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# FREE PRODUCTS OF CYCLIC GROUPS IN GROUPS OF INFINITE UNITRIANGULAR MATRICES

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Groups of infinite unitriangular matrices over associative unitary rings are considered. These groups naturally act on infinite dimensional free modules over underlying rings. They are profinite in case underlying rings are finite. Inspired by their connection with groups defined by finite automata the problem to construct faithful representations of free products of groups by banded infinite unitriangular matrices is considered.

For arbitrary prime $p$ a sufficient conditions on a finite set of banded infinite unitriangular matrices over unitary associative rings of characteristic $p$ under which they generate the free product of cyclic $p$-groups is given. The conditions are based on certain properties of the actions on finite dimensional free modules over underlying rings.

It is shown that these conditions are satisfied. For arbitrary free product of finite number of cyclic $p$-groups constructive examples of the sets of infinite unitriangular matrices over unitary associative rings of characteristic $p$ that generate given free product are presented. These infinite matrices are constructed from finite dimensional ones that are nilpotent Jordan blocks.

A few open questions concerning properties of presented examples and other types of faithful representations are formulated.

**1. Introduction.** Groups of infinite unitriangular matrices is a natural and interesting direction in modern algebra. The interest to them was raised due to at least two factors. The first is the possibility to construct in a relatively direct way free non-abelian groups generated by infinite unitriangular matrices with a simple structure defined by specially chosen finite dimensional matrices ([4, 1, 9]). The second argument is a close relation between groups defined by finite automata over an alphabet endowed with a structure of associative unitary ring and groups of infinite unitriangular matrices over this ring (see details in [5, 8]).

In this paper we mainly concentrate on the development of the first direction. Since free products are rich of free subgroups it is natural to describe concrete constructions of such products as subgroups of infinite unitriangular matrices. To continue and extends results of [6] for a unitary associative ring $R$ of prime characteristic $p$ and arbitrary finite tuple of powers of $p$ we construct finite sets of infinite unitriangular matrices over $R$ of a very simple structure such that they generate the free product of cyclic $p$-groups of given orders.

The paper is organized as follows. In Section  we recall required definitions and properties on infinite unitriangular matrices over associative unitary rings. In Section  we give a sufficient condition on a finite set of infinite unitriangular matrices over a unitary associative ring

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of prime characteristic $p$ under which this set generates a free product of cyclic $p$-groups. We construct explicit examples of such sets in Section  and conclude the paper with a few open questions in Section .

**2. Infinite unitriangular matrices.** Let $R$ be an associative unitary ring. Denote by $R^\infty$ the right $R$-module of all sequences over $R$.

An *infinite (upper) unitriangular matrix* over $R$ is a matrix

$$A = (a_{ij})_{i,j=1}^{\infty}, \quad a_{ij} \in R, i, j \geq 1,$$

such that $a_{ij} = 0$ provided $i > j \geq 1$ and $a_{ii} = 1$ for all $i \geq 1$. In other words such a matrix has the form

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \dots \\ 0 & 1 & a_{23} & a_{24} & \dots \\ 0 & 0 & 1 & a_{34} & \dots \\ . & . & . & . & \dots \end{pmatrix}, \tag{1}$$

where $a_{ij} \in R$ $(i = 1, 2, \dots, j = i+1, i+2, \dots)$. The sequence

$$d_i(A) = (a_{1,i}, a_{2,i+1}, a_{3,i+2}, \dots)$$

is called the *ith (upper) diagonal of matrix* $A$, $i \geq 0$. For a non-negative integer $m$ matrix $A$ is called *$m$-banded* if its $m$th diagonal is a non-zero sequence and for each $i > m$ the $i$th diagonal is the zero sequence. The only 0-banded matrix is the identity matrix. An infinite unitriangular matrix is called *banded* if it is $m$-banded for some $m > 0$.

The multiplication is well-defined for infinite unitriangular matrices over $R$. The set $UT(\infty, R)$ of all infinite unitriangular matrices over $R$ form a group. Its identity element is the identity matrix. For any infinite unitriangular matrix its inverse can be computed using standard Gaussian elimination algorithm. The product of banded matrices is banded. However, the inverse to a banded matrix can be not banded. Hence, the set $BUT(\infty, R)$ of all banded matrices form a submonoid in $UT(\infty, R)$.

The group $UT(\infty, R)$ acts by multiplication from the right on $R^\infty$. Indeed, the image of the sequence

$$\bar{x} = (x_1, x_2, \dots, x_n, \dots) \in R^\infty$$

under the action of matrix $A$ defined by (1) is

$$\bar{x}A = \left( x_1, x_2 + x_1 a_{12}, \dots, x_n + \sum_{i=1}^{n-1} x_i a_{in-1}, \dots \right). \tag{2}$$

In this way the group $UT(\infty, R)$ embeds into the semigroup of endomorphisms of $R^\infty$.

The group $UT(\infty, R)$ naturally arise as a projective limit of finite dimensional unitriangular groups over $R$. Let $UT(n, R)$ be the group of upper triangular matrices of size $n$ over $R$, $n \geq 1$. Denote by $\varphi_n$ the epimorphism from $UT(n+1, R)$ on $UT(n, R)$ that deletes the last row and column. Then the limit group of the inverse system

$$(UT(n, R), \varphi_n), n \geq 1$$

is exactly $UT(\infty, R)$. If the ring $R$ is finite the group $UT(\infty, R)$ is pro-finite. If the cardinality of the ring $R$ is a power of prime $p$ the group $UT(\infty, R)$ is a pro-$p$-group.

**3. Sufficient condition.** Let $p$ be a prime. Assume that the ring $R$ has charecteristic $p$, i.e. its unity 1 has order $p$ in the additive group of $R$.

For a square matrix $B$ of size $m$, $m \geq 1$, over $R$ denote by $U(B)$ the banded matrix of the form

$$\begin{pmatrix} I_m & B & & & \\ & I_m & B & & \\ & & I_m & B & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where $I_m$ denotes the identity matrix of size $m$.

**Lemma 1.** *For arbitrary $k \geq 1$ the following equality holds:*

$$(U(B))^k = \begin{pmatrix} I_m & \binom{k}{1}A & \dots & \binom{k}{k-1}A^{k-1} & A^k & & & \\ & I_m & \binom{k}{1}A & \dots & \binom{k}{k-1}A^{k-1} & A^k & & \\ & & I_m & \binom{k}{1}A & \dots & \binom{k}{k-1}A^{k-1} & A^k & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

*Proof.* The required equality is directly verified by induction on $k$. □

**Lemma 2.** *Let $B$ be a nilpotent matrix over $R$ of nilpotency index $p^n$ for some $n > 0$. Then the banded matrix $U(B)$ has order $p^n$.*

*Proof.* Since $p$ is a prime all binomial coefficients

$$\binom{p^n}{1}, \dots, \binom{p^n}{p^n - 1}$$

are divisible by $p$. Therefore all matrices

$$\binom{p^n}{1}A, \dots, \binom{p^n}{p^n - 1}A$$

are zero. Since $B$ is nilpotent of nilpotency index $p^n$ the statement immediately follows from Lemma 1. □

Let us describe a sufficient condition on matrices in $BUT(\infty, R)$ to generate the free product of finite number of cyclic $p$-groups. Denote by $p^{n_1}, \dots, p^{n_r}$, $r > 1$, the orders of cyclic groups and let

$$G(p^{n_1}, \dots, p^{n_r})$$

denotes the free product of cyclic groups of these orders.

Assume that for some positive integer $n$ the following 3 conditions hold.

(i) In the right $R$-module $R^n$ of vectors-rows there exist $r$ nonempty subsets $V_j \subset R^n$ $(1 \leq j \leq r)$ of non-zero vectors.

(ii) There exist nilpotent matrices $B_1, \dots, B_r$ of size $n$ and nilpotency indices $p^{n_1}, \dots, p^{n_r}$ correspondingly.

(iii) For arbitrary indices $i, j \in \{1, \dots, r\}$, $i \neq j$, arbitrary vector $v \in V_i$ and positive integer $l$ such that $l < p^{n_j}$ the image $vB_j^l$ belongs to the set $V_j$.

Consider banded matrices $A_1 = U(B_1), \dots, A_r = U(B_r)$ that correspond to nilpotent matrices $B_1, \dots, B_r$ described above.

Denote by $G$ the group generated by the set $\{A_1, \dots, A_r\}$.

**Theorem 1.** *The group $G$ is isomorphic to $G(p^{n_1}, \dots, p^{n_r})$.*

*Proof.* From condition (ii) and Lemma 2 it follows that the orders of matrices $A_1, \dots, A_r$ are $p^{n_1}, \dots, p^{n_r}$ correspondingly.

Hence, it is sufficient to show that the product of the form

$$A_{i_1}^{l_1} \dots A_{i_m}^{l_m}, \tag{3}$$

where $i_1, \dots, i_m \in \{1, \dots, r\}$, $i_j \neq i_{j+1}$, $1 \leq j < m$, and $1 \leq l_j < p^{i_j}$, $1 \leq j \leq m$, is not the identity matrix.

From Lemma 1 it implies that product (3) is banded matrix of the form

$$(U(B))^k = \begin{pmatrix} I_m & * & \dots & * & B_{i_1}^{l_1} \dots B_{i_m}^{l_m} & & & \\ & I_m & * & \dots & * & B_{i_1}^{l_1} \dots B_{i_m}^{l_m} & & \\ & & I_m & * & \dots & * & B_{i_1}^{l_1} \dots B_{i_m}^{l_m} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then it is sufficient to show that the product

$$B_{i_1}^{l_1} \dots B_{i_m}^{l_m}$$

is nonzero. Let $i \in \{1, \dots, r\}$ be an index such that $i \neq i$. Then for arbitrary $v \in V_i$ from condition (iii) we have

$$u_1 = vB_{i_1}^{l_1} \in V_{i_1}, \dots, u_m = vB_{i_1}^{l_1} \dots B_{i_m}^{l_m} \in V_{i_m}.$$

Since $u_m$ is nonzero by condition (iii) the required product is nonzero as well. The proof is complete. □

**4. Constructive examples.** Let $R$ be a unitary associative ring $R$ of prime characteristic $p$. For arbitrary positive integers $n_1, \dots, n_r$, $r > 1$, we use Theorem 1 to construct matrices that generate a group isomorphic to the free product

$$G(p^{n_1}, \dots, p^{n_r}).$$

Let $N_0 = 0, N_i = N_{i-1} + p^{n_i}, 1 \leq i \leq r$. Denote by

$$e_i = (\delta_{ij}, 1 \leq j \leq N_r), \quad 1 \leq i \leq N_r,$$

the standard basis from $R^{N_r}$. Define $r$ subsets of nonzero vectors

$$V_i = \{e_j : N_{i-1} + 1 \leq j \leq N_i\}, \quad 1 \leq i \leq r,$$

and $r$ matrices $B_1, \dots, B_r$ of size $N$ such that the element $(B_i)_{(k,l)}$ of the $i$th matrix, $1 \leq i \leq r$, on position $(k, l)$, $1 \leq k, l \leq N_r$, is zero except two cases when it is 1:

$$(B_i)_{(k,l)} = \begin{cases} 1, & \text{if } l = k + 1, N_{i-1} + 1 \leq k \leq N_i - 1 \text{ or } l = N_{i-1} + 2; \\ 1, & \text{if } l = N_{i-1} + 2 \text{ and } (k \leq N_{i-1} \text{ or } k > N_i); \\ 0, & \text{otherwise.} \end{cases}$$

In other words, matrix $B_i$ has the Jordan block of size $p^{n_i}$ with respect to 0 as a diagonal block on position $N_{i-1}+1$, the other elements of the $(N_{i-1}+2)$th column are 1, and all other elements are 0. Then

$$e_{N_{i-1}+1}B_i = e_{N_{i-1}+2}, \dots, e_{N_i-1}B_i = e_{N_i}, e_{N_i}B_i = 0.$$

and

$$e_jB_i = e_{N_{i-1}+2}, \quad j \leq N_{i-1} \text{ or } j \geq N_i.$$

It implies that for the sets $V_1, \dots, V_r$ and matrices $B_1, \dots, B_r$ conditions (i)–(iii) hold. Hence, Theorem 1 implies, that the group, generated by $U(B_1), \dots, U(B_r)$, is isomorphic to $G(p^{n_1}, \dots, p^{n_r})$, i.e. splits into the free product of cyclic groups of orders $p^{n_1}, \dots, p^{n_r}$. Note, that in fact, all constructed matrices are defined over the center of $R$.

Applying this method one obtains the following examples.

**Example 1.** Let $p = 2$. In order to generate the free product of 3 cyclic groups of order 2 one can take the following 3 matrices of size 6:

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}.$$

**Example 2.** Let $p = 3$. In order to generate the free product of 2 cyclic groups of order 3 one can take the following 2 matrices of size 6:

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}.$$

**5. Open questions.** Conditions (i)–(iii) can be used in a way that differs from the method proposed in Section 4. One can choose a set of nilpotent matrices of the same size and try to find suitable sets of vectors.

**Problem 1.** *Assume that a finite set of nilpotent matrices of size $n$ over an associative unitary ring $R$ of prime characteristic is chosen. Under what conditions this set can be supplied with sets of vectors from $R^n$ so that conditions (i)–(iii) are satisfied? Is there an algorithm to verify this property for finite $R$?*

Despite for finite $R$ the group $UT(\infty, R)$ is endowed with profinite topology our examples are discrete. However, the closures of these examples are interesting to investigate (cf. [3]).

**Problem 2.** *What are the closures of the groups defined in Theorem 1 in case the ring $R$ is finite? Are them self-normalizing?*

The other natural extension of the construction presented in this paper is inspired by connections between groups of infinite unitriangular matrices and groups defined by finite automata ([5, 8]). For the latter more general results on free products and free products with amalgamation are known ([2, 7]).

The restriction on orders of cyclic groups in Theorem 1 to be powers of the same prime looks redundant. It is also natural to ask about amalgamated free products of cyclic groups.

**Problem 3.** *Let $G$ be a free product of finite number of finite cyclic groups whose orders are not powers of the same prime or a free product of two cyclic groups with amalgamation over a nontrivial subgroup. Is there an associative unitary ring such that the group $UT(\infty, R)$ contains a subgroup isomorphic to $G$? Is it possible to choose generators so that each of them has the form $U(B)$ for a finite dimensional matrix $B$?*

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