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ON OPTIMIZATION OF CUBATURE FORMULAE FOR SOBOLEV CLASSES OF FUNCTIONS DEFINED ON STAR DOMAINS

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We find asymptotically optimal methods of recovery of the integration operator given values of the function at a finite number of points for a class of multivariate functions defined on a bounded star domain that have bounded in L_p norm of their distributional gradient. Thus we generalize the known solution of this optimization problem in the case, when the domain of the functions is convex. Let $Q\subset\mathbb{R}^d$, $d\in\mathbb{N}$, be a nonempty bounded open set. By $W^{1,p}(Q)$, $p\in[1,\infty]$, we denote the Sobolev space of functions $f\colon Q\to\mathbb{R}$ such that f and all their (distributional) partial derivatives of the first order belong to $L_p(Q)$. For $x=(x^1,\dots,x^d)\in\mathbb{R}^d$ and $q\in[1,\infty)$ set $|x|_q:=\Big(\sum_{k=1}^d|x^k|^q\Big)^{\frac1q}$, $|x|_\infty:=\max\{|x^k|\colon k\in\{1,\dots,d\}\}$, and $W^\infty_p(Q):=\{f\in W^{1,p}(Q)\colon \|\,|\nabla f|_1\,\|_{L_p(Q)}\le 1\}$, where $\nabla f=(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_d})$, $p\in[1,\infty]$. In particular we prove the following statement: Let $d\ge 2$, $p\in(d,\infty]$ and Q be a bounded star domain. Then $E_n\Big(W^\infty_p(Q)\Big)=c(d,p)\Big(\frac{\operatorname{meas}Q}{2^d}\Big)^{\frac1d+\frac1{p'}}\cdot\frac{1+o(1)}{n^{\frac1d}}$ $(n\to\infty)$, where $E_n(X):=\inf\Big\{\inf\big\{e(X,\Phi,x_1,\dots,x_n)\colon\ \Phi\colon\mathbb{R}^n\to\mathbb{R}\big\}\colon x_1,\dots,x_n\in Q\Big\}$, $e(X,\Phi,x_1,\dots,x_n):=\sup\Big\{\Big|\int\limits_Q f(x)dx-\Phi(f(x_1),\dots,f(x_n))\Big|\colon f\in X\Big\}$ for $X=W^\infty_p(Q)$, and $c(d,p)\in\mathbb{R}$ depends only on d and p.

1. Introduction. Let a bounded measurable set $Q\subset\mathbb{R}^d$, a class X of continuous on Q functions, and $n\in\mathbb{N}$ be given. An arbitrary function $\Phi\colon\mathbb{R}^n\to\mathbb{R}$ is called a method of recovery. For given points $x_1,\dots,x_n\in Q$ the error of recovery of the integral by the method Φ is

$$e(X,\Phi,x_1,\dots,x_n):=\sup\left\{\left|\int\limits_Q f(x)dx-\Phi(f(x_1),\dots,f(x_n))\right|\colon f\in X\right\}.$$

The problem of the optimal recovery of the integral is to find the best error of recovery

$$E_n(X):=\inf\Big\{\inf\big\{e(X,\Phi,x_1,\dots,x_n)\colon\ \Phi\colon\mathbb{R}^n\to\mathbb{R}\big\}\colon x_1,\dots,x_n\in Q\Big\},\tag{1}$$

the best method of recovery, and the best position of the informational set $x_1,\dots,x_n$ i.e., such method $\tilde\Phi\colon\mathbb{R}^n\to\mathbb{R}$ and points $\tilde x_1,\dots,\tilde x_n\in Q$, for which the infima in (1) are attained (if such a method and points exist).

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In many cases it is hard to find the best error of recovery and an optimal recovery method; in such situations it is interesting to find an asymptotically optimal method of recovery i.e., such sequence of methods $\Phi_n\colon \mathbb{R}^n \to \mathbb{R}$ and informational sets $\{x_1^n, \dots, x_n^n\}$, $n \in \mathbb{N}$, that

$$\lim_{n\to\infty} \frac{E_n(X)}{e(X, \Phi_n, x_1^n, \dots, x_n^n)} = 1.$$

The problem of optimal recovery and, in particular, the problem of optimization of cubature formulae has a rich history, see e.g. monographs [14]–[17].

Let $Q \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a nonempty bounded open set. By $W^{1,p}(Q)$, $p \in [1, \infty]$, we denote the Sobolev space of functions $f\colon Q \to \mathbb{R}$ such that f and all their (distributional) partial derivatives of the first order belong to $L_p(Q)$. As usually, for $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $q \in [1, \infty)$ set

$$|x|_q := \Big(\sum_{k=1}^d |x^k|^q\Big)^{\frac{1}{q}}, \qquad |x|_\infty := \max\{|x^k|\colon k \in \{1, \dots, d\}\}.$$

It is clear that for all $f \in W^{1,p}(Q)$ we have $\|\, |\nabla f|_1 \,\|_{L_p(Q)} < \infty$, where $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d})$. For $p \in [1, \infty]$ set

$$W_p^\infty(Q) := \{f \in W^{1,p}(Q)\colon \|\, |\nabla f|_1 \,\|_{L_p(Q)} \le 1\}.$$

Below we consider only the case $p > d \ge 2$. For $p > d$ imbedding of the class $W^{1,p}(Q)$ into the space of bounded continuous on Q functions holds, provided by some restrictions on the geometry of Q (sets Q for which the imbedding holds will be called admissible). For example, it is sufficient to require that Q satisfies the cone condition, see Chapter 4 and Theorem 4.12 in [1].

A bounded open set $Q \subset \mathbb{R}^d$ is called a star domain with respect to a ball B (or simply a star domain for brevity), if for all $x \in Q$ and $y \in B$ the segment xy belongs to Q. It is not hard to verify that the interior of the closure of a star domain Q coincides with Q. This implies that the closure of Q is an asymmetric star body according to [12, Definition I.2.2], hence its distance function is continuous (see [12, Theorem I.2.2]), and thus Q is Jordan measurable (see the proof of [12, Theorem I.1.5]). Moreover, it is easy to see that a star domain satisfies the cone condition, and hence each function f from $W_p^\infty(Q)$ has a continuous representation.

Everywhere below, for a finite set A, by $|A|$ we denote the number of its elements. For $x, y \in \mathbb{R}^d$ by (x, y) we denote the dot product of x and y. For a bounded set $Q \subset \mathbb{R}^d$ we set $\operatorname{diam} Q := \sup\{|a - b|_2\colon a, b \in Q\}$. We write $\operatorname{meas} Q$ to denote the Lebesgue measure of a measurable set Q.

The goal of the paper is to find asymptotically optimal cubature formulae on the class $W_p^\infty(Q)$, where Q is a bounded star domain. In the case, when Q is a convex bounded domain, such problem was solved in [2]. Solutions to some extremal problems for similar classes of functions can be found in [8, 9, 11, 6, 4, 3].

2. The main result.

2.1. Asymptotically optimal informational sets and methods. The construction of asymptotically optimal informational sets and recovery methods in the case when Q is a star domain is the same as in the case of a convex set Q. We adduce it below together with some properties that we will need, see [5, 2, 4].

For a given $h > 0$ consider the lattice Λ in $\mathbb{R}^d$ generated by the vectors $(2h, 0, 0, \dots, 0)$, $(0, 2h, 0, 0, \dots, 0), \dots,$ $(0, \dots, 0, 2h) \in \mathbb{R}^d$. Denote by $P_k(h)$, $k \in \mathbb{N}$, the cubes into which the lattice Λ divides $\mathbb{R}^d$; their volumes are equal to $(2h)^d$. Denote by $A(h)$ the set of all cubes $P_k(h)$ that are contained in Q, let $a(h)$ be the set of the centers of the cubes from

$A(h)$. Denote by $B(h)$ the set of all cubes $P_k(h)$ that have non-empty set of common with Q internal points. For each $n \in \mathbb{N}$ set

$$h_n := \frac{1}{2}\Big(\frac{\operatorname{meas} Q}{n}\Big)^{\frac{1}{d}}. \tag{2}$$

For each cube P from the set $B(3h_n)$ choose a point by the following rule: the center of the cube P, if it belongs to $a(h_n)$; else, arbitrary point from $P \cap a(h_n)$, if the intersection is not empty; else, arbitrary internal point of $Q \cap P$.

Denote by $S_1(n)$ the set of chosen points; by $S_2(n)$ arbitrary subset of the set $a(h_n)\setminus S_1(n)$ that contains $n - |S_1(n)|$ points (for large enough n this number is positive; if the set $a(h_n) \setminus S_1(n)$ contains less than $n - |S_1(n)|$ points, then we take all points of $a(h_n) \setminus S_1(n)$). Set

$$S(n) := S_1(n) \cup S_2(n). \tag{3}$$

Let $S(n) = \{x_1^*, \dots, x_{|S(n)|}^*\}$. For each $k \in \{1, \dots, |S(n)|\}$ define the set

$$V_k := \{x \in Q \cap P(3h_n; x_k^*)\colon |x - x_k^*|_\infty < |x - x_s^*|_\infty,\ s \neq k\},$$

where $P(3h_n; x_k^*)$ is the cube from $B(3h_n)$ that contains x_k^*. Then the sets V_k are pairwise disjoint, $\bigcup_{k=1}^{|S(n)|} V_k \subset Q$ and $\operatorname{meas}\Big(Q \setminus \bigcup_{k=1}^{|S(n)|} V_k\Big) = 0$. Set

$$c_k^* := \operatorname{meas} V_k, \qquad k \in \{1, \dots, |S(n)|\}. \tag{4}$$

The following lemma states some of the properties of the sets $S(n)$ and V_k, defined above, see [2, Lemma 4].

Lemma 1. *Let $Q \subset \mathbb{R}^d$ be a Jordan measurable set, $n \in \mathbb{N}$ be large enough and h_n be defined by (2). Then the following properties hold:*

1. *$S(n) \subset Q$ and $|S(n)| \le n$.*
2. *If $x \in V_k$ then $|x - x_k^*|_\infty \le 6h_n$.*
3. *For each cube $P \in B(h_n)$, $|P \cap S(n)| \le 1$.*
4. *Let R_n be the union of cubes $P \in A(h_n)$ with centers that belong to $S(n)$. Denote by $U_k := V_k \setminus R_n$. Then $\operatorname{meas}\bigcup_{k=1}^{|S(n)|} U_k = o(1)$, $n \to \infty$.*

2.2. Asymptotically optimal cubature formulae. The following theorem is the main result of the paper.

Theorem 1. *Let $d \ge 2$, $p \in (d, \infty]$ and a bounded star domain Q be given. Then*

$$E_n\Big(W_p^\infty(Q)\Big) = c(d,p)\Big(\frac{\operatorname{meas} Q}{2^d}\Big)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \quad n \to \infty,$$

where

$$c(d,p) := \frac{1}{d}\Big\| \frac{1}{|\cdot|_\infty^{d-1}} - |\cdot|_\infty \Big\|_{L_{p'}(\{x \in \mathbb{R}^d\colon |x|_\infty \le 1\})}.$$

The asymptotically optimal informational set is $S(n)$ defined by (3), and the optimal recovery method is

$$\tilde{\Phi}_n(f(x_1), \dots, f(x_{|S(n)|})) = \sum_{k=1}^{|S(n)|} c_k^* f(x_k),$$

where the weights c_k^* are defined by (4).

Remark 1. We note that the case $p = \infty$ was obtained (by different methods) in [5], see Theorem 3, in the case of an arbitrary Jordan measurable set Q.

It was proved, see [2, Lemma 5] that for arbitrary admissible Q

$$E_n\Big(W_p^\infty(Q)\Big) \geq c(d,p)\Big(\frac{\operatorname{meas} Q}{2^d}\Big)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \quad n \to \infty$$

and, see [2, Lemma 6]

$$E_n\Big(W_p^\infty(Q)\Big) \leq \sup_{f\in W_p^\infty(Q)} \left| \sum_{k=1}^{|S(n)|} \int\limits_{U_k} [f(x) - f(x_k^*)]dx \right| + c(d,p)\Big(\frac{\operatorname{meas} Q}{2^d}\Big)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}$$

as $n \to \infty$, where the sets U_k are defined in property 4 of Lemma 1. Thus in order to prove the theorem it is sufficient to prove the following lemma.

Lemma 2. *Let $d \in \mathbb{N}$, $p \in (d,\infty]$ and a bounded star domains Q be given. Then*

$$\sup_{f\in W_p^\infty(Q)} \sum_{k=1}^{|S(n)|} \int\limits_{U_k} \Big|f(x) - f(x_k^*)\Big|dx = o\Big(n^{-\frac{1}{d}}\Big), \quad n \to \infty. \tag{5}$$

A crucial tool in the proof of the main results is the following result, see [13, Ch. 6.9].

Lemma 3. *Suppose $p > d$ and $Q \subset \mathbb{R}^d$ is admissible. Let $f \in W^{1,p}(Q)$ and $x, y \in Q$ be such that the whole segment with ends at the points x and y belongs to Q. Then*

$$f(y) - f(x) = \int\limits_0^1 \Big(y - x, \nabla f[(1-t)x + ty]\Big)dt.$$

Observe that since the sets U_k, $k \in \{1,\dots,|S(n)|\}$ are generally speaking not convex, we can not directly apply Lemma 3 to the difference $f(x) - f(x_k^*)$ under the integral in (5). We define functions $p_k\colon U_k \to Q$, $k \in \{1,\dots,|S(n)|\}$, such that whole segments $xp_k(x)$ and $p_k(x)x_k^*$ belong to Q and they are "not much longer" than the segment xx_k^*. Once this is done we can write the inequality

$$|f(x) - f(x_k^*)| \leq |f(x) - f(p_k(x))| + |f(p_k(x)) - f(x_k^*)|,$$

and apply Lemma 3 to switch from the values of the function f to the values of its gradient. This will allow to obtain an estimate from above for the quantity (5) in terms of $\| \, |\nabla f|_1 \, \|_{L_p(Q)}$ and the total measure of the sets U_k, which in turn will imply Lemma 2.

3. Proof of the main result.

3.1. Auxiliary results. We need the following lemmas.

Lemma 4. *Let $T_k \subset U_k$ and functions $\phi_k\colon T_k \to Q$ be such that $\phi_k(T_k)$ is measurable, $k \in \{1,\dots,|S(n)|\}$. Assume that there exists a number $c > 0$ such that*

$$|\phi_k(x) - x_k^*|_\infty \leq c|x - x_k^*|_\infty$$

for all $x \in T_k$, $k \in \{1,\dots,|S(n)|\}$. Then there exists a number $C > 0$ that does not depend on n and such that for all integrable on Q functions g

$$\sum_{k=1}^{|S(n)|} \int\limits_{\phi_k(T_k)} |g(x)|dx \leq C \int\limits_Q |g(x)|dx.$$

Proof. If the numbers $1 \le k_1 < k_2 \le |S(n)|$ and the points $x \in T_{k_1}$ and $y \in T_{k_2}$ are such that $z = \phi_{k_1}(x) = \phi_{k_2}(y)$, then, using Property 2 of Lemma 1, we obtain

$$|x^*_{k_1} - x^*_{k_2}|_\infty \le |x^*_{k_1} - z|_\infty + |z - x^*_{k_2}|_\infty = |x^*_{k_1} - \phi_{k_1}(x)|_\infty + |\phi_{k_2}(y) - x^*_{k_2}|_\infty \le \le c|x^*_{k_1} - x|_\infty + c|y - x^*_{k_2}|_\infty \le 12h_n \cdot c. \tag{6}$$

It is now sufficient to prove that there exists a number $N \in \mathbb{N}$ that does not depend on n, and a partition of the set $\{T_1, \dots, T_{|S(n)|}\}$ into groups $\{T_{k^i_1}, \dots, T_{k^i_{m_i}}\}$, $i \in \{1, \dots, N\}$, such that for all $i \in \{1, \dots, N\}$ and different $j, s \in \{k^i_1, \dots, k^i_{m_i}\}$, $|x^*_j - x^*_s|_\infty > 12h_n \cdot c$. Really, if such partition is done, then, due to (6), the sets $\{\phi_{k^i_1}(T_{k^i_1}), \dots, \phi_{k^i_{m_i}}(T_{k^i_{m_i}})\}$ are pairwise disjoint for each $i \in \{1, \dots, N\}$. Hence we obtain

$$\sum_{k=1}^{|S(n)|} \int\limits_{\phi_k(T_k)} |g(x)|dx = \sum_{i=1}^{N} \sum_{s=1}^{m_i} \int\limits_{\phi_{k^i_s}(T_{k^i_s})} |g(x)|dx =$$

$$= \sum_{i=1}^{N} \int\limits_{\bigcup_{s=1}^{m_i} \phi_{k^i_s}(T_{k^i_s})} |g(x)|dx \le \sum_{i=1}^{N} \int\limits_{Q} |g(x)|dx = N \int\limits_{Q} |g(x)|dx.$$

To do such partition we associate an index $I_P \in \mathbb{Z}^d$ with each of the cubes $P \in B(h_n)$. The index I_P is equal to the coordinates of the "left bottom point" of P (i.e., the point of the cube P with minimal coordinates) in the basis of the lattice. Let M be an integer bigger than $6c + 1$. We divide all cubes from $B(h_n)$ into $N = M^d$ groups in such a way that two cubes $P_1, P_2 \in B(h_n)$ belong to the same group if and only if all the coordinates of $I_{P_1} - I_{P_2}$ are divisible by M.

Let two different cubes P_1, P_2 belong to one group and $x \in P_1$, $y \in P_2$. Then due to the definition of the group $|x - y|_\infty \ge (M - 1) \cdot 2h_n > 12ch_n$.

Now we construct a partition of the sets T_k, $k \in \{1, \dots, |S(n)|\}$. We put two sets T_k and T_j into one group if and only if x^*_k and x^*_j belong to cubes from the same group. Such partition is a desired one, since Property 3 of Lemma 1 holds. The lemma is proved. □

The following lemma follows from the so-called matrix determinant lemma (see [10, Lemma 1.1]), or from the Weinstein–Aronszajn identity; it can also be proved directly by induction on d. We omit the technical details.

Lemma 5. *Let two vectors $u = (u^1, \dots, u^d)$, $v = (v^1, \dots, v^d)$ and numbers $\alpha, \beta \in \mathbb{R}$ be given. Then*

$$\det\left(\alpha \cdot I + \beta \cdot \left\|u^i v^j\right\|_{i,j=1}^d\right) = \alpha^{d-1}(\alpha + \beta(u, v)),$$

where I denotes the identity matrix.

3.2. Auxiliary construction. Let Q be a star domain with respect to a ball $S^d_R(o)$ with the center $o \in Q$ and the radius larger than some positive number $R > 0$. For all $k \in \{1, \dots, |S(n)|\}$ and $0 < r \le R$ define a function $p_k(\cdot; r) \colon U_k \to \mathbb{R}^d$ by the following equation

$$p_k(x; r) := \frac{r \cdot (x + x^*_k) + |x - x^*_k|_2 \cdot o}{|x - x^*_k|_2 + 2r}. \tag{7}$$

Everywhere below we assume that the distance from o to the boundary ∂Q of the set Q is greater than R (otherwise we can decrease R). We also consider so large $n \in \mathbb{N}$ that all the points x_k^* (from V_k with non-empty U_k) and sets U_k are outside of the ball $S_R^d(o)$. As the value of r we will use either $r = R$, or $r = \frac{R}{8}$, so that r will be separated from 0 and independent of n.

Equality (7) has the following geometrical sense.

Lemma 6. *Assume that a point $x \in U_k$ is such that vectors $\overline{ox}$ and $\overline{ox_k^*}$ are not collinear. Consider the 2-dimensional space E^2 generated by these two vectors. Let $S_r^2(o)$ be the boundary circle of $E^2 \cap S_r^d(o)$. Let o_1o_2 be the diameter of the circle $S_r^2(o)$ parallel to the segment xx_k^*. Then $p_k(x;r)$ is the point where the diagonals of the convex hull of the points o_1, o_2, x and x_k^* intersect.*

This lemma will be proved below together with the following lemma.

Lemma 7. *Segments $xp_k(x)$ and $p_k(x)x_k^*$ are fully contained in Q and there exists a number C that does not depend on n such that*

$$\max\{|x - p_k(x;r)|_\infty, |x_k^* - p_k(x;r)|_\infty\} \leq C|x - x_k^*|_\infty.$$

Proof. Let p be the intersection of the diagonals xo_1 and $x_k^*o_2$. Then triangles xpx_k^* and o_1po_2 are similar. This, in particular, means that the points o, p and the middle m of the segment xx_k^* belong to a line and $\frac{|m-p|_2}{|o-p|_2} = \frac{|x-x_k^*|_2}{2r}$. Hence

$$p = \frac{1}{2r + |x - x_k^*|_2}(|x - x_k^*|_2 \cdot o + 2r \cdot m) = \frac{r \cdot (x + x_k^*) + |x - x_k^*|_2 \cdot o}{|x - x_k^*|_2 + 2r}.$$

This means that $p = p_k(x;r)$ and Lemma 6 is proved.

From the similarity of triangles xpx_k^* and o_1po_2 it also follows that $\frac{|x-p|_2}{|o_1-p|_2} = \frac{|x-x_k^*|_2}{2r}$. Hence

$$|x - p|_\infty \leq |x - p|_2 = \frac{|o_1 - p|_2 \cdot |x - x_k^*|_2}{2r} \leq \frac{\operatorname{diam} Q \cdot \sqrt{d} \cdot |x - x_k^*|_\infty}{2r}.$$

Hence the inequality in the statement of the lemma holds with the constant $C := \frac{\operatorname{diam} Q\sqrt{d}}{2r}$. The estimate for $|x_k^* - p_k(x,r)|_\infty$ can be obtained using the same arguments.

Segments $xp_k(x)$ and $p_k(x)x_k^*$ are fully inside Q, since the set Q is a star domain and they are subsegments of the segments xo_1 and $x_k^*o_2$. □

3.3. Some properties of the functions $p_k(\cdot;r)$. Below we state several properties of the functions $p_k(\cdot;r)$ defined in the previous paragraph. Everywhere in this paragraph we assume $k \in \{1, \dots, |S(n)|\}$ and $0 < r \leq R$ are fixed, and n is big enough, so that the sets U_k are outside the ball $S_R^d(o)$.

Lemma 8. *For each $t \in [0,1]$ there exists a partition $U_k = \bigcup_{i=1}^4 U_k^i(t)$ into measurable sets $U_k^i(t)$, $i \in \{1, \dots, 4\}$, such that the function*

$$\psi_k(x;r,t) := t \cdot x + (1-t) \cdot p_k(x;r) \tag{8}$$

is injective on each of the sets $U_k^1(t), \dots, U_k^4(t)$. For all $t \in (0,1]$ the functions

$$\varphi_k(x;r,t) := (1-t) \cdot x_k^* + t \cdot p_k(x;r), \tag{9}$$

are injective on each of the sets $U_k^1(0), \dots, U_k^4(0)$.

Proof. First of all note that for $x, y \in Q$ and $t \in (0, 1]$,

$$\varphi_k(x; r, t) = \varphi_k(y; r, t) \iff p_k(x; r) = p_k(y; r) \iff \psi_k(x; r, 0) = \psi_k(y; r, 0).$$

Thus the statement about functions (9) follows from the statement about functions (8).

Let $t \in [0, 1]$ and $q \in \psi_k(U_k; r, t)$ be fixed. Next we prove that the preimage $\psi_k^{-1}(q; r, t)$ consists of at most 4 points.

If the points q, o and x_k^* belong to one line (i.e., are linearly dependent), then all the points from the set $\psi_k^{-1}(q; r, t)$ also belong to this line and the proof of the lemma is analogous to the that below for the case when the points q, o and x_k^* are linearly independent.

Assume that the points q, o and x_k^* are linearly independent. Then they generate a 2-dimensional space. From the geometrical sense of $p_k(\cdot; r)$ it follows that the set $\psi_k^{-1}(q; r, t)$ is a subset of this 2-dimensional space. Let

$$x \in \psi_k^{-1}(q; r, t). \tag{10}$$

Consider a 2-dimensional Cartesian coordinate system in it such that the point o is on the ordinate axis and the point x and the point that is symmetric to the point x_k^* with respect to o are on the abscissa axis (to determine the coordinate system uniquely we can additionally require the point x_k^* to be in the upper half-plane with respect to the abscissa axis). Let $x_k^* = (x_*, 2y_*)$, $q = (x_q, y_q)$, and $x = (x_1, 0)$ in this coordinate system. Then $o = (0, y_*)$. By (7) and the definition of the function ψ_k we have that for arbitrary point $z = (x_z, y_z)$

$$\psi_k(z; r, t) = (1-t)\frac{r(x_z + x_*, y_z + 2y_*) + (0, y_*|x_k^* - z|_2)}{2r + |x_k^* - z|_2} + t(x_z, y_z) =$$
$$= \Big((1-t)\frac{r(x_z + x_*)}{2r + |x_k^* - z|_2} + tx_z, (1-t)y_* + (1-t)\frac{ry_z}{2r + |x_k^* - z|_2} + ty_z\Big).$$

From (10) we obtain that $y_q = (1-t)y_*$; hence $\psi_k^{-1}(q; r, t)$ is a subset of the abscissa axis, and for all $z = (x_z, 0) \in \psi_k^{-1}(q; r, t)$ we have

$$(1-t)\frac{r(x_z + x_*)}{2r + \sqrt{(x_* - x_z)^2 + 4y_*^2}} + tx_z = x_q.$$

All the solutions of the latter equation are also solutions of the equation

$$\Big[(1-t)r(x_z + x_*) - 2r(x_q - tx_z)\Big]^2 = (x_q - tx_z)^2((x_* - x_z)^2 + 4y_*^2),$$

which is an equation of not more than degree 4 with respect to the unknown x_z; hence the set $\psi_k^{-1}(q; r, t)$ contains at most 4 points.

Finally, we construct the required partition of the set U_k. Consider the function $\psi_k(\cdot; r, t)$ as a function defined on the closure $\overline{U_k}$ of U_k. It is easy to see that ψ_k is continuous, and hence by [7, Theorem 6.9.7] there exists a Borel set $B \subset \overline{U_k}$ such that $\psi_k(B; r, t) = \psi_k(\overline{U_k}; r, t)$ and $\psi_k(\cdot; r, t)$ is injective on B. Set $U_k^1 = B \cap U_k$. The set $\psi_k^{-1}(q; r, t) \setminus U_k^1$ consists of at most 3 points for any $q \in \psi_k(U_k; r, t)$. Repeating the same arguments we obtain measurable sets $U_k^2 \subset U_k \setminus U_k^1$ and $U_k^3 \subset U_k \setminus (U_k^1 \cup U_k^2)$ such that ψ_k is injective on each of them, and is injective on the measaruble set $U_k^4 := U_k \setminus (U_k^1 \cup U_k^2 \cup U_k^3)$. □

Below for a function $\phi\colon \mathbb{R}^d \to \mathbb{R}^d$ by $\frac{D\phi}{Dx}$ we denote its Jacobian matrix, $J_\phi(x) := \det \frac{D\phi}{Dx}(x)$ and I is the identity matrix. For a point or a vector $y \in \mathbb{R}^d$ and $s \in \{1, \dots, d\}$ by y^s we denote its s-th coordinate.

Lemma 9. *Assume a point $x \in U_k$ is given. Let m be the middle of the segment xx_k^*, $\overline{\Delta x} = \frac{\overline{x_k^* x}}{|x - x_k^*|_2}$. Then*

$$\frac{Dp_k(\cdot; r)}{Dx}(x) = \frac{r}{2r + |x - x_k^*|_2} I + \frac{2r}{(2r + |x - x_k^*|_2)^2} \cdot \left\| \overline{mo}^i \cdot \overline{\triangle x}^j \right\|_{i,j=1}^d.$$

Proof. Let $x = (x^1, \dots, x^d)$, $x_k^* = (x_k^1, \dots, x_k^d)$, $o = (o^1, \dots, o^d)$. Let also $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. Equality (7) can be rewritten in the coordinates

$$p_k(x; r) := \frac{r \cdot (x^1 + x_k^1, \dots, x^d + x_k^d) + \sqrt{\sum_{s=1}^d (x^s - x_k^s)^2} \cdot (o^1, \dots, o^d)}{\sqrt{\sum_{s=1}^d (x^s - x_k^s)^2} + 2r},$$

$$\frac{\partial p_k(x; r)^i}{\partial x^j} = \frac{(r\delta_{ij} + \frac{x^j - x_k^j}{|x - x_k^*|_2} o^i) \cdot (2r + |x - x_k^*|_2) - (r(x^i + x_k^i) + |x - x_k^*|_2 o^i) \frac{x^j - x_k^j}{|x - x_k^*|_2}}{(2r + |x - x_k^*|_2)^2} =$$

$$= \frac{r(2r + |x - x_k^*|_2)\delta_{ij} + 2r \frac{x^j - x_k^j}{|x - x_k^*|_2} o^i - r \frac{x^j - x_k^j}{|x - x_k^*|_2}(x^i + x_k^i)}{(2r + |x - x_k^*|_2)^2} =$$

$$= \frac{r}{2r + |x - x_k^*|_2} \delta_{ij} + \frac{2r}{(2r + |x - x_k^*|_2)^2} \cdot \frac{x^j - x_k^j}{|x - x_k^*|_2} \cdot \left(o^i - \frac{x^i + x_k^i}{2}\right).$$

□

From Lemmas 9 and 5 we immediately get the following lemma.

Lemma 10. *Let the function $\varphi_k(x; r, t)$ be defined in* (8)*. Then for all $t \in [0, 1]$*

$$J_{\varphi_k(\cdot; r, t)}(x) = t^d \frac{r^d}{(2r + |x - x_k^*|_2)^{d+1}} \left(2r + |x - x_k^*|_2 + 2\left(\overline{mo}, \overline{\triangle x}\right)\right).$$

Lemma 11. *Let the function $\psi_k(x; r, t)$ be defined in* (8)*. Then for all $t \in [0, 1]$*

$$J_{\psi_k(\cdot; r, t)}(x) =$$

$$= \left(\frac{r(1 - t)}{2r + |x - x_k^*|_2} + t\right)^{d-1} \cdot \left(\frac{2r^2 + 3r|x - x_k^*|_2 + |x - x_k^*|_2^2 - 2r\left(\overline{mo}, \overline{\triangle x}\right)}{(2r + |x - x_k^*|_2)^2}(t - 1) + 1\right).$$

Proof. To prove this lemma it is sufficient to apply Lemmas 9 and 5 and notice that

$$\frac{r(1 - t)}{2r + |x - x_k^*|_2} + t + \frac{2r(1 - t)}{(2r + |x - x_k^*|_2)^2}\left(\overline{mo}, \overline{\triangle x}\right) =$$

$$= \frac{2r^2 + 3r|x - x_k^*|_2 + |x - x_k^*|_2^2 - 2r\left(\overline{mo}, \overline{\triangle x}\right)}{(2r + |x - x_k^*|_2)^2}(t - 1) + 1.$$

□

The next lemma justifies possibility to make a substitution $y = \psi_k(x; r, t)$ in the integrals considered below.

Lemma 12. *For each fixed* $0 < r < R$ *the function* $\psi_k(x; r, t)$, $x \in U_k$, $t \in [0, 1]$ *is continuous on* $U_k \times [0, 1]$. *The set* $\Theta := \{(x, t) \in U_k \times [0, 1] \colon J_{\psi_k(\cdot; r,t)}(x) = 0\}$ *has measure zero in* $\mathbb{R}^{d+1}$.

Proof. Continuity of ψ_k follows from the definition. The set Θ is a piece of the plot of the function

$$t(x) = 1 - \frac{(2r + |x - x_k^*|_2)^2}{2r^2 + 3r|x - x_k^*|_2 + |x - x_k^*|_2^2 - 2r\Big(\overline{mo}, \overline{\triangle x}\Big)}, \quad x \in U_k$$

on which $t(x) \in [0, 1]$. There exists $\varepsilon > 0$ such that

$$\Theta \cap \Big\{(x, t) \in U_k \times [0, 1] \colon \Big|2r^2 + 3r|x - x_k^*|_2 + |x - x_k^*|_2^2 - 2r\Big(\overline{mo}, \overline{\triangle x}\Big)\Big| < \varepsilon\Big\} = \varnothing.$$

Thus on the set $\{x \in U_k \colon t(x) \in [0, 1]\}$ the function t is uniformly continuous, and hence its plot has zero measure. □

In the next two paragraphs we use the following notation. The symbol C stands for a positive number that does not depend on n. This number may be different in left and right parts of equality or inequality.

3.4. Estimate for $\sum\limits_{k=1}^{|S(n)|} \int\limits_{T_k} |f(x) - f(p_k(x; r))| dx$.

Lemma 13. *Assume* $T_k \subset U_k$, $k \in \{1, \ldots, |S(n)|\}$, *are measurable sets and*

$$\max_{k\in\{1,\ldots,|S(n)|\}} \operatorname{diam} U_k \le r \le R. \tag{11}$$

Then there exists a number C *that does not depend on* n *and such that for all* $f \in W^{1,p}(Q)$

$$\sum_{k=1}^{|S(n)|} \int\limits_{T_k} |f(x) - f(p_k(x; r))| dx \le C h_n \Big(\operatorname{meas} \bigcup_{k=1}^{|S(n)|} T_k\Big)^{\frac{1}{p'}} \| \, |\nabla f|_1 \, \|_{L_p(Q)}.$$

Proof. Using Lemma 3 and Lemma 7 we obtain

$$\sum_{k=1}^{|S(n)|} \int\limits_{T_k} |f(x) - f(p_k(x; r))| dx \le \sum_{k=1}^{|S(n)|} \int\limits_0^1 \int\limits_{T_k} |p_k(x; r) - x|_\infty |\nabla f((1 - t)p_k(x; r) + tx)|_1 dx dt \le$$

$$\le C h_n \sum_{k=1}^{|S(n)|} \int\limits_0^1 \int\limits_{T_k} |\nabla f(\psi_k(x; r, t))|_1 dx dt, \tag{12}$$

where the functions $\psi_k(\cdot; r, t)$ are defined in (8). Due to Lemma 8 for each $t \in [0, 1]$ and $k \in \{1, \ldots, |S(n)|\}$ there exists a partition $T_k = \bigcup_{i=1}^4 T_k^i(t)$ such that the function $\psi_k(\cdot; r, t)$ is injective on each of $T_k^i(t)$, $i \in \{1, \ldots, 4\}$. Hence for all $k \in \{1, \ldots, |S(n)|\}$

$$\int\limits_0^1 \int\limits_{T_k} |\nabla f(\psi_k(x; r, t))|_1 dx dt = \sum_{i=1}^4 \int\limits_0^1 \int\limits_{T_k^i(t)} |\nabla f(\psi_k(x; r, t))|_1 dx dt, \tag{13}$$

and for each $i \in \{1, \dots, 4\}$, making a substitution $y = \psi_k(x; r, t)$ in the internal integral, and applying Holder's inequality, we can write

$$\int_0^1 \int_{T_k^i(t)} |\nabla f(\psi_k(x;r,t))|_1 dxdt = \int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |\nabla f(y)|_1 |J_{\psi_k^{-1}(\cdot;r,t)}(y)| dydt \leq$$

$$\leq \left(\int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |\nabla f(y)|_1^p dydt \right)^{\frac{1}{p}} \left(\int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |J_{\psi_k^{-1}(\cdot;r,t)}(y)|^{p'} dydt \right)^{\frac{1}{p'}}. \tag{14}$$

Applying the inverse function theorem and making a substitution $x = \psi_k^{-1}(y; r, t)$, we obtain

$$\int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |J_{\psi_k^{-1}(\cdot;r,t)}(y)|^{p'} dydt = \int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |J_{\psi_k(\cdot;r,t)}(\psi_k^{-1}(y;r,t))|^{-p'} dydt =$$

$$= \int_0^1 \int_{T_k^i(t)} |J_{\psi_k(\cdot;r,t)}(x)|^{1-p'} dxdt \leq \int_{T_k} \int_0^1 |J_{\psi_k(\cdot;r,t)}(x)|^{1-p'} dtdx. \tag{15}$$

Since $p > d \geq 2$ we have $\frac{1}{p'} = 1 - \frac{1}{p} > \frac{1}{2}$ and hence $0 \geq 1 - p' > -1$. Note that for all $t \in [0, 1]$

$$\frac{r(1-t)}{2r + |x - x_k^*|_2} + t = \frac{r(1+t) + t|x - x_k^*|_2}{2r + |x - x_k^*|_2} > \frac{r}{2r + r} = \frac{1}{3},$$

$$-\frac{\operatorname{diam} Q}{2r} \leq \frac{2r^2 + 3r|x - x_k^*|_2 + |x - x_k^*|_2^2 - 2r\left(\overline{mo}, \overline{\triangle x}\right)}{(2r + |x - x_k^*|_2)^2} \leq$$

$$\leq \frac{2r^2 + 3r^2 + r^2 + 2r \cdot \operatorname{diam} Q}{4r^2} = \frac{3}{2} + \frac{\operatorname{diam} Q}{2r}.$$

Using Lemma 11, the latter inequalities and $0 \geq 1 - p' > -1$ we obtain

$$\int_0^1 |J_{\psi_k(\cdot;r,t)}(x)|^{1-p'} dt \leq 3^{(d-1)(p'-1)} \sup_A \int_0^1 (A(t-1) + 1)^{1-p'} dt \leq C, \tag{16}$$

where the supremum is taken over the segment

$$A \in \left[-\frac{\operatorname{diam} Q}{2r}, \frac{3}{2} + \frac{\operatorname{diam} Q}{2r} \right]$$

and the number C is finite and depends on $\operatorname{diam} Q$ and r and does not depend on h_n. Using (16), (15), (14), (13) and (12), we deduce

$$\sum_{k=1}^{|S(n)|} \int_{T_k} |f(x) - f(p_k(x;r))| dx \leq C h_n \sum_{i=1}^{4} \sum_{k=1}^{|S(n)|} \left(\int_0^1 \int_{\psi_k(T_k^i(t);r,t)} |\nabla f(y)|_1^p dydt \right)^{\frac{1}{p}} (\operatorname{meas} T_k)^{\frac{1}{p'}} \leq$$

$$\leq C h_n \sum_{i=1}^{4} \left(\int_0^1 \sum_{k=1}^{|S(n)|} \int_{\psi_k(T_k^i(t);r,t)} |\nabla f(y)|_1^p dydt \right)^{\frac{1}{p}} \left(\sum_{k=1}^{|S(n)|} \operatorname{meas} T_k \right)^{\frac{1}{p'}}. \tag{17}$$

Note that for all $t \in [0,1]$ and each $i \in \{1,\dots,4\}$ the sets $T_k^i(t)$ and the functions $\psi_k(\cdot;r,t)$ satisfy the conditions of Lemma 4, because for all $x \in T_k^i(t)$ due to Lemma 7

$$|x_k^* - \psi_k(x;r,t)|_\infty = |x_k^* - (1-t)p_k(x;r) - tx|_\infty \le t|x_k^* - x|_\infty + \\ +(1-t)|x_k^* - p_k(x;r)|_\infty \le |x_k^* - x|_\infty + |x_k^* - p_k(x;r)|_\infty \le C|x_k^* - x|_\infty$$

with some number C independent of n (and t). To finish the proof of the lemma it is sufficient to apply Lemma 4 to (17). □

3.5. Estimate for $\sum_{k=1}^{|S(n)|}\int_{T_k}|f(x_k^*) - f(p_k(x;r))|dx$.

Lemma 14. *Assume* (11) *holds, and measurable sets $T_k \subset U_k$, $k \in \{1,\dots,|S(n)|\}$, are such that there exists a number $c > 0$ that does not depend on n for which $|J_{\varphi_k(\cdot;r,1)}(x)| > c$ for all $x \in T_k$. Then there exists a number C that does not depend on n, such that for all $f \in W^{1,p}(Q)$*

$$\sum_{k=1}^{|S(n)|}\int_{T_k}|f(x_k^*) - f(p_k(x;r))|dx \le Ch_n\Big(\operatorname{meas}\bigcup_{k=1}^{|S(n)|}T_k\Big)^{\frac{1}{p'}}\|\,|\nabla f|_1\,\|_{L_p(Q)}.$$

Proof. We may assume that each function $\varphi_k(\cdot;r,t)$ is injective on T_k, $k \in \{1,\dots,|S(n)|\}$. Otherwise, due to Lemma 8, we can divide each of the sets T_k into four subsets, so that the functions $\varphi_k(\cdot;r,t)$ are injective on each of the subsets, and apply arguments below to each of the subsets. Using Lemma 3 we obtain

$$\sum_{k=1}^{|S(n)|}\int_{T_k}|f(p_k(x;r)) - f(x_k^*)|dx \le \sum_{k=1}^{|S(n)|}\int_0^1\int_{T_k}|p_k(x;r) - x_k^*|_\infty|\nabla f(\varphi_k(x;r,t))|_1 dxdt \le$$

$$\le \sum_{k=1}^{|S(n)|}\int_0^1\left(\int_{T_k}|p_k(x;r) - x_k^*|_\infty^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{T_k}|\nabla f(\varphi_k(x;r,t))|_1^p dx\right)^{\frac{1}{p}}dt \le$$

$$\overset{\text{by Lemma 7}}{\le} Ch_n\int_0^1\sum_{k=1}^{|S(n)|}(\operatorname{meas}T_k)^{\frac{1}{p'}}\left(\int_{T_k}|\nabla f(\varphi_k(x;r,t))|_1^p dx\right)^{\frac{1}{p}}dt \le$$

$$\le Ch_n\left(\sum_{k=1}^{|S(n)|}\operatorname{meas}T_k\right)^{\frac{1}{p'}}\int_0^1\left(\sum_{k=1}^{|S(n)|}\int_{T_k}|\nabla f(\varphi_k(x;r,t))|_1^p dx\right)^{\frac{1}{p}}dt =$$

$$= Ch_n\left(\sum_{k=1}^{|S(n)|}\operatorname{meas}T_k\right)^{\frac{1}{p'}}\int_0^1\left(\sum_{k=1}^{|S(n)|}\int_{\varphi_k(T_k;r,t)}|\nabla f(y)|_1^p|J_{\varphi_k^{-1}(\cdot;r,t)}(y)|dy\right)^{\frac{1}{p}}dt =$$

$$= Ch_n\left(\sum_{k=1}^{|S(n)|}\operatorname{meas}T_k\right)^{\frac{1}{p'}}\int_0^1\left(\sum_{k=1}^{|S(n)|}\int_{\varphi_k(T_k;r,t)}\frac{|\nabla f(y)|_1^p dy}{|J_{\varphi_k(\cdot;r,t)}(\varphi_k^{-1}(y;r,t))|}\right)^{\frac{1}{p}}dt.$$

Due to Lemma 10 for all $k \in \{1,\dots,|S(n)|\}$, $t \in [0,1]$ and $x \in T_k$, $J_{\varphi_k(\cdot;r,t)}(x) = t^d J_{\varphi_k(\cdot;r,1)}(x)$. Hence, using the conditions of the lemma we get

$$\sum_{k=1}^{|S(n)|}\int_{T_k}|f(p_k(x;r)) - f(x_k^*)|dx \le$$

$$\leq Ch_n\Bigg(\sum_{k=1}^{|S(n)|}\operatorname{meas}T_k\Bigg)^{\frac{1}{p'}}\int_0^1 t^{-\frac{d}{p}}\Bigg(\sum_{k=1}^{|S(n)|}\int_{\varphi_k(T_k;r,t)}\frac{|\nabla f(y)|_1^p dy}{|J_{\varphi_k(\cdot;r,1)}(\varphi_k^{-1}(y;r,t))|}\Bigg)^{\frac{1}{p}}dt\leq$$

$$\leq Ch_n\Bigg(\sum_{k=1}^{|S(n)|}\operatorname{meas}T_k\Bigg)^{\frac{1}{p'}}\int_0^1 t^{-\frac{d}{p}}\Bigg(\sum_{k=1}^{|S(n)|}\int_{\varphi_k(T_k;r,t)}\frac{|\nabla f(y)|_1^p dy}{c}\Bigg)^{\frac{1}{p}}dt.$$

Note that the sets T_k and the functions $\varphi_k(\cdot;r,t)$ satisfy the conditions of Lemma 4, since by Lemma 7 for all $k\in\{1,\dots,|S(n)|\}$, $t\in[0,1]$, and $x\in T_k$

$$|x_k^*-\varphi_k(x;r,t)|_\infty=t|p_k(x;r)-x_k^*|_\infty\leq|p_k(x;r)-x_k^*|_\infty\leq C|x-x_k^*|_\infty.$$

To finish the proof of the lemma it is sufficient to apply Lemma 4 and recall that $p>d$, and hence the integral $\int_0^1 t^{-\frac{d}{p}}dt$ converges. □

3.6. Proof of Lemma 2. For each $k=1,\dots,|S(n)|$, we divide the set U_k into three subsets $W_k^1:=\{x\in U_k\colon(\overline{mo},\overline{\triangle x})<-2R\}$, $W_k^2:=\{x\in U_k\colon(\overline{mo},\overline{\triangle x})\in[-2R,-\frac{R}{2}]\}$ and $W_k^3:=\{x\in U_k\colon(\overline{mo},\overline{\triangle x})>-\frac{R}{2}\}$.

Let n be so large that for all $k\in\{1,\dots,|S(n)|\}$ and $x\in U_k$, $|x-x_k^*|_2<\frac{R}{8}$. For all $x\in W_k^1$ we have $2R+|x-x_k^*|_2+2(\overline{mo},\overline{\triangle x})<2R+\frac{R}{8}-4R<-R$, for all $x\in W_k^3$ we have $2R+|x-x_k^*|_2+2(\overline{mo},\overline{\triangle x})>2R-R=R$, and hence by Lemma 10 for all $x\in W_k^1\cup W_k^3$ we have

$$\Big|J_{\varphi_k(\cdot;R,1)}(x)\Big|=\frac{R^d}{(2R+|x-x_k^*|_2)^{d+1}}\Big|2R+|x-x_k^*|_2+2\Big(\overline{mo},\overline{\triangle x}\Big)\Big|>\frac{R^d}{(3R)^{d+1}}R=\frac{1}{3^{d+1}}.$$

For all $x\in W_k^2$ we have $2\cdot\frac{R}{8}+|x-x_k^*|_2+2(\overline{mo},\overline{\triangle x})<\frac{2R}{8}+\frac{R}{8}-R<-\frac{R}{2}$, and hence by Lemma 10 for all $x\in W_k^2$ we have

$$\Big|J_{\varphi_k\left(\cdot;\frac{R}{8},1\right)}(x)\Big|=\frac{\left(\frac{R}{8}\right)^d}{\left(\frac{2R}{8}+|x-x_k^*|_2\right)^{d+1}}\Big|\frac{2R}{8}+|x-x_k^*|_2+2\Big(\overline{mo},\overline{\triangle x}\Big)\Big|>$$

$$>\frac{8R^d}{(2R+R)^{d+1}}\cdot\frac{R}{2}=\frac{4}{3^{d+1}}.$$

Finally let us return to the proof of (5).

$$\sum_{k=1}^{|S(n)|}\int_{U_k}\Big|f(x)-f(x_k^*)\Big|dx=\sum_{k=1}^{|S(n)|}\int_{W_k^1\cup W_k^3}\Big|f(x)-f(x_k^*)\Big|dx+\sum_{k=1}^{|S(n)|}\int_{W_k^2}\Big|f(x)-f(x_k^*)\Big|dx\leq$$

$$\leq\sum_{k=1}^{|S(n)|}\int_{W_k^1\cup W_k^3}\Big|f(x)-f(p_k(x;R))\Big|dx+\sum_{k=1}^{|S(n)|}\int_{W_k^1\cup W_k^3}\Big|f(p_k(x;R))-f(x_k^*)\Big|dx+$$

$$+\sum_{k=1}^{|S(n)|}\int_{W_k^2}\Big|f(x)-f\Big(p_k\Big(x;\frac{R}{8}\Big)\Big)\Big|dx+\sum_{k=1}^{|S(n)|}\int_{W_k^2}\Big|f\Big(p_k\Big(x;\frac{R}{8}\Big)\Big)-f(x_k^*)\Big|dx.$$

To prove (5) we need to apply Lemmas 13, 14 and note that $\operatorname{meas}\bigcup_{k=1}^{|S(n)|}U_k=o(1)$ as $n\to\infty$ due to property 4 of Lemma 1. □

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