac{1}{F_1^{(k)}-a(z)}\right) + (\varepsilon+O(1))T(r,F).$$

In view of Lemma 4 with $s = 1$, and Lemma 10, applying to the function F, we obtain

$$(n+m-\overline{\gamma}_p)T(r,f) \le \overline{N}(r,f) + \overline{N}(r,H[f]) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P(f)}\right)+$$
$$+N_{k+1}\left(r,\frac{1}{H[f]}\right) + \overline{N}\left(r,\frac{1}{g^nP(g)H[g]-a}\right) + (\varepsilon+O(1))T(r,f) \le$$
$$\le \overline{N}(r,f) + \overline{N}(r,H[f]) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P(f)}\right)+$$
$$+Q\overline{N}(r,f) + \overline{\gamma}_pN_{k+l+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g^nP(g)H[g]-a}\right) + (\varepsilon+O(1))T(r,f) \le$$
$$\le (k+m+2Q+\overline{\gamma}_p(k+l+1)+2)T(r,f)+$$
$$+(n+m+\overline{\gamma}_p)(k+1)T(r,g) + (\varepsilon+O(1))T(r,f).$$

A quick calculation reveals that

$$(n-k-2Q-\overline{\gamma}_p(k+l)-2)T(r,f) \le (n+m+\overline{\gamma}_p)(k+1)T(r,g) + (\varepsilon+O(1))T(r,f).$$

Since $n > k + 2Q + \overline{\gamma}_p(k + l) + 2$, taking $\varepsilon > 1$, we obtain $T(r, f) = O(T(r, g))$. Similarly, we can prove that $T(r, g) = O(T(r, f))$. □

Lemma 12. *Let $f(z)$, $g(z)$ be two nonconstant finite-order meromorphic functions such that the poles of $f(z)$ are not zeros of $H[f]$ and the poles of $g(z)$ are not zeros of $H[g]$, F, G and V are defined as in (3) and (5), let $P(w)$ and $H[f]$ be defined as in definitions (1) and (2), and $k(> 0)$, $n(> 3)$, $m(\ge 0)$ Q be positive integers. Also let $c_j(j = 1, 2, .., l)$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_j. If $V \not\equiv 0$, F and, in addition, G share the values 1 and ∞ IM, then*

$$(n+m+k-2Q-3)\overline{N}(r,f) \le 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}\left(r,\frac{1}{G}\right) + S(r,f) + S(r,g),$$

and

$$(n+m+k-2Q-3)\overline{N}(r,g) \le 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}\left(r,\frac{1}{G}\right) + S(r,f) + S(r,g).$$

Proof. Let the poles of $f(z)$ and $g(z)$ be not zeros of $H[f]$ and $H[g]$, respectively. If z_0 is a pole of $f(z)$ and $g(z)$ of order p and q, respectively, then z_0 must be a pole of F and G of order $(n+m)p+k$ and $(n+m)q+k$, respectively.

Thus z_0 is a zero of $\frac{F'}{F-1} - \frac{F'}{F}$ of order $(n+m)p+k-1 \geq n+m+k-1$. Moreover, z_0 is a zero of $\frac{G'}{G-1} - \frac{G'}{G}$ of order $(n+m)q+k-1 \geq n+m+k-1$. Hence, z_0 is a zero of V of order at least $n+m+k-1$. Therefore, we obtain

$$(n+m+k-1)\overline{N}(r,f) \leq N\left(r, \frac{1}{V}\right) \tag{8}$$

and

$$(n+m+k-1)\overline{N}(r,g) \leq N\left(r, \frac{1}{V}\right). \tag{9}$$

By the Lemma on the logarithmic derivative, we get $m(r,V) = S(r,f) + S(r,g)$. We now consider

$$N\left(r, \frac{1}{V}\right) \leq T(r,V) \leq m(r,V) + N(r,V) \leq N(r,V) + S(r,f) + S(r,g). \tag{10}$$

Since $F(z)$ and $G(z)$ share the value 1 IM, the zeros of $F(z)-1$ and the zeros of $G(z)-1$ with different multiplicities contribute to the poles of V. Furthermore, since $F(z)$ and $G(z)$ share the value 1 IM, the poles of $F(z)$ and $G(z)$ with different multiplicities contribute to the zeros of V. Thus, it follows from (8) and (10) that

$$\begin{aligned} N\left(r, \frac{1}{V}\right) &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \\ &+ \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r,f) + S(r,g). \end{aligned} \tag{11}$$

Since F and G share 1 IM, by Lemma 9 and (11), we get

$$\begin{aligned} N\left(r, \frac{1}{V}\right) &\leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + \\ &+ \overline{N}(r,F) + \overline{N}(r,G) + S(r,f) + S(r,g). \end{aligned} \tag{12}$$

By Lemma 3, we obtain

$$\begin{aligned} \overline{N}(r,F) &= \overline{N}\left(r, \frac{[f^n P(f) H[f]]^{(k)}}{a(z)}\right) \leq \\ &\leq \overline{N}(r,f) + \overline{N}(r,H[f]) + S(r,f) \leq (Q+1)\overline{N}(r,f) + S(r,f). \end{aligned} \tag{13}$$

Similarly,

$$\overline{N}(r,G) \leq (Q+1)\overline{N}(r,g) + S(r,g). \tag{14}$$

In view of (12)–(14) and the fact that $f(z)$ and $g(z)$ share ∞ IM, we find

$$N\left(r, \frac{1}{V}\right) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + (1+Q)\overline{N}(r,f) + (Q+1)\overline{N}(r,g) + S(r,f) +$$

$$+S(r,g) \le 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + 2(Q+1)\overline{N}(r,f) + S(r,f) + S(r,g). \tag{15}$$

It follows from (8) and (15) that

$$(n+m+k-1)\overline{N}(r,f) \le 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + 2(Q+1)\overline{N}(r,f) + S(r,f) + S(r,g),$$

i.e.,

$$(n+m+k-2Q-3)\overline{N}(r,f) \le 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r,f) + S(r,g).$$

Similarly,

$$(n+m+k-2Q-3)\overline{N}(r,g) \le 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r,f) + S(r,g).$$

□

Lemma 13. *Let $f(z)$ be a transcendental finite-order meromorphic function, k, n, m, $\overline{\gamma}_p$, Q be positive integers satisfying the inequality $n > k + 2Q + \overline{\gamma}_p(k+l) + 2$ and $c_j (j = 1, 2..., l)$ be a nonzero complex constant such that $f(z)$ is not a periodic function of period c_j, let $P(w)$ and $H[f]$ be defined as in definitions (1) and (2). Suppose that $a(z)(\not\equiv 0, \infty)$ is a small function with respect to $f(z)$. Then $[f^n P(f) H[f]]^{(k)} - a(z)$ has infinitely many zeros.*

Proof. Suppose $[f^n P(f) H[f]]^{(k)} - a(z)$ has only finitely many zeros. Let $F_1 = f^n P(f) H[f]$ and $F = F_1^{(k)}$. By the Second Fundamental Theorem, we obtain

$$T(r, F^{(k)}) \le \overline{N}\left(r, \frac{1}{F_1^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F_1^{(k)} - a}\right) + \overline{N}(r, F_1^{(k)}) + S(r, F_1) \le$$
$$\le T(r, F_1^{(k)}) - T(r, F_1) + \overline{N}_{k+1}\left(r, \frac{1}{F_1}\right) + \overline{N}(r, F_1) + S(r, F_1),$$

it reveals that

$$T(r, F) \le \overline{N}_{k+1}\left(r, \frac{1}{F_{k+1}}\right) + \overline{N}(r, F_1) + S(r, F_1). \tag{16}$$

Hence we have $T(r,f) = T(r,f) + S(r,f)$. Therefore, we obtain

$$\overline{N}_{k+1}\left(r, \frac{1}{F_1}\right) = \overline{N}_{k+1}\left(r, \frac{1}{f^n P(f) H[f]}\right) \le$$
$$\le \overline{N}_{k+1}\left(r, \frac{1}{f^n}\right) + \overline{N}_{k+1}\left(r, \frac{1}{P(f)}\right) + N_{k+1}\left(r, \frac{1}{H[f]}\right) + S(r,f) \le$$
$$\le (k+1)\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_{k+1}\left(r, \frac{1}{P(f)}\right) + Q\overline{N}(r,f) + \overline{\gamma}_p N_{k+l+1}\left(r, \frac{1}{f}\right) + S(r,f) \le$$
$$\le (k+m+Q+\overline{\gamma}_p(k+l+1)+1)T(r,f) + S(r,f). \tag{17}$$

and

$$\overline{N}(r, F_1) = \overline{N}(r, f^n P(f) H[f]) \le (Q+1)T(r,f) + S(r,f). \tag{18}$$

By Lemma 10, using (17) and (18), from (16) we obtain

$$(n+m-\overline{\gamma}_p)T(r,f) \le (k+m+2Q+\overline{\gamma}_p(k+l+1)+2)T(r,f) + S(r,f).$$

Which contradicts to $n > k + 2Q + \overline{\gamma}_p(k+l) + 2$. □

Lemma 14. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions, $P(w)$ and $H[f]$ be defined as in definitions (1) and (2) and k, n, m, $\overline{\gamma}_p$, Q be positive integers satisfying the inequality $n > m + 3\overline{\gamma}_p + Q + 2k + 1$, and let c_j $(j = 1, 2, ..., l)$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_j $(j = 1, 2, ..., l)$. If*

$$[f^n P(f)H[f]]^{(k)} \equiv [g^n P(g)H[g]]^{(k)},$$

then

$$f^n P(f)H[f] \equiv g^n P(g)H[g].$$

Proof. Let $[f^n P(f)H[f]]^{(k)} \equiv [g^n P(g)H[g]]^{(k)}$. Integrating above, k time we get

$$f^n P(f)H[f] \equiv g^n P(g)H[g] + Q(z),$$

where $Q(z)$ is a polynomial of degree at most $k-1$. If $R(z) \not\equiv 0$, This equation can be expressed as

$$\frac{f^n P(f)H[f]}{R} = \frac{g^n P(g)H[g]}{R} + 1.$$

Then from the above equation and Lemma 6, we have

$$T\left(r, \frac{f^n P(f)H[f]}{R}\right) \leq \overline{N}\left(r, \frac{f^n P(f)H[f]}{R}\right) + \overline{N}\left(r, \frac{R}{f^n P(f)H[f]}\right) +$$
$$+ \overline{N}\left(r, \frac{R}{g^n P(g)H[g]}\right) + S(r, f).$$

Using the above equation, we obtain

$$T(r, f^n P(f)H[f]) \leq \overline{N}\Big(r, f^n P(f)H[f]\Big) + \overline{N}\left(r, \frac{1}{f^n P(f)H[f]}\right) +$$
$$+ \overline{N}\left(r, \frac{1}{g^n P(g)H[g]}\right) + 2(k-1)\log r + S(r, f) \leq$$
$$\leq \overline{N}(r, f) + \overline{N}(r, H[f]) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right) +$$
$$+ \overline{N}\left(r, \frac{1}{H[f]}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) +$$
$$+ \overline{N}\left(r, \frac{1}{H[g]}\right) + 2(k-1)\log r + S(r, f).$$

Using the aforementioned equation and Lemma 10, we get

$$(n + m - \overline{\gamma}_p)T(r, f) \leq (m + Q + \overline{\gamma}_p + 2)T(r, f) + (m + \overline{\gamma}_p + 1)T(r, g) +$$
$$+ 2(k-1)\log r + S(r, f) + S(r, g). \tag{19}$$

Similarly, we obtain

$$(n + m - \overline{\gamma}_p)T(r, g) \leq (m + Q + \overline{\gamma}_p + 2)T(r, g) + (m + \overline{\gamma}_p + 1)T(r, f) +$$
$$+ 2(k-1)\log r + S(r, f) + S(r, g). \tag{20}$$

Since f and g are nonconstant, we have

$$T(r,f) \geq \log r + S(r,f), \qquad T(r,g) \geq \log r + S(r,g). \tag{21}$$

It follows from (19), (20) and (21) that

$$(n+m-\overline{\gamma}_p)\{T(r,f)+T(r,g)\} \leq$$
$$\leq (2k+2m+2\overline{\gamma}_p+Q+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which contradicts $n > m+3\overline{\gamma}_p+Q+2k+1$. Thus we have $Q(z) \equiv 0$ and therefore, we obtain

$$f^nP(f)H[f] \equiv g^nP(g)H[g].$$

□

Lemma 15. *Let $f(z)$ and $g(z)$ be two nonconstant finite-order meromorphic functions, let $c_j (j=1,2,...,l)$ be a nonzero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c_j, and let $k(>0)$ be an integer satisfying $n > k+1$. Also let $P(w)$ and $H[f]$ be defined as in definitions (1) and (2). Suppose that $a(z)$ ($\not\equiv 0,\infty$) is a small function with respect to $f(z)$ with finitely many zeros and poles. If*

$$[f^nP(f)H[f]]^{(k)} \equiv [g^nP(g)H[g]]^{(k)}, \quad [f^nP(f)H[f]] \equiv [g^nP(g)H[g]].$$

and, in addition, $f(z)$ and $g(z)$ share 1 IM, then $P(w)$ reduces to a nonzero monomial, namely, $P(w) = a_iw^i \not\equiv 0$ for some $i \in 0,1,...,m$.

Proof. Using the same reasoning as in Lemma 3.13 [9], we can easily obtain Lemma 15. □

4. Proof of the main results.

Proof of Theorem 3. Let F, G, H and V be defined as in (3), (4) and (5). We suppose that $F_1 = f^nP(f)H[f]$ and $G_1 = g^nP(g)H[g]$. By the assumption of the result, $[f^nP(f)H[f]]^{(k)}$ and $[g^nP(g)H[g]]^{(k)}$ share a small function $a(z)$ and 1 IM, hence F and G share the values 1 and ∞ IM. Suppose that $H \not\equiv 0$. It is easy to see that $F \not\equiv G$. We must have $V \not\equiv 0$. It follows from Lemmas 7 and 8 that

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+$$
$$+7\overline{N}(r,F)+S(r,F)+S(r,G). \tag{22}$$

By Lemma 4 with $s=2$, Lemma 3 and (22), we obtain

$$T(r,F_1) \leq N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+$$
$$+N_{k+2}\left(r,\frac{1}{F_1}\right)+7\overline{N}(r,F)+S(r,F)+S(r,G) \leq$$
$$\leq N_{k+2}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)+2N_{k+1}\left(r,\frac{1}{F_1}\right)+2k\overline{N}(r,F_1)+N_{k+1}\left(r,\frac{1}{G_1}\right)+$$
$$+k\overline{N}(r,G_1)+N_{k+2}\left(r,\frac{1}{F_1}\right)+7\overline{N}(r,F)+S(r,F)+S(r,G) \leq$$

$$\leq (k+2)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{P(g)}\right)+N_{k+2}\left(r,\frac{1}{H[g]}\right)+k(Q+1)\overline{N}(r,g)+$$
$$+2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{P(f)}\right)+2N_{k+1}\left(r,\frac{1}{H[f]}\right)+2k(Q+1)\overline{N}(r,f)+$$
$$+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{P(g)}\right)+N_{k+1}\left(r,\frac{1}{H[g]}\right)+k(Q+1)\overline{N}(r,g)+$$
$$+(k+2)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{P(f)}\right)+N_{k+2}\left(r,\frac{1}{H[f]}\right)+7(Q+1)\overline{N}(r,f)+$$
$$+S(r,f)+S(r,g)\leq$$
$$\leq (k+2)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{P(g)}\right)+Q\overline{N}(r,g)+\overline{\gamma}_pN_{k+l+2}\left(r,\frac{1}{g}\right)+$$
$$+k(Q+1)\overline{N}(r,g)+2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{P(f)}\right)+$$
$$+2Q\overline{N}(r,f)+2\overline{\gamma}_pN_{k+l+1}\left(r,\frac{1}{f}\right)+2k(Q+1)\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+$$
$$+N\left(r,\frac{1}{P(g)}\right)+Q\overline{N}(r,g)+\overline{\gamma}_pN_{k+l+1}\left(r,\frac{1}{g}\right)+k(Q+1)\overline{N}(r,g)+$$
$$+(k+2)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{P(f)}\right)+Q\overline{N}(r,f)+\overline{\gamma}_pN_{k+l+2}\left(r,\frac{1}{f}\right)+$$
$$+7(Q+1)\overline{N}(r,f)+S(r,f)+S(r,g).$$

Therefore, we have

$$T(r,F_1)\leq (3k+4)\overline{N}\left(r,\frac{1}{f}\right)+(2k+3)\overline{N}\left(r,\frac{1}{g}\right)+3N\left(r,\frac{1}{P(f)}\right)+2N\left(r,\frac{1}{P(g)}\right)+$$
$$+\overline{\gamma}_p(3k+3l+4)N\left(r,\frac{1}{f}\right)+\overline{\gamma}_p(2k+2l+3)N\left(r,\frac{1}{g}\right)+$$
$$+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,f)+S(r,f)+S(r,g).$$

By Lemma 10, the above inequality can be reduced as

$$\begin{aligned}(n+m-\overline{\gamma}_p)T(r,f)\leq{}&(3k+3m+\overline{\gamma}_p(3k+3l+4)+4)T(r,f)+\\&+(2k+2m+\overline{\gamma}_p(2k+2l+3)+3)T(r,g)+\\&+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,f)+S(r,f)+S(r,g).\end{aligned}\tag{23}$$

Similarly, we obtain

$$\begin{aligned}(n+m-\overline{\gamma}_p)T(r,g)\leq{}&(3k+3m+\overline{\gamma}_p(3k+3l+4)+4)T(r,g)+\\&+(2k+2m+\overline{\gamma}_p(2k+2l+3)+3)T(r,f)+\\&+\{(4k+7)(Q+1)+5Q\}\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned}\tag{24}$$

Combining (23) and (24), we obtain

$$(n+m-\overline{\gamma}_p)\{T(r,f)+T(r,g)\}\leq (5k+5m+\overline{\gamma}_p(5k+5l+7)+7)\{T(r,f)+T(r,g)\}+$$
$$+\{(4k+7)(Q+1)+5Q\}\{\overline{N}(r,f)+\overline{N}(r,g)\}+S(r,f)+S(r,g).$$

Thus we have

$$\begin{aligned}&(n-5k-4m-\overline{\gamma}_p(5k+5l+6)-7)\{T(r,f)+T(r,f)\}\leq\\&\leq 2((4k+7)(Q+1)+5Q)\overline{N}(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{25}$$

Since $V\not\equiv 0$ and F and G share the values 1 and ∞ IM, by Lemma 11, we obtain

$$(n+m+k-2Q-3)\overline{N}(r,f)\leq 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g). \tag{26}$$

By Lemma 4 with $s=1$, (26) takes from

$$\begin{aligned}&(n+m+k-2Q-3)\overline{N}(r,f)\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{P(f)}\right)+2N_{k+1}\left(r,\frac{1}{H[f]}\right)+\\&+2k\overline{N}(r,f)+2k\overline{N}(r,H[f])+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{P(g)}\right)+2N_{k+1}\left(r,\frac{1}{H[g]}\right)+\\&+2k\overline{N}(r,g)+2k\overline{N}(r,H[g])+S(r,f)+S(r,g)\leq\\&\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{P(f)}\right)+2Q\overline{N}(r,f)+2\overline{\gamma}_pN_{k+l+1}\left(r,\frac{1}{f}\right)+2k\overline{N}(r,f)+\\&+2k\overline{N}(r,H[f])+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{P(g)}\right)+2Q\overline{N}(r,g)+\\&+2\overline{\gamma}_pN_{k+l+1}\left(r,\frac{1}{g}\right)+2k\overline{N}(r,g)+2k\overline{N}(r,H[g])+S(r,f)+S(r,g)+\\&\leq 2(k+m+\overline{\gamma}_p(k+l+1)+1)T(r,f)+\\&+2(k+m+\overline{\gamma}_p(k+l+1)+1)T(r,g)+\{4k(Q+1)+4Q\}\overline{N}(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Thus we have

$$\begin{aligned}&(n+m-k(4Q+3)-6Q-3)\overline{N}(r,f)\leq 2(k+m+\overline{\gamma}_p(k+l+1)+1)\{T(r,f)+T(r,g)\}+\\&+S(r,f)+S(r,g).\end{aligned} \tag{27}$$

Since $\overline{N}(r,f)=\overline{N}(r,g)$, combining (25) and (27), it follows that

$$\begin{aligned}&(n-5k-4m-\overline{\gamma}_p(5k+5l+6)-7)(n+m-k(4Q+3)-6Q-3)-\\&-4((4k+7)(Q+1)+5Q)(k+m+\overline{\gamma}_p(k+l+1)+1)\{T(r,f)+T(r,g)\}\leq\\&\leq S(r,f)+S(r,g).\end{aligned}$$

Which contradicts

$$n>4m+4k(Q+2)+\overline{\gamma}_p(5k+5l+6)+8Q+11.$$

As in the proof of Lemma 5 applied to the functions F and G, we obtain the following cases:

(i) $T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,F)+S(r,F)+S(r,G)$,

(ii) $FG=1$,

(iii) $F\equiv G$.

By the condition imposed on n, case (i) is impossible. By Lemma 15, case (ii) is impossible. Hence, we get only the case (iii), i.e.,

$$[f^n P(f)H[f]]^{(k)} \equiv [g^n P(g)H[g]]^{(k)}.$$

Thus, by Lemma 14, we obtain $[f^n P(f)H[f]] \equiv [g^n P(g)H[g]]$, i.e.,

$$f^n\Big(a_m f^m + a_{m-1}f^{m-1} + ... + a_1 f + a_0\Big)\sum_{i=1}^{m} M_i[f] \equiv$$
$$\equiv g^n(a_m g^m + a_{m-1}g^{m-1} + ... + a_1 g + a_0)\sum_{i=1}^{m} M_i[g]. \tag{28}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (28), we deduce

$$[a_1 g^{n+m}(h^{n+m+\overline{\gamma}_p}-1)+a_{m-1}g^{n+m+\overline{\gamma}_p-1}(h^{n+m+\overline{\gamma}_p-1})+...+a_1 g^{n+\gamma}(h^{n+\sigma}-1)+a_0 g^n(h^n-1)]H[g].$$

Therefore, since g is nonconstant, so we must have $h^d = 1$, where

$$d = GCD(n+m+\overline{\gamma}_p, ..., n+m+\overline{\gamma}_p-1, ..., n+1)$$

and $a_{m-i} \neq 0$ for $i = 0, 1, ..., m$. Thus $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$.
If h is nonconstant, then $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + ... + a_0)H[w_1]-$$
$$-w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + ... + a_0)H[w_2].$$

□

Proof of Theorem 4. Substituting $a_1 = a_2 = ... = a_m = 0$ in $P(w)$ and proceeding as in the proof of Theorem 3, we complete the proof of Theorem 4. □

5. Some applications.

Application 1 (Application of uniqueness proofs). As a practical application of Theorem 3, one can establish a proof for the result presented in [15]. In their work, Xiao-Guang Qi, Lian-Zhong Yang, and Kai Liu demonstrated the uniqueness problems related to the difference polynomials of entire functions of the form $f^n f(z+c)$ sharing specific values. This uniqueness result is valid for polynomials with $n > 6$, assuming $f = e^z$ and $g = e^{-z}$. It is evident that $f^n f(z+c)$ and $g^n g(z+c)$ share a common 1-counting multiplicity for any positive integer n and constant c. Further, based on the findings of Kai Liu, Xin-Ling Liu, and Ting-Bin Cao in [16], the investigation delves into the zero distribution of derivatives of difference polynomials of entire functions sharing a common value. This analysis is conducted under the condition $[f^n f(z+c)]^{(k)}$ with $n \geq 2k+6$, considering functions $f(z) = c_e^{Cz}$ and $g(z) = c_2 e^{-Cz}$, where c_1, c_2, and C satisfy the condition

$$(-1)^k(c_1c_2)^n[(n+1)C]^{2k} = 1.$$

Additionally, in [3], the research focuses on the uniqueness problems related to difference-differential polynomials of finite-order meromorphic functions that share a small function (disregarding multiplicities). This investigation pertains to expressions such as

$$[f^n P(f) f(z+c)]^{(k)} \text{ and } [g^n P(g) g(z+c)]^{(k)},$$

which share the function $a(z)$ in terms of identity multiplicity (IM), while $f(z)$ and $g(z)$ share the value ∞ IM. These results are valid under the condition $n > 4m + 13k + 19$. It is worth noting that as an outcome of applying Theorem 3, these results can be easily extended to involve certain differential polynomials of degree γ. Furthermore, the variable Q is defined as the maximum value obtained from the sum of certain coefficients $n_{i0} + n_{i1} + ... + ln_{il}$, where it relates to the function

$$[f^n P(f) H[f]]^{(k)}$$

under the same conditions applied to $H[f]$.

Application 2 (Properties of meromorphic functions). The second application of Theorem 3 centers on exploring the properties of meromorphic functions, particularly those that satisfy specific differential or recurrence equations. This theorem provides a structured approach for comprehending how functions can be interrelated based on their derivatives and values at specific points.

Application 3. The obtained result presented in the proof can find a practical application in studying the relationships between meromorphic functions. Specifically, it deals with the equation

$$f^n(a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0) \sum_{i=1}^{m} M_i[f] \equiv$$
$$\equiv g^n(a_m g^m + a_{m-1} g^{m-1} + \ldots + a_1 g + a_0) \sum_{i=1}^{m} M_i[g].$$

Let us explore two scenarios:
(i) If the ratio of f to g is constant, denoted as h, then this equation implies that f is a constant multiple of g where t is a constant such that $t^d = 1$, and d is calculated as:

$$d = GCD(n + m + \overline{\gamma}_p, \ldots, n + m + \overline{\gamma}_p - 1, \ldots, n + 1)$$

This result shows that if h is constant, then $f(z)$ is proportional to $g(z)$.
(ii) If h is not constant, it means that $f(z)$ and $g(z)$ satisfy an algebraic difference equation denoted as $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \ldots + a_0) H[w_1] - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \ldots + a_0) H[w_2].$$

This analysis demonstrates the relationship between $f(z)$ and $g(z)$ under more general conditions.

In summary, the theorem provides insights into the possible relationships between meromorphic functions and the conditions under which they may be proportional or satisfy an algebraic difference equation, as expressed in the provided equation.

Application 4. The lemmas, such as Lemma 13, provide additional insights and conditions that contribute to the proof of the main result. For instance, Lemma 13 asserts that under certain conditions, a particular expression involving derivatives of a meromorphic function has infinitely many zeros.

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REFERENCES

1. W.K. Hayman, Meromorphic functions, Oxford University Press, 1964.
2. A.A. Goldberg, I.V. Ostrovskii, Value Distribution of Meromorphic Functions, American Mathematical Soc., **236**, 2008.
3. R.G Halburd, R.J Korhonen, *Difference analogue of the lemma on the* log *arithmic derivative with applications to difference equations*, J. Math. Anal. Appl., **314** (2006), 477–87.
4. R.G. Halburd, R.J. Korhonen, *Nevanlinna theory for the difference operator*, Annales Academiae Scientiarum Fennicae. Mathematica, **31** (2006), 463–478.
5. Y.F. Wang, *On Mues conjecture and Picard values*, Science in China, Ser. A, **36** (1993), 28–35.
6. M.L. Fang, *Uniqueness and value-sharing of entire functions*, Computers and Mathematics with Applications, **44** (2002), 828–831.
7. X.Y. Zhang, J.F. Chen, W.C. Lin, *Entire or meromorphic functions sharing one value*, Computers and Mathematics with Applications, **56** (2008), 1876–1883.
8. K. Liu, X.L. Liu, T.B. Cao, *Some results on zeros and uniqueness of difference-differential polynomials*, Appl. Math. (A Journal of Chinese Universities), **27** (2012), 94–104.
9. R.S. Dyavanal, M.M. Mathai, *Uniqueness of difference-differential polynomials of meromporphic functions*, Ukr. Math. J., **71** (2019), 1032–1042.
10. R.S. Dyavanal, S.B. Kalakoti, *Normality and uniqueness of homogeneous differential polynomials*, Mat. Stud., **59** (2023), 168–177.
11. P. Sahoo, G. Biswas, *Meromorphic function that share a set of small functions with its derivative*, Mat. Stud., **46** (2016), 130–136.
12. P. Sahoo, B. Saha, *Entire functions that share a polynomial with finite weight*, Mat. Stud., **44** (2015), 36–44.
13. A. Banerjee, S. Bhattacharyya, *Meromorphic function sharing three sets with its shift, q-shift, q-difference and analogous operators*, Mat. Stud., **52** (2019), 38–47.
14. A. Banerjee, A. Roy, *A note on power of meromorphic function and its shift operator of certain hyper-order sharing one small function and a value*, Mat. Stud., **55** (2021), 57–63.
15. Xiao-Guang Qi, Lian-Zhong Yang, Kai Liu, *Uniqueness and periodicity of meromorphic functions concerning the difference operator*, Computers and Mathematics with Applications, **60** (2010), 1739–1746.
16. Kai Liu, Xin-ling Liu, Ting-bin Cao, *Some results on zeros and uniqueness of difference-differential polynomials*, Appl. Math. (A Journal of Chinese Universities), **27** (2012), 94–104.
17. S. Mallick, *On the distribution of unique range sets and its elements over the extended complex plane*, Mat. Stud., **60** (2023), 40–54.
18. W. Bergweiler, J. K. Langley, *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philosoph. Soc., **142** (2007), 133–147.
19. Yik-Man Chiang, Shao-Ji Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, The Ramanujan Journal, **16** (2008), 105–129.
20. Xudan Luo, Wei-Chuan Lin, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl., **377** (2011), 441–449.
21. Hong Xun Yi, *Uniqueness of meromorphic functions and a question of C.C. Yang*, Complex Variables, Theory and Application: an International Journal, **14** (1990), 169–176.
22. Chung-Chun Yang, Xinhou Hua, *Uniqueness and value-sharing of meromorphic functions*, Annales Academiae Scientiarum Fennicae, Ser. A1, Math., **22** (1997), 395–406.

23. A. Banerjee, *Meromorphic functions sharing one value*, International J. Math. and Math. Sc., **22** (2005), 3587–3598.
24. Hong-Xun Yi, *Meromorphic functions that share three sets*, Kodai Math. J., **20** (1997), 22–32.
25. Molla Basir Ahamed, Goutam Haldar, *Uniqueness of difference-differential polynomials of meromorphic functions sharing a small function IM*, J. Anal., **30** (2022), 147–174.

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