УДК 517.98, 517.5

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TRANSFORMATION OPERATORS FOR IMPEDANCE STURM–LIOUVILLE OPERATORS ON THE LINE

M. Kazanivskiy, Ya. Mykytyuk, N. Sushchyk. *Transformation operators for impedance Sturm–Liouville operators on the line*, Mat. Stud. **60** (2023), 79–98.

In the Hilbert space $H := L_2(\mathbb{R})$, we consider the impedance Sturm–Liouville operator $T : H \to H$ generated by the differential expression $-p\frac{d}{dx}\frac{1}{p^2}\frac{d}{dx}p$, where the function $p : \mathbb{R} \to \mathbb{R}_+$ is of bounded variation on $\mathbb{R}$ and $\inf_{x\in\mathbb{R}} p(x) > 0$. Existence of the transformation operator for the operator T and its properties are studied.

In the paper, we suggest an efficient parametrization of the impedance function p in term of a real-valued bounded measure $\mu \in \boldsymbol{M}$ via $p_\mu(x) := e^{\mu([x,\infty))}, x \in \mathbb{R}$. For a measure $\mu \in \boldsymbol{M}$, we establish existence of the transformation operator for the Sturm–Liouville operator T_μ, which is constructed with the function p_μ. Continuous dependence of the operator T_μ on μ is also proved. As a consequence, we deduce that the operator T_μ is unitarily equivalent to the operator $T_0 := -d^2/dx^2$.

1. Introduction. In the Hilbert space $H := L_2(\mathbb{R})$, we consider the Sturm–Liouville operator T in the impedance form generated by the differential expression

$$\mathfrak{t}(f) := -\left(p\frac{d}{dx}\frac{1}{p^2}\frac{d}{dx}p\right) f, \tag{1}$$

where the function p belongs to the class $\mathscr{P}$ consisting of all functions $p\colon \mathbb{R} \to \mathbb{R}_+$ of bounded variation on $\mathbb{R}$ such that $\inf_{x\in\mathbb{R}} p(x) > 0$.

We define the domain of the differential expression (1) as the set

$$\operatorname{dom}\mathfrak{t} := \{f \in L_{1,\mathrm{loc}}(\mathbb{R}) \mid \ pf \in AC(\mathbb{R}),\ \ p^{-2}(pf)' \in AC(\mathbb{R})\}.$$

Here and hereafter, $AC(\mathbb{R})$ is the linear space of all locally absolutely continuous functions $f\colon \mathbb{R} \to \mathbb{C}$. The differential expression (1) generates an operator

$$Tf := \mathfrak{t}(f)$$

that acts in H on the domain

$$\operatorname{dom} T := \{f \in H \mid \ pf \in W_2^1(\mathbb{R}),\ \ p^{-2}(pf)' \in W_2^1(\mathbb{R})\}.$$

Here, $W_2^1(\mathbb{R})$ is the standard Sobolev space. For every $p \in \mathscr{P}$, the operator T is self-adjoint and non-negative in H. Indeed, let us denote by S_0 and P self-adjoint operators acting in the space H by the formulas

$$S_0 f := if', \quad f \in \operatorname{dom} S_0 := W_2^1(\mathbb{R}),$$

2020 *Mathematics Subject Classification:* 34L25, 34L40, 47L10.
Keywords: impedance Sturm–Liouville operator; transformation operators; bounded measures.
doi:10.30970/ms.60.1.79-98

$$Pf := pf, \quad f \in \operatorname{dom} P := H.$$

For the operator P there exists an inverse continuous operator P^{-1}. Thus the operator

$$S := P^{-1} S_0 P$$

is similar to the operator S_0. This implies that the operator S is a closed operator and the equality $S^* = PS_0P^{-1}$ holds. It is easy to see that $T = S^*S$. In view of von Neumann's theorem (see [1, Sect. V.3.7]), the operator $T = S^*S$ is a self-adjoint non-negative operator.

Our goal is to prove the existence of transformation operators for the operator T and to study their properties. It is well known (see, e.g., [2]) that transformation operators for Sturm–Liouville operators play an important role in solving inverse problems. For this reason, the authors consider this paper as a step towards solving the inverse problem of scattering theory (ISP) for the operator T. Let us point out two partial cases, where ISP for the operator T is solved. In the first case, a function $p \in \mathscr{P}$ is absolutely continuous, and this problem comes down to a well-studied ISP for a Dirac operator with an integrable potential (see, e.g., [3]). In the second case, a function $p \in \mathscr{P}$ is piecewise-constant and its points of discontinuity belong to the lattice $\mathbb{Z}$, and ISP for this case is studied in detail in [4].

Note that the parametrization of the operators T using functions $p \in \mathscr{P}$ is inconvenient. Firstly, each function from the subclass

$$\mathscr{P}(p_0) := \{p \in \mathscr{P} \mid \exists c > 0 \quad \forall x \in \mathbb{R} \quad p(x-0) = cp_0(x-0)\} \quad (p_0 \in \mathscr{P})$$

generates the same operator T, secondly, the set $\mathscr{P}$ has a difficult structure. To circumvent these drawbacks, we suggest the following approach.

We denote by $\boldsymbol{M}$ the real Banach space of all real-valued bounded measures on $\mathbb{R}$. Clearly, for an arbitrary measure $\mu \in \boldsymbol{M}$, the function

$$p_\mu(x) := e^{\mu([x,\infty))}, \quad x \in \mathbb{R},$$

belongs to the class $\mathscr{P}$ and is left-continuous. We also denote by P_μ, S_μ and T_μ the operators P, S and T, which are constructed with the function p_μ. It is easy to check that the class of the operators T, generated by functions $p \in \mathscr{P}$, coincides with the class $\{T_\mu \mid \mu \in \boldsymbol{M}\}$. Therefore, we can take a measure $\mu \in \boldsymbol{M}$ as a parameter on which the operator T depends. The advantage of this choice is that, in contrast to $\mathscr{P}$, the space $\boldsymbol{M}$ is a Banach space, moreover, it is a well-studied Banach space.

Denote by $\boldsymbol{M}_0$ the subspace in $\boldsymbol{M}$ consisting of all measures $\mu \in \boldsymbol{M}$ that are absolutely continuous with respect to the Lebesgue measure m with the density $u = \frac{d\mu}{dm}$ that belongs to the space $\mathscr{D}$ of all real-valued functions from $C^\infty(\mathbb{R})$ with compact support.

If $\mu \in \boldsymbol{M}_0$ and $u = \frac{d\mu}{dm}$, then T_μ can be presented in the potential form

$$T_\mu = -\frac{d^2}{dx^2} + v, \quad \operatorname{dom} T_\mu = W_2^2(\mathbb{R}),$$

where $v = u' + u^2$. Since $v \in \mathscr{D}$, there exists the right transformation operator for the operator T_μ (see, e.g., [2]), which we denote by U_μ. The operator U_μ acts continuously in all spaces $L_p(\mathbb{R})$ $(1 \le p \le \infty)$ by the formula

$$(U_\mu f)(x) := f(x) + \int_x^\infty K(x,t) f(t)\, dt, \quad x \in \mathbb{R}. \tag{2}$$

Unless otherwise noted, we will consider U_μ as an element of the Banach algebra $\mathcal{B}(H)$ of all linear continuous operators in H.

The kernel K in (2) is a smooth function in $\Omega := \{(x,t) \in \mathbb{R}^2 \mid x \le t\}$, which is defined unambiguously by the fact that at all $\lambda \in \overline{\mathbb{C}}_+$ the formula

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,dt, \quad x \in \mathbb{R},$$

defines the right Jost solution of the equation $-y'' + vy = \lambda^2 y$ (see Section 4). It follows from the results of [2] that for $\mu \in \boldsymbol{M}_0$ the operator U_μ is invertible in the algebra $\mathcal{B}(H)$ and $T_\mu = U_\mu T_0 U_\mu^{-1}$.

The main result of this paper reads as follows.

Theorem 1. *The mapping* $\boldsymbol{M}_0 \ni \mu \mapsto U_\mu \in \mathcal{B}(H)$ *has a unique extension to the mapping*

$$\boldsymbol{M} \ni \mu \mapsto \mathcal{U}_\mu \in \mathcal{B}(H),$$

which is sequentially continuous if $\boldsymbol{M}$ *is equipped with the weak topology and* $\mathcal{B}(H)$ *is equipped with the strong operator topology. Moreover, for every* $\mu \in \boldsymbol{M}$ *the operator* $\mathcal{U}_\mu$ *is invertible in the algebra* $\mathcal{B}(H)$ *and* $T_\mu = \mathcal{U}_\mu T_0 \mathcal{U}_\mu^{-1}$.

The following corollary obviously follows from Theorem 1 and Lemma 21.

Corollary 1. *For an arbitrary* $\mu \in \boldsymbol{M}$ *the operator* T_μ *is unitarily equivalent to the operator* T_0.

Remark 1. To avoid possible confusion, we use different (but similar) notations for the function $\mu \mapsto U_\mu$ and its extension $\mu \mapsto \mathcal{U}_\mu$.

This paper is organized as follows. In the next section, we introduce necessary definitions and prove auxiliary propositions. In Section 3, we study properties of transformation operators U_μ in the smooth case. In Section 4, we prove Theorems 3 and 4 to describe properties of transformation operators $\mathcal{U}_\mu$ when $\mu \in \boldsymbol{M}$. In Section 5, we prove Theorem 1. The proofs in this section are analogous to that in Sections 3 and 4, thus they are presented in abbreviated form. Finally, Appendix A contains auxiliary lemmas that were used in the proofs.

2. Preliminaries.

2.1. The spaces L_q and $L_{q,\mathrm{loc}}$. We use the abbreviation $L_q := L_q(\mathbb{R})$, $1 \le q \le \infty$, for the Banach spaces of Lebesgue measurable functions with the standard norms

$$\|f\|_q := \Big(\int_{\mathbb{R}} |f(x)|^q\,dx\Big)^{1/q}, \quad \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

We distinguish the Hilbert space $H := L_2$ among the spaces L_q $(1 \le q \le \infty)$ and denote by $(\cdot \mid \cdot)_H$ the inner product in H. We also denote by $L_{q,\mathrm{loc}}$ the linear space of all functions $f\colon \mathbb{R} \to \mathbb{C}$ locally belonging to L_q.

2.2. The Sobolev space W_q^n. For any positive integer n and $q \in [1,\infty]$, we denote by W_q^n the Sobolev space, i.e.

$$W_q^n := \{f \in C^{n-1}(\mathbb{R}) \mid f^{(n-1)} \in AC(\mathbb{R}),\ f, f', \dots f^{(n)} \in L_q\}.$$

The space W_q^n will be endowed with the norm

$$\|f\|_{W_q^n} := \sum_{j=0}^n \|f^{(j)}\|_q.$$

2.3. The spaces $C_b(\mathbb{R})$ and $\mathscr{D}$. We denote by $C_b(\mathbb{R})$ the real Banach space of all continuous bounded real functions on $\mathbb{R}$ with the norm $\|f\|_\infty$. We also denote by $\mathscr{D}$ the linear space of all real-valued infinitely differentiable functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ with compact support.

2.4. The space $\boldsymbol{M}$. We denote by $\boldsymbol{M}$ the real Banach space of all real-valued bounded Borel measures on $\mathbb{R}$. We shall denote by $|\mu|$ the total variation of a measure $\mu \in \boldsymbol{M}$. The space $\boldsymbol{M}$ is endowed with the following norm

$$\|\mu\| := |\mu|(\mathbb{R}), \quad \mu \in \boldsymbol{M}.$$

We distinguish the following subspaces in $\boldsymbol{M}$:

- the space $\boldsymbol{M}_{ac}$ of all measures that are absolutely continuous with respect to the Lebesgue measure;
- the space $\boldsymbol{M}_0$ of all $\mu \in \boldsymbol{M}_{ac}$ with the density $\frac{d\mu}{dm}$ belonging to $\mathscr{D}$.

In the space $\boldsymbol{M}$, besides the strong topology generated by the norm, we consider also the weak topology. Let us recall some facts about weak convergence of measures.

A sequence $(\mu_n)_{n=1}^{\infty}$ in the space $\boldsymbol{M}$ *converges weakly* to $\mu \in \boldsymbol{M}$ (the notation $\mu_n \overset{w}{\to} \mu$) if for every $f \in C_b(\mathbb{R})$ the equality

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\, d\mu_n = \int_{\mathbb{R}} f\, d\mu$$

holds.

A sequence $(\mu_n)_{n=1}^{\infty}$ in the space $\boldsymbol{M}$ is called *uniformly dense* if for an arbitrary $\varepsilon > 0$ there exists a bounded interval $[a, b]$ such that $|\mu_n|(\mathbb{R} \setminus [a, b]) < \varepsilon$ for all $n \in \mathbb{N}$.

For every measure $\mu \in \boldsymbol{M}$, we denote by h_μ the function

$$h_\mu(x) := \mu([x, \infty)), \quad x \in \mathbb{R}.$$

The following proposition follows from theorems 1.4.7 and 1.7.2 of [5].

Proposition 1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \overset{w}{\to} \mu \in \boldsymbol{M}$. Then:*
1) $\|\mu\| \le \sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$; 2) *the sequence $(\mu_n)_{n=1}^{\infty}$ is uniformly dense;*
3) *on each interval $[a, b]$ the sequence $(h_{\mu_n})_{n\in\mathbb{N}}$ converges in measure to h_μ (with respect to the Lebesgue measure);*
4) *from the sequence $(h_{\mu_n})_{n\in\mathbb{N}}$ one can choose a subsequence, which converges almost everywhere to h_μ.*

Let us recall (see [5]) that the convolution of measures $\mu, \nu \in \boldsymbol{M}$ is a measure $\mu * \nu$ defined by the formula

$$\int_{\mathbb{R}} f\, d(\mu * \nu) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x + y)\, \mu(dx)\, \nu(dy), \quad f \in C_b(\mathbb{R}).$$

It is well known that

$$\|\mu * \nu\| \le \|\mu\|\|\nu\|, \quad \mu, \nu \in \boldsymbol{M}.$$

Let us fix a non-negative function $\theta \in \mathscr{D}$ such that

$$\operatorname{supp} \theta \subset [-1, 0], \qquad \int_{\mathbb{R}} \theta(t)\, dt = 1.$$

Put

$$\theta_n(x) := n\theta(nx), \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

and denote by ω_n the measure in $\boldsymbol{M}_0$, for which $d\omega_n/dm = \theta_n$.

Let $\mu \in \boldsymbol{M}$. A sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\boldsymbol{M}$ is called a *θ-sequence* for μ if $\nu_n := \mu_n * \omega_n$, where the measures μ_n are defined by the formula

$$\mu_n(A) := \mu(A \cap [-n, n]), \quad A \in B(\mathbb{R}).$$

Here, $B(\mathbb{R})$ is the algebra of all Borel subsets of $\mathbb{R}$.

Lemma 1. *Let $\mu \in M, n \in \mathbb{N}$ and $(\nu_k)_{k\in\mathbb{N}}$ be a θ-sequence for μ. Then:*
1) *the sequence $(\nu_k)_{k\in\mathbb{N}}$ belongs to $\boldsymbol{M}_0$ and $\nu_k \xrightarrow{w} \mu$;*
2) *if $\operatorname{supp}\mu \subset (-\rho,\rho)$ $(\rho > 0)$, then $\operatorname{supp}\nu_n \subset (-1/n-\rho, 1/n+\rho)$;*
3) *$\|\nu_n\| \le \|\mu\|$ and $h_{|\nu_n|} \le h_{|\mu|}$.*

Proof. The proof of (1) and (2) is obvious.

Let us prove part (3). Since for every $n \in \mathbb{N}$ $\|\omega_n\| = 1$ and $\|\mu_n\| \le \|\mu\|$, we get that

$$\|\nu_n\| = \|\mu_n * \omega_n\| \le \|\mu_n\|\|\omega_n\| \le \|\mu\|, \quad n \in \mathbb{N}.$$

It follows from the definition of the measures ν_n that

$$\left|\frac{d\nu_n}{dm}(t)\right| = \left|\int_{\mathbb{R}} \theta_n(t-\xi)\,d\mu_n(\xi)\right| \le \int_{\mathbb{R}} \theta_n(t-\xi)\,d|\mu|(\xi), \quad t \in \mathbb{R}.$$

Thus

$$h_{|\nu_n|}(x) = |\nu_n|([x,\infty)) \le \int_{\mathbb{R}} \left(\int_x^\infty \theta_n(t-\xi)\,dt\right) d|\mu|(\xi), \quad x \in \mathbb{R}.$$

Using the last inequality, it is easy to obtain that

$$h_{|\nu_n|}(x) \le \int_x^\infty d|\mu|(\xi) = h_{|\mu|}(x), \quad x \in \mathbb{R}.$$

□

2.5. The algebra $\mathcal{B}(H)$. We denote by $\mathcal{B}(X)$ the algebra of all linear everywhere defined continuous operators acting on a topological vector space X. If X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

In this paper, we mainly make use of the algebra $\mathcal{B}(H)$. Let us agree that $A_n \xrightarrow{s} A$ (or $A = \operatorname*{s\text{-}lim}_{n\to\infty} A_n$) means that a sequence $(A_n)_{n\in\mathbb{N}}$ converges to an operator A in the strong operator topology of the algebra $\mathcal{B}(H)$.

2.6. The operators P_μ. Let us recall that $P_\mu \colon H \to H$ is a multiplication operator by the function $p_\mu(x) := e^{h_\mu(x)}$, where $h_\mu(x) = \mu([x,\infty))$.

Lemma 2. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Then $P_{\mu_n} \xrightarrow{s} P_\mu$.*

Proof. Let the conditions of the lemma be satisfied. In view of Proposition 1,

$$\|h_\mu\|_\infty, \|h_{\mu_n}\|_\infty \le \sup_{k\in\mathbb{N}} \|\mu_k\| = c < \infty, \quad n \in \mathbb{N},$$

and on an arbitrary interval $[a,b]$ the sequence $(h_{\mu_n})_{n\in\mathbb{N}}$ converges in measure to h_μ (with respect to the Lebesgue measure). Thus $\|p_\mu\|_\infty, \|p_{\mu_n}\|_\infty \le c_1 := e^c$, $n \in \mathbb{N}$, and the sequence $(p_{\mu_n})_{n\in\mathbb{N}}$ on $[a,b]$ converges in measure to p_μ. For an arbitrary function $f \in H$ and $n \in \mathbb{N}$, we have

$$\|(P_\mu - P_{\mu_n})f\|^2 = \int_{\mathbb{R}} |p_\mu(x) - p_{\mu_n}(x)|^2 |f(x)|^2\,dx$$

and $|p_\mu(x) - p_{\mu_n}(x)|^2|f(x)|^2 \le 4c_1^2|f(x)|^2$, $x \in \mathbb{R}$. Thus, using the Lebesgue dominated convergence theorem, we obtain that $\lim_{n\to\infty} \|(P_\mu - P_{\mu_n})f\|^2 = 0$.

Therefore, $P_{\mu_n} \xrightarrow{s} P_\mu$. □

2.7. Chain of orthoprojectors E_ξ. Denote by E_ξ the orthogonal projector in H defined by the formula

$$E_\xi f := \chi_\xi f, \quad \xi \in \mathbb{R},$$

where χ_ξ is the indicator function of the half-line $(-\infty, \xi]$. The set $\mathfrak{E} := \{E_\xi \mid \xi \in \mathbb{R}\}$ forms a chain of orthoprojectors in the algebra $\mathcal{B}(H)$.

An operator $A \in \mathcal{B}(H)$ is called an upper-triangular (lower-triangular) operator with respect to the chain $\mathfrak{E}$ if (see [6])

$$E_\xi^\perp A E_\xi = 0 \quad (E_\xi A E_\xi^\perp = 0), \quad \xi \in \mathbb{R},$$

where $E_\xi^\perp := I - E_\xi$ and I is the identity operator.

The set $\mathcal{B}^+(H)$ $(\mathcal{B}^-(H))$ of all upper-triangular (lower-triangular) operators $A \in \mathcal{B}(H)$ is a subalgebra closed in the strong operator topology.

We denote by H^+ the linear subspace in $L_{2,\mathrm{loc}}$ consisting of all functions $f \in L_{2,\mathrm{loc}}$ such that

$$\int_0^\infty |f(x)|^2\, dx < \infty.$$

It is clear that if $f \in H^+$, then for all $a \in \mathbb{R}$ the integral $\int_a^\infty |f(x)|^2\, dx$ is also convergent. Let us introduce the topology in the space H^+ generated by seminorms

$$\rho_n(f) := \Big(\int_{-n}^\infty |f(x)|^2\, dx\Big)^{1/2}, \quad n \in \mathbb{N}.$$

The topological vector space H^+ is a Fréchet space, i.e., a complete locally convex metrizable space (see [7]).

Remark 2. The space H is everywhere dense in H^+. Indeed, every element $f \in H^+$ is a limit of the sequence of the following elements from H:

$$f_k(x) = \begin{cases} f(x), & \text{if } x \geq -k; \\ 0, & \text{if } x < -k, \end{cases} \qquad k \in \mathbb{N}.$$

It is not difficult to verify that every $A \in \mathcal{B}^+(H)$ can be uniquely extended to a continuous operator in H^+. Let us agree to identify the extended operator with the original one, i.e., for every operator $A \in \mathcal{B}^+(H)$ we will keep the same notation for its extension to an element of $\mathcal{B}(H^+)$.

For every $\lambda \in \mathbb{C}_+$, we denote by e_λ the function

$$e_\lambda(x) = \mathrm{e}^{i\lambda x}, \quad x \in \mathbb{R}.$$

Obviously, $e_\lambda \in H^+$ for an arbitrary $\lambda \in \mathbb{C}_+$.

Lemma 3. *The linear span* $\mathcal{E} := \mathrm{lin}\{e_\lambda \mid \lambda \in \mathbb{C}_+\}$ *is everywhere dense in* H^+.

Proof. First, we prove that $\mathcal{E}$ is everywhere dense in every space $L_2(-n, \infty)$ $(n \in \mathbb{N})$. Assume this statement to be false. Then there exist $n \in \mathbb{N}$ and a nonzero function $f \in L_2(-n, \infty)$ such that

$$\int_{-n}^\infty e^{i\lambda x} f(x)\, dx = 0, \quad \lambda \in \mathbb{C}_+.$$

It follows that for a fixed number $\varepsilon > 0$, $\int_{-n}^\infty e^{i\xi x} e^{-\varepsilon x} f(x)\, dx = 0$, $\xi \in \mathbb{R}$. Using properties of the Fourier transform, we obtain that $e^{-\varepsilon x} f(x) = 0$ for almost all $x \in (-n, \infty)$, hence $f = 0$. We have come across a contradiction. Hence $\mathcal{E}$ is everywhere dense in every space $L_2(-n, \infty)$ $(n \in \mathbb{N})$.

Let $f \in H^+$. It follows from the above established that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\rho_n(f - \varphi_n) \leq n^{-1}$, $n \in \mathbb{N}$. Obviously, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to f in the topology of the space H^+. □

3. Transformation operators. The smooth case. In this section we derive explicit formulas for transformation operators in the case of $\mu \in \boldsymbol{M}_0$. Our approach differs from the classical one (see [2]).

For $u \in \mathscr{D}$ and $n \in \mathbb{N}$, we denote by $K_{u,n}$ the linear operator acting in the space $L_{1,\mathrm{loc}}$ by the formula

$$(K_{u,n}f)(x) := (-1)^n \int_{\Pi_n(x)} u(t_1)\cdot \ldots \cdot u(t_n) f(2\xi_n(t) + (-1)^n x)\, dt_1 \cdot \ldots \cdot dt_n, \quad x \in \mathbb{R}, \quad (3)$$

where $\xi_n(t) := \sum_{k=1}^n (-1)^{k+1} t_k$, $\Pi_n(x) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \colon x \le t_n \le \cdots \le t_1\}$.

It is easy to see that $K_{u,n} f \in C(\mathbb{R})$ for arbitrary functions $f \in L_{1,\mathrm{loc}}$, that is the operator $K_{u,n}$ maps the space $L_{1,\mathrm{loc}}$ into the space $C(\mathbb{R})$.

Denote by Δ_ρ the interval $[-\rho, \rho]$.

Lemma 4. *Let $u \in \mathscr{D}$, $n \in \mathbb{N}$ and $f \in L_{1,\mathrm{loc}}$. Then:*
(1) *if* $\operatorname{supp} u \subset (-\infty, \eta]$, *then* $\operatorname{supp} K_{u,n} f \subset (-\infty, \eta]$;
(2) *if* $\operatorname{supp} f \subset (-\infty, \eta]$, *then* $\operatorname{supp} K_{u,n} f \subset (-\infty, \eta]$;
(3) *if* $\operatorname{supp} u \subset \Delta_\rho$ *and* $\operatorname{supp} f \subset \Delta_\rho$, *then* $\operatorname{supp} K_{u,n} f \subset \Delta_{5\rho}$;
(4) *if* $\operatorname{supp} u \subset \Delta_\rho$ *and* $\operatorname{supp} f \subset \mathbb{R} \setminus \Delta_{5\rho}$, *then* $\operatorname{supp} K_{u,n} f \subset \mathbb{R} \setminus \Delta_\rho$.

Proof. Part (1) is obvious. To prove part (2), we note that $2\xi_n(t) + (-1)^n x \ge x$, $t \in \Pi_n(x)$. Thus, in view of the formula (3), we obtain that $(K_{u,n} f)(x) = 0$ for $x > \eta$ if $\operatorname{supp} f \subset (-\infty, \eta]$.

Let the conditions of part (3) be satisfied. It follows from the formula (3) that

$$\operatorname{supp} K_{u,n} f \subset \{x \in \mathbb{R} \mid \exists t \in \Delta_\rho^n \colon (2\xi_n(t) + (-1)^n x) \in \Delta_\rho\}.$$

It is easy to see that $|2\xi_n(t)| \le 4\rho$ if $t \in \Delta_\rho^n$. Since $|x| \le |2\xi_n(t)| + |2\xi_n(t) + (-1)^n x|$, we conclude that $\operatorname{supp} K_{u,n} f \subset \Delta_{5\rho}$.

Finally, let the conditions of part (4) be satisfied. Similar arguments as in part (3) give

$$\operatorname{supp} K_{u,n} f \subset \{x \in \mathbb{R} \mid \exists t \in \Delta_\rho^n \colon (2\xi_n(t) + (-1)^n x) \notin \Delta_{5\rho}\}.$$

Since for $x \in \Delta_\rho$ and $t \in \Delta_\rho^n$ the estimate $|2\xi_n(t) + (-1)^n x| \le |2\xi_n(t)| + |x| \le 5\rho$ holds, we get that $\operatorname{supp} K_{u,n} f \subset \mathbb{R} \setminus \Delta_\rho$. □

Lemma 5. *For $u \in \mathscr{D}$, $n \in \mathbb{N}$ and $p \in [1, \infty]$ the operator $K_{u,n}$ belongs to the algebra $\mathcal{B}(L_p)$ and*

$$\|K_{u,n}\|_{\mathcal{B}(L_p)} \le \frac{1}{(n)!} \|u\|_1^n. \quad (4)$$

Proof. In view of the interpolation theorem (see, e.g., [9]), it suffices to prove the estimate (4) for $p = 1$ and $p = \infty$. In these cases, for all $f \in L_p$

$$\|K_{u,n} f\|_p \le \|f\|_p \int_{t_n \le \cdots \le t_1} |u(t_1)| \cdots |u(t_n)|\, dt = \frac{1}{(n)!} \|u\|_1^n \|f\|_p.$$

□

Set

$$K_u := \sum_{n=1}^\infty K_{u,n}, \quad u \in \mathscr{D}. \quad (5)$$

Lemma 6. *Let $p \in [1, \infty]$. For $u \in \mathscr{D}$ the operator K_u belongs to the algebra $\mathcal{B}(L_p)$ and*

$$\|I + K_u\|_{\mathcal{B}(L_p)} \le \exp\{\|u\|_1\}.$$

Moreover, if $u_1, u_2 \in \mathscr{D}$ and $r := \|u_1\|_1 + \|u_2\|_1$, then

$$\|K_{u_1} - K_{u_2}\|_{\mathcal{B}(L_p)} \le e^r \|u_1 - u_2\|_1. \quad (6)$$

Proof. The first part of the lemma clearly follows from Lemma 5. It remains to prove the estimate (6). In view of the interpolation theorem, is suffices to prove it for $p = 1$ and $p = \infty$. It is easy to see that in these cases

$$\|K_{u_1,n} - K_{u_2,n}\|_{\mathcal{B}(L_p)} \le \int_{t_n \le \ldots \le t_1} \Big| \prod_{j=1}^n u_1(t_j) - \prod_{j=1}^n u_2(t_j) \Big| \, dt. \tag{7}$$

Put $u(x) := \max\{|u_1(x)|, |u_2(x)|\}$. Using the identity

$$\prod_{j=1}^n u_1(t_j) - \prod_{j=1}^n u_2(t_j) = \sum_{k=1}^n \Big[\Big(\prod_{j<k} u_1(t_j) \Big) (u_1(t_k) - u_2(t_k)) \Big(\prod_{s>k} u_2(t_s) \Big) \Big],$$

we obtain that

$$\int_{t_n \le \ldots \le t_1} \Big| \prod_{j=1}^n u_1(t_j) - \prod_{j=1}^n u_2(t_j) \Big| \, dt \le$$
$$\le \int_{-\infty}^{\infty} |u_1(\xi) - u_2(\xi)| \, d\xi \int_{t_{n-1} \le \ldots \le t_1} u(t_1) \cdot \ldots \cdot u(t_{n-1}) \, dt_1 \cdot \ldots \cdot dt_{n-1} \le$$
$$\le \|u_1 - u_2\|_1 \frac{\|u\|_1^{n-1}}{(n-1)!} \le \|u_1 - u_2\|_1 \frac{r^{n-1}}{(n-1)!}.$$

Thus, taking into account (7) and (5), we derive the estimate (6). □

Lemma 7. *Let $u \in \mathscr{D}$ and $f \in L_\infty$. Then the function $K_u f$ is continuous and*

$$|(K_u f)(x)| \le \mathrm{e}^{\|u\|_1} \|f\|_\infty \int_x^\infty |u(y)| \, dy, \quad x \in \mathbb{R}.$$

Proof. It follows from the formula (3) that for an arbitrary $n \in \mathbb{N}$ the function $K_{u,n} f$ is continuous and

$$|(K_{u,n} f)(x)| \le \|f\|_\infty \int_{\Pi_n(x)} |u(t_1)| \cdots |u(t_n)| \, dt \le$$
$$\le \frac{\|f\|_\infty}{(n)!} \Big(\int_x^\infty |u(y)| \, dy \Big)^n \le \frac{\|f\|_\infty \|u\|_1^{n-1}}{(n-1)!} \int_x^\infty |u(y)| \, dy, \quad x \in \mathbb{R}.$$

Then, in view of (5), we obtain the result of Lemma 7. □

Lemma 8. *Let $u \in \mathscr{D}$ and $f \in W_2^1$. Then the function $g := (I + K_u) f$ belongs to W_2^1 and $g' - ug = (I + K_{-u}) f'$.*

Proof. First, we consider the case of $f \in \mathscr{D}_\mathbb{C} := \mathscr{D} + i\mathscr{D}$. In this case, for an arbitrary $n \in \mathbb{N}$ the function $K_{u,n} f$ is continuously differentiable. By part (3) of Lemma 4, it is a compactly supported function. Straightforward calculations give that

$$(K_{u,n} f)' = (-1)^n K_{u,n} f' + u K_{u,n-1} f \quad (K_{u,0} := I). \tag{8}$$

It follows from the estimate (4) that the series $\sum_{n=1}^\infty ((-1)^n K_{u,n} f' + u K_{u,n-1} f)$ converges in the space H. Thus the series $\sum_{n=0}^\infty (K_{u,n} f)'$ converges in the space H, hence the series $\sum_{n=0}^\infty K_{u,n} f$ converges in the space W_2^1, and $\sum_{n=0}^\infty K_{u,n} f = f + K_u f = g$. It follows from the equalities (8) that

$$g' = \sum_{n=0}^\infty (K_{u,n} f)' = f' + \sum_{n=1}^\infty ((-1)^n K_{u,n} f' + u K_{u,n-1} f) = \sum_{n=0}^\infty K_{-u,n} f' + u \sum_{n=1}^\infty K_{u,n-1} f =$$

$$= (I + K_{-u})f' + ug,$$

that is $g' - ug = (I + K_{-u})f'$.

Now we claim that $f \in W_2^1$. Since the set $\mathscr{D}_\mathbb{C}$ is everywhere dense in W_2^1, there exists a sequence $(f_n)_{n=1}^\infty$ in $\mathscr{D}_\mathbb{C}$, which converges to f in the space W_2^1. The proved above implies that for every $n \in \mathbb{N}$ the function $g_n := (I + K_u)f_n$ belongs to W_2^1 and

$$g_n' = u(I + K_u)f_n + (I + K_{-u})f_n'.$$

It follows from the continuity of the operators K_u and K_{-u} that:

(a) the sequence $(g_n)_{n\in\mathbb{N}}$ converges to $g = (I + K_u)f$ in the space H;

(b) the sequence $(g_n')_{n\in\mathbb{N}}$ converges to $ug + (I + K_{-u})f'$ in the space H.

This yields the conclusion that the sequence $(g_n)_{n\in\mathbb{N}}$ converges to g in the space W_2^1 and $g' - ug = (I + K_{-u})f'$. □

For an arbitrary $\mu \in \boldsymbol{M}_0$, we put by the definition

$$U_\mu := I + K_u \quad \left(u = \frac{d\mu}{dm}\right).$$

Remark 3. If $\mu \in \boldsymbol{M}_0$, $u = d\mu/dm$, then

$$S_\mu\varphi = i(\varphi' - u\varphi), \quad T_\mu f = -\left(\frac{d}{dx} + u\right)\left(\frac{d}{dx} - u\right) f, \tag{9}$$

where $\varphi \in \operatorname{dom} S_\mu = W_2^1$, $f \in \operatorname{dom} T_\mu = W_2^2$.

Moreover, it follows from part (2) of Lemma 4 that U_μ belongs to $\mathcal{B}^+(H)$.

By Lemmas 4 and 6, we obtain the following result.

Corollary 2. *Let $\mu, \widetilde{\mu} \in \boldsymbol{M}_0$. Then*

$$\|U_\mu\|_{\mathcal{B}(H)} \le \exp\{\|\mu\|\}, \quad \|U_\mu - U_{\widetilde{\mu}}\|_{\mathcal{B}(H)} \le \exp(\|\mu\| + \|\widetilde{\mu}\|)\|\mu - \widetilde{\mu}\|. \tag{10}$$

In view of Lemma 6, by using Remark 3, we arrive at the following corollary.

Corollary 3. *Let $\mu \in \boldsymbol{M}_0$. Then $U_\mu \in \mathcal{B}^+(H)$, moreover:*
1) *if $f \in \operatorname{dom} S_0$, then $U_\mu f$ belongs to $\operatorname{dom} S_\mu$ and $S_\mu U_\mu f = U_{-\mu} S_0 f$;*
2) *if $f \in \operatorname{dom} T_0$, then $U_\mu f$ belongs to $\operatorname{dom} T_\mu$ and $T_\mu U_\mu f = U_\mu T_0 f$.*

Denote by G_n the orthogonal projector in H given by the formula

$$G_n := E_n - E_{-n}, \quad n \in \mathbb{N},$$

and let $G_n^\perp := I - G_n$. Recall that the projector E_t was introduced in Subsection 2.7.

By the statements (3) and (4) of Lemma 4, we obtain the following corollary.

Corollary 4. *Let $\mu \in \boldsymbol{M}_0$, $\rho > 0$ and $\operatorname{supp} \mu \subset \Delta_\rho$. Then for an arbitrary natural $k > \rho$ the equalities $G_{5k}^\perp U_\mu G_k = G_k U_\mu G_{5k}^\perp = 0$ hold.*

Lemma 9. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Then for arbitrary $f \in H$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that*

$$\|G_m^\perp U_{\mu_n} f\| \le \varepsilon, \quad \|G_m^\perp U_{\mu_n}^* f\| \le \varepsilon \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. Let $f \in H$ and $\varepsilon \in (0,1)$. Since the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly, this sequence is bounded and uniformly dense. Set

$$c := \sup_{n\in\mathbb{N}}(2\|\mu_n\|+1), \quad \gamma := \frac{\varepsilon}{e^c(\|f\|_2+1)}.$$

Let us choose a compactly supported function $\widetilde{f} \in H$ such that $\|f-\widetilde{f}\|_2 \le \gamma$. Since $(\mu_n)_{n\in\mathbb{N}}$ is uniformly dense, there exists a sequence $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ in $\boldsymbol{M}_0$ such that the supports of measures $\widetilde{\mu}_n$ are contained in some interval Δ_ρ and $\|\mu_n-\widetilde{\mu}_n\| \le \gamma$. Assume that the support of the function $\widetilde{f}$ is also contained in Δ_ρ. It follows from Corollary 4 that if $k \in \mathbb{N}$, $k > \rho$ and $m = 5k$, then

$$G_m^\perp U_{\widetilde{\mu}_n}\widetilde{f} = 0 = G_m^\perp U_{\widetilde{\mu}_n}^*\widetilde{f}, \quad n \in \mathbb{N}. \tag{11}$$

Let us prove that $\|G_m^\perp U_{\mu_n} f\| \le \varepsilon$ for an arbitrary $n \in \mathbb{N}$. Using the equality (11), we have

$$\|G_m^\perp U_{\mu_n} f\| = \|G_m^\perp U_{\mu_n} f - G_m^\perp U_{\widetilde{\mu}_n}\widetilde{f}\| \le$$
$$\le \|U_{\mu_n} f - U_{\widetilde{\mu}_n}\widetilde{f}\| \le \|U_{\mu_n} - U_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)}\|f\|_2 + \|U_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)}\|f-\widetilde{f}\|_2.$$

The estimates (10) imply that

$$\|U_{\mu_n} - U_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \le e^c\gamma, \quad \|U_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \le e^c, \quad n \in \mathbb{N}.$$

Thus $\|G_m^\perp U_{\mu_n} f\| \le e^c\gamma\|f\|_2 + e^c\gamma = \varepsilon, \quad n \in \mathbb{N}.$

We prove similarly that $\|G_m^\perp U_{\mu_n}^* f\| \le \varepsilon$ for all $n \in \mathbb{N}$. □

Lemma 10. *Let $\mu \in \boldsymbol{M}_0$, $u = d\mu/dm$ and $q = u' + u^2$. Then for an arbitrary $\lambda \in \mathbb{C}_+$ the function $e_{\lambda,\mu}(x) := (U_\mu e_\lambda)(x)$, $x \in \mathbb{R}$, is the right Jost solution of the equation $-y''+qy = \lambda^2 y$ and*

$$|e^{-i\lambda x}e_{\lambda,\mu}(x) - 1| \le e^{\|\mu\|}h_{|\mu|}(x), \quad x \in \mathbb{R}. \tag{12}$$

Proof. In view of Corollary 3, $U_\mu \in \mathcal{B}^+(H)$, and, hence (see Remark 2) $U_\mu \in \mathcal{B}(H^+)$. Fix $\lambda \in \mathbb{C}_+$. Obviously, for an arbitrary $a \in \mathbb{R}$ there exists $g \in W_2^2$ such that $g(x) = e_\lambda(x)$ for $x \ge a$. Since $U_\mu \in \mathcal{B}^+(H)$, we get

$$e_{\lambda,\mu}(x) = (U_\mu e_\lambda)(x) = (U_\mu g)(x), \quad x > a.$$

By (9) and Corollary 3, we have that for $x > a$

$$-e''_{\lambda,\mu}(x) + q(x)e_{\lambda,\mu}(x) = [T_\mu U_\mu g](x) = [U_\mu T_0 g](x) = \lambda^2[U_\mu e_\lambda](x) = \lambda^2 e_{\lambda,\mu}(x).$$

Then, in view of part (1) of Lemma 4, for big enough numbers $\eta \in \mathbb{R}$

$$e_{\lambda,\mu}(x) = e_\lambda(x), \quad x \in (\eta,\infty).$$

Therefore, $e_{\lambda,\mu}$ is the right Jost solution of the equation $-y''+qy = \lambda^2 y$.

Let us prove the estimate (12). Fix $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}_+$, and consider the function

$$f(t) := \begin{cases} e^{i\lambda(t-x)}, & \text{if } t \ge x;\\ 0, & \text{if } t < x.\end{cases}$$

The function f belongs to L_∞ and $\|f\|_\infty = 1$. Lemma 7 implies that the function $K_u f$ is continuous and

$$|(K_u f)(x)| \le \mathrm{e}^{\|u\|_1}\int_x^\infty |u(t)|\,dt, \quad x \in \mathbb{R}.$$

Thus, using the equalities

$$e^{-i\lambda x}e_{\lambda,\mu}(x) - 1 = (K_u f)(x) \quad (x \in \mathbb{R}),\ \|u\|_1 = \|\mu\|,\ \int_x^\infty |u(t)|\,dt = |\mu|([x,\infty)) = h_{|\mu|}(x),$$

we obtain the estimate (12). □

Lemma 11. *Let $\mu \in \boldsymbol{M}_0$, $u = d\mu/dm$ and $q = u' + u^2$. Then U_μ is a classical transformation operator for Sturm–Liouville operator $T = -d^2/dx^2 + q$.*

Proof. Note that the classical transformation operator U for the operator $-d^2/dx^2 + q$ with a compactly supported potential q is continuous in all L_p and acts by the formula

$$(Uf)(x) = f(x) + \int_x^\infty K(x,t)f(t)\,dt, \quad x \in \mathbb{R},$$

where the function K is continuous on the set $\Omega = \{(x,t) \in \mathbb{R}^2 \mid x \le t\}$. In particular, U belongs to the algebra $\mathcal{B}^+(H)$, and it also belongs to the algebra $\mathcal{B}(H^+)$. Moreover, for an arbitrary $\lambda \in \mathbb{C}_+$ the formula

$$e(x,\lambda) := e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,dt$$

defines the right Jost solution of the equation $-y'' + qy = \lambda^2 y$. In this case, the right Jost solution is defined unambiguously, thus (see Lemma 10)

$$Ue_\lambda = e_{\lambda,\mu} = U_\mu e_\lambda, \quad \lambda \in \mathbb{C}_+.$$

Since (see Lemma 3) the linear span of the set $\{e_\lambda \mid \lambda \in \mathbb{C}_+\}$ is everywhere dense in the space H^+, we conclude that $U = U_\mu$. □

The following theorem describes some important properties of the operator U_μ in the smooth case.

Theorem 2. *Let $\mu \in \boldsymbol{M}_0$. Then the operator U_μ is invertible in $\mathcal{B}(H)$ and $\|U_\mu^{-1}\| \le \sqrt{2}$.*

Proof. Let $\mu \in \boldsymbol{M}_0$, $u = d\mu/dm$ and $q = u' + u^2$. In view of Lemma 11, the operator U_μ can be represented in the form $U_\mu = I + \mathbb{K}$, where $\mathbb{K}$ acts by the formula

$$(\mathbb{K}f)(x) = \int_x^\infty K(x,t)f(t)\,dt, \quad x \in \mathbb{R}, \quad f \in H.$$

Note that (see [2]) the function K is a solution of Gelfand-Levitan-Marchenko equation:

$$F(x+t) + K(x,t) + \int_x^\infty K(x,y)F(y+t)\,dy = 0, \quad x \le t. \tag{13}$$

In this case, the operator T_μ is non-negative, thus the real-valued function F is defined by the formula (see [2])

$$F(x) := \frac{1}{2\pi}\int_{\mathbb{R}} r(\lambda)\mathrm{e}^{-i\lambda x}\,d\lambda, \quad x \in \mathbb{R},$$

where r is the reflection coefficient. It follows from the results of [8] that r is the Fourier transform of a function from L_1 and $\|r\|_\infty < 1$. Hence $F \in L_1$. Denote by $\mathbb{F}$ the operator in the space H acting by the formula

$$(\mathbb{F}f)(x) := \int_{\mathbb{R}} F(x+t)f(t)\,dt, \quad x \in \mathbb{R}.$$

It is easy to see that the operator $\mathbb{F}$ is self-adjoint and $\|\mathbb{F}\| \le \|r\|_\infty < 1$. Fix an arbitrary $\xi \in \mathbb{R}$ and consider the auxiliary operators

$$\mathbb{K}_\xi := E_\xi^\perp \mathbb{K} E_\xi^\perp, \quad \mathbb{F}_\xi := E_\xi^\perp \mathbb{F} E_\xi^\perp.$$

The results of [2, Sect. III] imply that the operator $\mathbb{K}_\xi$ is compact. Moreover, as shown in [2, Sect. III], by (13), we obtain the equality

$$(I + \mathbb{K}_\xi)(I + \mathbb{F}_\xi)(I + \mathbb{K}_\xi^*) = I. \tag{14}$$

In view of (14), we have that $\operatorname{ran}(I+\mathbb{K}_\xi)=H$. Since the operator $\mathbb{K}_\xi$ is compact, the operator $(I+\mathbb{K}_\xi)$ is invertible. Thus we can rewrite (14) in the form

$$(I+\mathbb{K}_\xi)^{-1}[(I+\mathbb{K}_\xi)^{-1}]^*=I+\mathbb{F}_\xi,$$

which implies $\|(I+\mathbb{K}_\xi)^{-1}\|^2\le\|I+\mathbb{F}_\xi\|\le 1+\|\mathbb{F}\|<2$. Hence $\|(I+\mathbb{K}_\xi)^{-1}\|\le\sqrt{2}$, $\xi\in\mathbb{R}$.

Let us consider the sequence of the operators $A_n:=I+\mathbb{K}_{-n}$, $n\in\mathbb{N}$. Clearly, this sequence converges to the operator $A=I+\mathbb{K}$ in the strong operator topology, and the sequence $(A_n^*)_{n\in\mathbb{N}}$ converges to the operator A^* in the strong operator topology. Moreover, $\|A_n^{-1}\|\le\sqrt{2}$ for all $n\in\mathbb{N}$. Thus, in view of Lemma 19, the operator $I+\mathbb{K}$ is invertible and $(I+\mathbb{K})^{-1}\le\sqrt{2}$. □

4. Transformation operators. The general case.

Let $\mu\in\boldsymbol{M}$. We introduce the notation

$$\mathfrak{t}_\mu(f):=-\left(p_\mu\frac{d}{dx}\frac{1}{p_\mu^2}\frac{d}{dx}p_\mu\right)f. \tag{15}$$

We define the domain of the differential expression (15) as the set (see the introduction)

$$\operatorname{dom}\mathfrak{t}_\mu:=\{f\in L_{1,\mathrm{loc}}(\mathbb{R})\mid\ p_\mu f\in AC(\mathbb{R}),\ \ p_\mu^{-2}(p_\mu f)'\in AC(\mathbb{R})\}.$$

Let $\mu\in\boldsymbol{M}$ and $\lambda\in\mathbb{C}_+$. A solution y of the equation $\mathfrak{t}_\mu(f)=\lambda^2 f$ is called the *right* (*left*) *Jost solution* if

$$y(x)=e^{i\lambda x}(1+o(1)),\quad x\to+\infty\quad\big(y(x)=e^{-i\lambda x}(1+o(1)),\quad x\to-\infty\big).$$

It what follows, we denote the right and left Jost solutions by $e_{\lambda,\mu}$ and $e^-_{\lambda,\mu}$, respectively.

The main results of this section are the following two theorems.

Theorem 3. *For $\mu\in\boldsymbol{M}$ and $\lambda\in\mathbb{C}_+$, the equation $\mathfrak{t}_\mu(f)=\lambda^2 f$ has unique right and left Jost solutions $e_{\lambda,\mu}$ and $e^-_{\lambda,\mu}$. Moreover, for $e_{\lambda,\mu}$ the estimate*

$$|e^{-i\lambda x}e_{\lambda,\mu}(x)-1|\le e^{\|\mu\|}h_{|\mu|}(x),\quad x\in\mathbb{R}, \tag{16}$$

holds.

Theorem 4. *For each $\mu\in\boldsymbol{M}$, there exists a unique operator $\mathcal{U}_\mu\in\mathcal{B}^+(H)$ such that*

$$\mathcal{U}_\mu e_\lambda=e_{\lambda,\mu},\qquad\lambda\in\mathbb{C}_+.$$

Moreover: 1. *$\mathcal{U}_\mu$ is invertible in the algebra $\mathcal{B}(H)$, and $\|\mathcal{U}_\mu^{-1}\|_{\mathcal{B}(H)}\le\sqrt{2}$;* 2. *the equality $T_\mu=\mathcal{U}_\mu T_0\mathcal{U}_\mu^{-1}$ holds.*

First, we prove two auxiliary lemmas.

Lemma 12. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}_0$ and $\mu_n\xrightarrow{w}\mu\in\boldsymbol{M}$. Then from the sequence $(\mu_n)_{n\in\mathbb{N}}$ one can choose a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that there exist the limits*

$$U=\operatorname*{s-lim}_{k\to\infty}U_{\mu_{n_k}},\quad U_-=\operatorname*{s-lim}_{k\to\infty}U_{-\mu_{n_k}}.$$

Moreover: 1) *if $f\in\operatorname{dom}S_0$, then $Uf\in\operatorname{dom}S_\mu$ and $S_\mu Uf=U_-S_0f$;* 2) *if $f\in\operatorname{dom}S_0$, then $U_-f\in\operatorname{dom}S_{-\mu}$ and $S_{-\mu}U_-f=US_0f$;* 3) *if $f\in\operatorname{dom}T_0$, then $Uf\in\operatorname{dom}T_\mu$ and $T_\mu Uf=UT_0f$.*

Proof. Since $\mu_n \xrightarrow{w} \mu$, we obtain $\sup\limits_{n\in\mathbb{N}} \|\mu_n\| = \alpha < \infty$. Thus (see Corollary 2)

$$\sup_{n\in\mathbb{N}} \|U_{\pm\mu_n}\|_{\mathcal{B}(H)} \le e^{\alpha}.$$

Also (see Lemma 2),

$$P_{\mu_n} \xrightarrow{s} P_\mu, \quad P_{-\mu_n} \xrightarrow{s} P_{-\mu}. \tag{17}$$

Let $\varphi \in W_2^1$. In view of Corollary 3, $U_{\mu_n}\varphi \in \operatorname{dom} S_{\mu_n}$ and $S_{\mu_n}U_{\mu_n}\varphi = U_{-\mu_n}S_0\varphi$, $n \in \mathbb{N}$.

Since $S_{\mu_n} = P_{\mu_n}^{-1} S_0 P_{\mu_n}$, we get

$$S_0 P_{\mu_n} U_{\mu_n}\varphi = P_{\mu_n} U_{-\mu_n} S_0\varphi, \quad \varphi \in W_2^1. \tag{18}$$

Consider the auxiliary operators

$$V_{\mu_n} := P_{\mu_n} U_{\mu_n}, \quad n \in \mathbb{N}.$$

Let us fix $\varphi \in W_2^1$ and show that the set $\{V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in the space H. The equality (18) implies that $V_{\mu_n}\varphi \in W_2^1$ for all $n \in \mathbb{N}$ and $(V_{\mu_n}\varphi)' = P_{\mu_n}U_{-\mu_n}\varphi'$.

Since the operator sequences $(U_{\pm\mu_n})_{n\in\mathbb{N}}$ and $(P_{\mu_n})_{n\in\mathbb{N}}$ are bounded in $\mathcal{B}(H)$, the sequences $(V_{\mu_n}\varphi)_{n\in\mathbb{N}}$ and $([V_{\mu_n}\varphi]')_{n\in\mathbb{N}}$ are bounded in H. Therefore, the set $\{V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is bounded in W_2^1. Note that for an arbitrary $m \in \mathbb{N}$ the operator

$$W_2^1 \ni f \mapsto G_m f \in H$$

is compact. Thus the set $\{G_m V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in H for an arbitrary $m \in \mathbb{N}$. In view of Lemma 9, for an arbitrary $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|G_m^{\perp} V_{\mu_n}\varphi\| \le \varepsilon$, $n \in \mathbb{N}$. Thus the set $\{V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in H.

It follows from the above result that for an arbitrary $\varphi \in W_2^1$ and for an arbitrary sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ from the vector sequence $(V_{\mu_{n_k}}\varphi)_{k\in\mathbb{N}}$ one can choose a convergent subsequence. Observing that the space H is separable and the set W_2^1 is everywhere dense in H, by Lemma 18, we conclude that from the sequence $(V_{\mu_n})_{n\in\mathbb{N}}$ one can choose the convergent subsequence $(V_{\mu_{n_k}})_{k\in\mathbb{N}}$, which converges in the strong operator topology. Taking into account (17) and the equalities

$$U_{\mu_{n_k}} = P_{\mu_{n_k}}^{-1} V_{\mu_{n_k}} = P_{-\mu_{n_k}} V_{\mu_{n_k}}, \quad k \in \mathbb{N},$$

we obtain that the sequence $(U_{\mu_{n_k}})_{k\in\mathbb{N}}$ converges in the strong operator topology. The established results imply that from the sequence $(\mu_n)_{n\in\mathbb{N}}$ one can choose a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that both limits exist

$$U = \operatorname*{s-lim}_{k\to\infty} U_{\mu_{n_k}}, \quad U_- = \operatorname*{s-lim}_{k\to\infty} U_{-\mu_{n_k}}. \tag{19}$$

Let us prove part (1). Fix an arbitrary $\varphi \in W_2^1$. From (18), we deduce that

$$S_0 P_{\mu_{n_k}} U_{\mu_{n_k}}\varphi = P_{\mu_{n_k}} U_{-\mu_{n_k}} S_0\varphi.$$

Taking into account (17), (19) and the fact that the operator S_0 is closed, by passing to the limit, we obtain that $P_\mu U\varphi \in \operatorname{dom} S_0$ and $S_0 P_\mu U\varphi = P_\mu U_- S_0\varphi$. Thus $U\varphi \in \operatorname{dom} S_\mu$ and $S_\mu U\varphi = U_- S_0\varphi$. Therefore, part (1) is proved.

Obviously, part (2) follows from part (1). Let us prove part (3). In view of (1) and (2), by using the equality $T_\mu = S_{-\mu}S_\mu$, we obtain that for an arbitrary $f \in W_2^2$ the element Uf belongs to $\operatorname{dom} T_\mu$ and

$$T_\mu U f = S_{-\mu} S_\mu U f = S_{-\mu} U_- S_0 f = U S_0^2 f = U T_0 f.$$

□

Lemma 13. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}_0$ and $\mu_n \overset{w}{\to} \mu \in \boldsymbol{M}$. Assume that $U_{\mu_n} \overset{s}{\to} U$. Then:* 1) *the operator U belongs to the algebra $\mathcal{B}^+(H)$ and is invertible in the algebra $\mathcal{B}(H)$, moreover, $\|U^{-1}\|_{\mathcal{B}(H)} \le \sqrt{2}$;* 2) *for an arbitrary $\lambda \in \mathbb{C}_+$ the function Ue_λ is the right Jost solution of the equation $\mathfrak{t}_\mu(f) = \lambda^2 f$.*

Proof. Let us prove part (1). Since $\mu_n \in \boldsymbol{M}_0$ $(n \in \mathbb{N})$, the sequence $(U_{\mu_n})_{n\in\mathbb{N}}$ belongs to the algebra $\mathcal{B}^+(H)$ (see Corollary 3). Therefore, the operator U also belongs to $\mathcal{B}^+(H)$.

Theorem 2 implies that the operators U_{μ_n} are invertible in $\mathcal{B}(H)$ and $|U_{\mu_n}^{-1}\|_{\mathcal{B}(H)} \le \sqrt{2}$, $n \in \mathbb{N}$. Thus, in view of Lemma 19, to prove invertibility of the operator U, it suffices to show that from the sequence $(U^*_{\mu_n})_{n\in\mathbb{N}}$ one can choose a subsequence, which converges in the strong operator topology in the algebra $\mathcal{B}(H)$. It follows from Lemma 2 that $P_{\pm\mu_n} \overset{s}{\to} P_{\pm\mu}$ as $n \to \infty$, and the sequences $(U_{\mu_n})_{n\in\mathbb{N}}$ and $(U_{-\mu_n})_{n\in\mathbb{N}}$ are bounded. Replacing the measures μ_n in (18) with $-\mu_n$, we obtain that for an arbitrary $\varphi, \psi \in W_2^1$

$$(S_0 P_{-\mu_n} U_{-\mu_n} \varphi \mid \psi)_H = (P_{-\mu_n} U_{\mu_n} S_0 \varphi \mid \psi)_H,$$

and, therefore,

$$(\varphi \mid U^*_{-\mu_n} P_{-\mu_n} S_0 \psi)_H = (S_0 \varphi \mid U^*_{\mu_n} P_{-\mu_n} \psi)_H, \quad \varphi, \psi \in W_2^1.$$

The last equality implies that the functional

$$W_2^1 \ni \varphi \mapsto (S_0 \varphi \mid U^*_{\mu_n} P_{-\mu_n} \psi)_H$$

is continuous in H. Thus $U^*_{\mu_n} P_{-\mu_n} \psi \in \operatorname{dom} S_0$ and

$$S_0 U^*_{\mu_n} P_{-\mu_n} \psi = U^*_{-\mu_n} P_{-\mu_n} S_0 \psi. \tag{20}$$

Let us consider the auxiliary operators

$$V_{\mu_n} := U^*_{\mu_n} P_{-\mu_n}, \quad n \in \mathbb{N}.$$

Let us fix $\psi \in W_2^1$ and show that the set $\{V_{\mu_n}\psi\}_{n\in\mathbb{N}}$ is relatively compact in H.

It follows from the equality (20) that $V_{\mu_n}\psi \in W_2^1$ for all $n \in \mathbb{N}$ and

$$(V_{\mu_n}\psi)' = U^*_{-\mu_n} P_{-\mu_n} \psi'.$$

Since the operator sequences $(U_{\pm\mu_n})_{n\in\mathbb{N}}$ and $(P_{-\mu_n})_{n\in\mathbb{N}}$ are bounded in $\mathcal{B}(H)$, the sequences $(V_{\mu_n}\psi)_{n\in\mathbb{N}}$ and $([V_{\mu_n}\psi]')_{n\in\mathbb{N}}$ are bounded in H. Hence the set $\{V_{\mu_n}\psi\}_{n\in\mathbb{N}}$ is bounded in the space W_2^1. Thus for an arbitrary $m \in \mathbb{N}$ the set $\{G_m V_{\mu_n}\psi\}_{n\in\mathbb{N}}$ is relatively compact in H. In view of Lemma 9, for an arbitrary $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|G_m^\perp U^*_{\mu_n} P_{-\mu}\psi\| \le \varepsilon$, $n \in \mathbb{N}$. Thus

$$\|G_m^\perp V_{\mu_n}\psi\| = \|G_m^\perp U^*_{\mu_n} P_{-\mu_n}\psi\| \le \|G_m^\perp U^*_{\mu_n} P_{-\mu}\psi\| +$$
$$+ \|G_m^\perp U^*_{\mu_n}(P_{-\mu} - P_{-\mu_n})\psi\| \le \varepsilon + C\|(P_{-\mu} - P_{-\mu_n})\psi\|, \quad n \in \mathbb{N},$$

where $C = \sup_{n\in\mathbb{N}} \|U^*_{\mu_n}\|$. Therefore, taking into account $P_{-\mu_n} \overset{s}{\to} P_{-\mu}$, we obtain that the set $\{V_{\mu_n}\psi\}_{n\in\mathbb{N}}$ is relatively compact in H.

It follows from the above proof that for an arbitrary $\psi \in W_2^1$ and for an arbitrary sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ from the sequence $(V_{\mu_{n_k}}\psi)_{k\in\mathbb{N}}$ one can choose a convergent subsequence. Observing Lemma 18, we conclude that from the sequence $(V_{\mu_n})_{n\in\mathbb{N}}$ one can choose a subsequence $(V_{\mu_{n_k}})_{k\in\mathbb{N}}$ that converges in the strong operator topology. Taking into account the equalities

$$U^*_{\mu_{n_k}} = V_{\mu_{n_k}} P_{\mu_{n_k}}, \quad k \in \mathbb{N},$$

and the convergence $P_{\mu_n} \overset{s}{\to} P_\mu$, we obtain that the sequence $(U^*_{\mu_{n_k}})_{k\in\mathbb{N}}$ also converges in the strong operator topology. Therefore, invertibility of the operator U is proved.

We now prove part (2). Fix $\lambda \in \mathbb{C}_+$ and let $\varphi = Ue_\lambda$. It is obvious that for an arbitrary $a \in \mathbb{R}$ there exists a function $g \in W_2^2$ such that $g(x) = e_\lambda(x)$ for $x > a$. Since $U \in \mathcal{B}^+(H)$,

$$\varphi(x) = (Ue_\lambda)(x) = (Ug)(x), \quad x > a.$$

Thus, in view of Lemma 12 (part (3)), for $x > a$

$$[\mathfrak{t}_\mu(\varphi)](x) = [T_\mu Ug](x) = [UT_0 g](x) = \lambda^2[Ue_\lambda](x) = \lambda^2\varphi(x).$$

Since a is arbitrary, we conclude that φ is a solution of the equation $\mathfrak{t}_\mu(y) = \lambda^2 y$. Let us show that

$$\varphi(x) = e^{i\lambda x}(1 + o(1)), \quad x \to +\infty. \tag{21}$$

According to Lemma 10, we obtain that for an arbitrary $n \in \mathbb{N}$

$$e_{\lambda,\mu_n}(x) = (U_{\mu_n} e_\lambda)(x) = (U_{\mu_n} g)(x), \quad x > a,$$

and

$$|e^{-i\lambda x} e_{\lambda,\mu_n}(x) - 1| \leq e^{\|\mu_n\|} h_{|\mu_n|}(x), \quad x \in \mathbb{R}. \tag{22}$$

Since $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$, we obtain (see Proposition 1) that $\sup_{n\in\mathbb{N}} \|\mu_n\| = c < \infty$ and the sequence $(\mu_n)_{n=1}^{\infty}$ is uniformly dense. Thus for an arbitrary $\varepsilon > 0$ there exists $a_\varepsilon > a$ such that $h_{|\mu_n|}(x) \leq \varepsilon e^{-c}, n \in \mathbb{N}, x > a_\varepsilon$. It follows from the established above that

$$|e^{-i\lambda x}(U_{\mu_n} g)(x) - 1| \leq \varepsilon, \quad x > a_\varepsilon.$$

Thus, taking into account $\lim\limits_{n\to\infty} \|Ug - U_{\mu_n} g\| = 0$ and $(Ug)(x) = \varphi(x)$, we get that the inequality

$$|e^{-i\lambda x}\varphi(x) - 1| \leq \varepsilon \tag{23}$$

holds almost everywhere on the half-line (a_ε, ∞). Note that the function φ is left continuous. Indeed, since $\varphi \in \operatorname{dom} T_\mu$, the function $p_\mu\varphi$ is continuous on $\mathbb{R}$. And since p_μ is left continuous, the function φ is also left continuous. Thus the inequality (23) holds for all $x > a$. Hence, since ε is arbitrary, the asymptotic (21) holds. Therefore, Ue_λ is the right Jost solution of the equation $\mathfrak{t}_\mu(f) = \lambda^2 f$. □

Remark 4. Since the space $\boldsymbol{M}_0$ is everywhere dense in $\boldsymbol{M}$ in the weak topology, in view of Lemmas 12 and 13, for arbitrary $\mu \in \boldsymbol{M}$ and $\lambda \in \mathbb{C}_+$ the equation $\mathfrak{t}_\mu(f) = \lambda^2 f$ has the right Jost solution.

Proof of Theorem 3. Denote by J the reflection operator given in the space $L_{1,\mathrm{loc}}$ by the formula

$$(Jf)(x) = f(-x), \quad x \in \mathbb{R}.$$

It is an involution, and acts in the space H as a unitary operator.

It is easy to see that for an arbitrary measure $\mu \in \boldsymbol{M}$ there exists the unique measure $\mu^\flat \in \boldsymbol{M}$, which satisfies the equality

$$p_{\mu^\flat}(-x) = p_\mu(x)$$

at all points of continuity of the function p_μ. Therefore, we obtain the mapping $\boldsymbol{M} \ni \mu \mapsto \mu^\flat \in \boldsymbol{M}$, which is an involution. It is easy to see that for $\mu \in \boldsymbol{M}$

$$JP_\mu J = P_{\mu^\flat}, \quad JS_\mu J = -S_{\mu^\flat}, \quad JT_\mu J = T_{\mu^\flat}, \quad J\mathfrak{t}_\mu J = \mathfrak{t}_{\mu^\flat}.$$

Let $\mu \in \boldsymbol{M}$. It follows from the above result that if φ is the right Jost solution of the equation $\mathfrak{t}_{\mu^\flat}(y) = \lambda^2 y$, then $J\varphi$ is the left Jost solution of the equation $\mathfrak{t}_\mu(y) = \lambda^2 y$. Thus, by Remark 4, we get that for arbitrary $\mu \in \boldsymbol{M}$ and $\lambda \in \mathbb{C}_+$ the equation $\mathfrak{t}_\mu(f) = \lambda^2 f$ has right and left Jost solutions.

It follows from the above that to prove uniqueness of the Jost solutions, it suffices to prove only uniqueness of the right Jost solution. Assume that for some $\mu \in \boldsymbol{M}$ and $\lambda \in \mathbb{C}_+$ the equation $\mathfrak{t}_\mu(y) = \lambda^2 y$ has two different right Jost solutions φ_1 and φ_2. These solutions are certainly linearly independent. Otherwise, in view of the asymptotic on $+\infty$, they would be equal. Denote by φ a left Jost solution of the equation $\mathfrak{t}_\mu(y) = \lambda^2 y$. Linear independence of φ_1 and φ_2 implies that φ is a linear combination of the solutions φ_1 and φ_2. It means that φ is the eigenfunction of the operator T_μ corresponding to the eigenvalue λ^2. But this is impossible, since $T_\mu \geq 0$ and $\lambda^2 \notin [0, \infty)$. Therefore, uniqueness of the right Jost solution is proved.

It remains to prove the estimate (16) for $e_{\lambda,\mu}$. Fix an arbitrary $\mu \in \boldsymbol{M}$ and $\lambda \in \mathbb{C}_+$, and let $(\nu_n)_{n\in\mathbb{N}}$ be a θ-sequence for μ. Since (see Lemma 1) $\nu_n \xrightarrow{w} \mu \in \boldsymbol{M}$ and $\|\nu_n\| \leq \|\mu\|$, $h_{|\nu_n|} \leq h_{|\mu|}$, for all $n \in \mathbb{N}$, we find (see (22)) that $|e^{-i\lambda x} e_{\lambda,\mu_n}(x) - 1| \leq e^{\|\mu\|} h_{|\mu|}(x)$, $x \in \mathbb{R}$.

Taking into account the above inequality, following the logic on the final part of the proof of Lemma 13, by passing to the limit, we obtain

$$|e^{-i\lambda x} e_{\lambda,\mu}(x) - 1| \leq e^{\|\mu\|} h_{|\mu|}(x), \quad x \in \mathbb{R}.$$

□

Proof of Theorem 4. Let $\mu \in M$ and $(\nu_k)_{k\in\mathbb{N}}$ be a θ-sequence for μ. Then $\mu_n \xrightarrow{w} \mu$ and $\{\nu_n\}_{n\in\mathbb{N}} \subset \boldsymbol{M}_0$. Combining the results of Lemmas 12 and 13, we conclude that there exists the operator $U \in \mathcal{B}^+(H)$, which possess the following properties: (a) $Ue_\lambda = e_{\lambda,\mu}$ for all $\lambda \in \mathbb{C}_+$; (b) U is invertible in the algebra $\mathcal{B}(H)$, and $\|U^{-1}\|_{\mathcal{B}(H)} \leq \sqrt{2}$; (c) if $f \in \operatorname{dom} T_0$, then $Uf \in \operatorname{dom} T_\mu$ and $T_\mu U f = U T_0 f$.

In view of Remark 2 and Lemma 3, the condition (a) determines operator U unambiguously. Thus it only remains to prove the equality $T_\mu = U T_0 U^{-1}$. Since the operator T_μ is self-adjoint, the operator $T_0' := U^{-1} T_\mu U$ is similar to a self-adjoint operator. The condition (c) implies the inclusion $T_0 \subset T_0'$. But it is possible only if $T_0' = T_0$. Therefore, $U^{-1} T_\mu U = T_0$, that is $T_\mu = U T_0 U^{-1}$. □

5. Proof of Theorem 1.

To prove Theorem 1, we repeat the consideration of Section 4 and partly Section 3 with some modifications. In the case when proofs are the same, we will only make a corresponding reference.

The following lemma is a counterpart of Corollaries 2 and 4.

Lemma 14. *Let $\mu, \widetilde{\mu} \in \boldsymbol{M}$. Then*

$$\|\mathcal{U}_\mu\|_{\mathcal{B}(H)} \leq \exp\{\|\mu\|\}, \quad \|\mathcal{U}_\mu - \mathcal{U}_{\widetilde{\mu}}\|_{\mathcal{B}(H)} \leq \exp(\|\mu\| + \|\widetilde{\mu}\|)\|\mu - \widetilde{\mu}\|. \tag{24}$$

Moreover, if $\operatorname{supp}\mu \subset (-k, k)$ *for some* $k \in \mathbb{N}$, *then*

$$G_{5k}^\perp \mathcal{U}_\mu G_k = 0. \tag{25}$$

Proof. Let $\mu, \widetilde{\mu} \in \boldsymbol{M}$, and $(\nu_n)_{n=1}^\infty$, $(\widetilde{\nu}_n)_{n=1}^\infty$ be θ-sequences for the measures μ and $\widetilde{\mu}$ respectively. Due to Lemma 1, these sequences belong to $\boldsymbol{M}_0$ and converge weakly to the measures μ and $\widetilde{\mu}$ respectively. Moreover, $\|\nu_n\| \leq \|\mu\|$, $\|\widetilde{\nu}_n\| \leq \|\widetilde{\mu}\|$, $n \in \mathbb{N}$.

Note, $(\nu_n - \widetilde{\nu}_n)_{n=1}^\infty$ is a θ-sequence for the measure $\mu - \widetilde{\mu}$, thus $\|\nu_n - \widetilde{\nu}_n\| \leq \|\mu - \widetilde{\mu}\|$, $n \in \mathbb{N}$. Taking into account the above inequalities and Corollary 2, we obtain that

$$\|U_{\nu_n}\|_{\mathcal{B}(H)} \leq \exp\{\|\mu\|\}, \quad \|U_{\nu_n} - U_{\widetilde{\nu}_n}\|_{\mathcal{B}(H)} \leq \exp(\|\mu\| + \|\widetilde{\mu}\|)\|\mu - \widetilde{\mu}\|. \tag{26}$$

By passing in (26) to the limit as $n \to \infty$, we obtain the estimate (24)

Let $\operatorname{supp} \mu \subset (-k, k)$ for some $k \in \mathbb{N}$. It follows from part (2) of Lemma 1 that $\operatorname{supp} \nu_n \subset (-k, k)$, $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Thus part (4) of Lemma 4 implies that $G_{5k}^{\perp} U_{\nu_n} G_k = 0$, $n \geq n_0$. By passing to the limit as $n \to \infty$, we obtain the equality (25). □

The following lemma is an analog of Lemma 9.

Lemma 15. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Then for arbitrary $f \in H$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|G_m^{\perp} \mathcal{U}_{\mu_n} f\| \leq \varepsilon$, $n \in \mathbb{N}$.*

Proof. Let $f \in H$, $\varepsilon \in (0, 1)$ and

$$c := \sup_{n\in\mathbb{N}} (2\|\mu_n\| + 1), \;\; \gamma := \frac{\varepsilon}{e^c(\|f\|_2 + 1)}.$$

Let us choose a compactly supported function $\widetilde{f} \in H$ such that $\|f - \widetilde{f}\|_2 \leq \gamma$. Since the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly dense, there exists a sequence $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ in $\boldsymbol{M}$ such that all supports of the measures $\widetilde{\mu}_n$ lie in some interval Δ_ρ and $\|\mu_n - \widetilde{\mu}_n\| \leq \gamma$, $n \in \mathbb{N}$.

We can also assume that $\operatorname{supp} \widetilde{f} \subset \Delta_\rho$. It follows from Lemma 14 that there exists $m \in \mathbb{N}$ such that $G_m^{\perp} \mathcal{U}_{\widetilde{\mu}_n} \widetilde{f} = 0$ for all $n \in \mathbb{N}$. Thus, we obtain that

$$\begin{gathered}
\|G_m^{\perp} \mathcal{U}_{\mu_n} f\| = \|G_m^{\perp} \mathcal{U}_{\mu_n} f - G_m^{\perp} \mathcal{U}_{\widetilde{\mu}_n} \widetilde{f}\| \leq \\
\leq \|\mathcal{U}_{\mu_n} f - \mathcal{U}_{\widetilde{\mu}_n} \widetilde{f}\| \leq \|\mathcal{U}_{\mu_n} - \mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f\|_2 + \|\mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f - \widetilde{f}\|_2, \\
\|G_m^{\perp} \mathcal{U}^*_{\mu_n} f\| = \|G_m^{\perp} U^*_{\mu_n} f - G_m^{\perp} \mathcal{U}^*_{\widetilde{\mu}_n} \widetilde{f}\| \leq \\
\leq \|\mathcal{U}^*_{\mu_n} f - \mathcal{U}^*_{\widetilde{\mu}_n} \widetilde{f}\| \leq \|\mathcal{U}_{\mu_n} - \mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f\|_2 + \|\mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f - \widetilde{f}\|_2.
\end{gathered}$$

The estimates (24) imply that

$$\|\mathcal{U}_{\mu_n} - \mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \leq e^c \gamma, \quad \|\mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \leq e^c, \quad n \in \mathbb{N}.$$

Thus $\|\mathcal{U}_{\mu_n} - \mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f\|_2 + \|\mathcal{U}_{\widetilde{\mu}_n}\|_{\mathcal{B}(H)} \|f - \widetilde{f}\|_2 \leq e^c \gamma \|f\|_2 + e^c \gamma = \varepsilon$. Hence $\|G_m^{\perp} \mathcal{U}_{\mu_n} f\| \leq \varepsilon$ for all $n \in \mathbb{N}$. □

Lemma 16. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Then from the sequence $(\mu_n)_{n\in\mathbb{N}}$ one can choose a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that there exist the limits $\mathcal{U} = \operatorname*{s-lim}_{k\to\infty} \mathcal{U}_{\mu_{n_k}}$, $\mathcal{U}_- = \operatorname*{s-lim}_{k\to\infty} \mathcal{U}_{-\mu_{n_k}}$. Moreover:*

1) *if $f \in \operatorname{dom} S_0$, then $\mathcal{U} f \in \operatorname{dom} S_\mu$ and $S_\mu \mathcal{U} f = \mathcal{U}_- S_0 f$;*
2) *if $f \in \operatorname{dom} S_0$, then $\mathcal{U}_- f \in \operatorname{dom} S_{-\mu}$ and $S_{-\mu} \mathcal{U}_- f = \mathcal{U} S_0 f$;*
3) *if $f \in \operatorname{dom} T_0$, then $\mathcal{U} f \in \operatorname{dom} T_\mu$ and $T_\mu \mathcal{U} f = \mathcal{U} T_0 f$.*

Proof. Since $\mu_n \xrightarrow{w} \mu$, we obtain $\sup_{n\in\mathbb{N}} \|\mu_n\| = \alpha < \infty$. Therefore (see Lemma 14),

$$\sup_{n\in\mathbb{N}} \|\mathcal{U}_{\pm\mu_n}\|_{\mathcal{B}(H)} \leq e^\alpha.$$

Let $\varphi \in W_2^1$. In view of Lemma 4, $\mathcal{U}_{\mu_n} \varphi \in \operatorname{dom} S_\mu$ and $S_{\mu_n} \mathcal{U}_{\mu_n} \varphi = \mathcal{U}_{-\mu_n} S_0 \varphi$, $n \in \mathbb{N}$. Since $S_{\mu_n} = P_{\mu_n}^{-1} S_0 P_{\mu_n}$,

$$S_0 P_{\mu_n} \mathcal{U}_{\mu_n} \varphi = P_{\mu_n} \mathcal{U}_{-\mu_n} S_0 \varphi, \quad \varphi \in W_2^1. \tag{27}$$

Consider the auxiliary operators

$$V_{\mu_n} := P_{\mu_n}\mathcal{U}_{\mu_n}, \quad n \in \mathbb{N}.$$

Let us fix $\varphi \in W_2^1$ and show that the set $\{V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in the space H. The equality (27) implies that $V_{\mu_n}\varphi \in W_2^1$ for all $n \in \mathbb{N}$ and $(V_{\mu_n}\varphi)' = P_{\mu_n}U_{-\mu_n}\varphi'$. Repeating the same consideration as in the proof of Lemma 12, we show that for an arbitrary $m \in \mathbb{N}$ the set $\{G_m V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in H.

By Lemma 15, for an arbitrary $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|G_m^\perp V_{\mu_n} f\| \le \varepsilon$, $n \in \mathbb{N}$. Thus the set $\{V_{\mu_n}\varphi\}_{n\in\mathbb{N}}$ is relatively compact in H. Next, the proof follows the considerations of the proof of Lemma 12. □

Lemma 17. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Assume that $\mathcal{U}_{\mu_n} \xrightarrow{s} \mathcal{U}$. Then $\mathcal{U} = \mathcal{U}_\mu$.*

Proof. According to Theorem 4, $\mathcal{U}_{\mu_n} \in \mathcal{B}^+(H)$ for all $n \in \mathbb{N}$. And, thus the limited operator $\mathcal{U}$ also belongs to $\mathcal{B}^+(H)$. Fix $\lambda \in \mathbb{C}_+$ and let $\varphi = \mathcal{U}e_\lambda$. For an arbitrary $a \in \mathbb{R}$ there exists the function $g \in W_2^2$ such that $g(x) = e_\lambda(x)$ for $x > a$. Since $\mathcal{U} \in \mathcal{B}^+(H)$,

$$\varphi(x) = (\mathcal{U}e_\lambda)(x) = (\mathcal{U}g)(x), \quad x > a.$$

Thus, in view of part (3) of Lemma 16, for $x > a$

$$[\mathfrak{t}_\mu(\varphi)](x) = [T_\mu\mathcal{U}g](x) = [\mathcal{U}T_0 g](x) = \lambda^2[\mathcal{U}e_\lambda](x) = \lambda^2\varphi(x).$$

Since a is arbitrary, we have that $\mathfrak{t}_\mu(\varphi) = \lambda^2\varphi$. Let us show that

$$\varphi(x) = e^{i\lambda x}(1 + o(1)), \quad x \to +\infty. \tag{28}$$

Using Theorems 3 and 4, we obtain that for an arbitrary $n \in \mathbb{N}$

$$e_{\lambda,\mu_n}(x) = (U_{\mu_n}e_\lambda)(x) = (U_{\mu_n}g)(x), \quad x > a,$$

and

$$|e^{-i\lambda x}e_{\lambda,\mu_n}(x) - 1| \le e^{\|\mu_n\|}h_{|\mu_n|}(x), \quad x \in \mathbb{R}.$$

Hence

$$|e^{-i\lambda x}(U_{\mu_n}g)(x) - 1| \le e^{\|\mu_n\|}h_{\mu_n}(x), \quad x > a.$$

Since $\mu_n \xrightarrow{w} \mu$, we obtain that $\sup_{n\in\mathbb{N}}\|\mu_n\| = c < \infty$ and the sequence $(\mu_n)_{n=1}^\infty$ is uniformly dense. Thus for an arbitrary $\varepsilon > 0$ there exists $a_\varepsilon > a$ such that $h_{|\mu_n|}(x) \le \varepsilon e^{-c}$, $n \in \mathbb{N}$, $x > a_\varepsilon$. It follows that

$$|e^{-i\lambda x}(U_{\mu_n}g)(x) - 1| \le \varepsilon, \quad x > a_\varepsilon.$$

Thus, taking into account that $\lim\limits_{n\to\infty}\|Ug - U_{\mu_n}g\| = 0$ and $(Ug)(x) = \varphi(x)$, we conclude that the inequality

$$|e^{-i\lambda x}\varphi(x) - 1| \le \varepsilon \tag{29}$$

holds almost everywhere on the half-line (a_ε, ∞). Thus similar considerations as in the proof of Lemma 13 establish that the inequality (29) holds for all $x > a$. Since ε is arbitrary, we establish the asymptotic (28). Therefore, $\mathcal{U}e_\lambda$ is the right Jost solution of the equation $\mathfrak{t}_\mu(f) = \lambda^2 f$. Hence, we proved that $\mathcal{U}e_\lambda = e_{\lambda,\mu}$ for all $\lambda \in \mathbb{C}_+$. Thus, in view of Theorem 4, $\mathcal{U} = \mathcal{U}_\mu$. □

Proof of Theorem 1. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{M}$ and $\mu_n \xrightarrow{w} \mu \in \boldsymbol{M}$. Combining the results of Lemmas 16 and 17, we conclude that:

(*a*) from each subsequence of the sequence $(\mathcal{U}_{\mu_n})_{n\in\mathbb{N}}$ one can choose a subsequence, which converges in the strong operator topology;

(*b*) if the subsequences $(\mathcal{U}_{\mu_{n_k}})_{k\in\mathbb{N}}$ of the sequence $(\mathcal{U}_{\mu_n})_{n\in\mathbb{N}}$ converges in the strong operator topology, then $\operatorname*{s-lim}_{k\to\infty}\mathcal{U}_{\mu_{n_k}} = \mathcal{U}_\mu$.

Thus, by Lemma 20, the sequence $(\mathcal{U}_{\mu_n})_{n\in\mathbb{N}}$ is convergent in the strong operator topology and $\mathcal{U}_{\mu_n} \xrightarrow{s} \mathcal{U}_\mu$. Therefore, the mapping $\boldsymbol{M} \ni \mu \mapsto \mathcal{U}_\mu \in \mathcal{B}(H)$ is sequentially continuous if $\boldsymbol{M}$ is equipped with the weak topology and $\mathcal{B}(H)$ is equipped with the strong operator topology. It follows from Lemma 10 that $U_\mu e_\lambda = e_{\lambda,\mu}$, $\lambda \in \mathbb{C}_+$. Thus (see Theorem 4) $U_\mu = \mathcal{U}_\mu$ for $\mu \in \boldsymbol{M}_0$. Since $\boldsymbol{M}_0$ is everywhere dense in $\boldsymbol{M}$ (in the weak topology), the mapping $\mu \mapsto \mathcal{U}_\mu$ is the unique continuous extension of the mapping $\mu \mapsto U_\mu$. It also follows from Theorem 4 that for every $\mu \in \boldsymbol{M}$ the operator $\mathcal{U}_\mu$ is invertible in $\mathcal{B}(H)$ and $T_\mu = \mathcal{U}_\mu T_0 \mathcal{U}_\mu^{-1}$. □

Appendix. Some auxiliary results. Here we will give auxiliary lemmas.

Lemma 18. *Let G be a separable infinite-dimensional Hilbert space, Φ be an everywhere dense set in G, and the sequence $(A_n)_{n\in\mathbb{N}}$ possess in the algebra $\mathcal{B}(G)$ the following properties:* 1) $\sup_{n\in\mathbb{N}}\|A_n\| < \infty$; 2) *for an arbitrary $\varphi \in \Phi$ and for an arbitrary subsequence $(A_{n_k})_{k\in\mathbb{N}}$ one can choose a G-convergent subsequence of the sequence $(A_{n_k}\varphi)_{k\in\mathbb{N}}$. Then from $(A_n)_{n\in\mathbb{N}}$ one can choose a subsequence which converges in the strong operator topology.*

Proof. To prove this lemma we use Cantor's diagonal argument. Since G is separable, there exists a countable subset $\{\varphi_s\}_{s\in\mathbb{N}}$ in Φ which is everywhere dense in G. Using (2), one can construct by induction a sequence of the subsequences $(A_n^{\{j\}})_{n\in\mathbb{N}}$, $j \in \mathbb{N}$, such that for each j: (*a*) the sequence $(A_n^{\{j\}}\varphi_j)_{n\in\mathbb{N}}$ is convergent in the space G; (*b*) the sequence $(A_n^{\{j+1\}})_{n\in\mathbb{N}}$ is a subsequence of the sequence $(A_n^{\{j\}})_{n\in\mathbb{N}}$. Then the diagonal sequence $(A_n^{\{n\}})_{n\in\mathbb{N}}$ converges on each φ_j. Since the sequence $(A_n)_{n\in\mathbb{N}}$ is bounded, and the set $\{\varphi_s\}_{s\in\mathbb{N}}$ is everywhere dense in G, the sequence $(A_n^{\{n\}})_{n\in\mathbb{N}}$ converges in the strong operator topology. □

Lemma 19. *Let G be a Hilbert space, $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(G)$, and $A_n \xrightarrow{s} A \in \mathcal{B}(G)$. Assume that:* 1) *all A_n are invertible in the algebra $\mathcal{B}(G)$ and $\sup_{n\in\mathbb{N}}\|A_n^{-1}\| = \alpha < \infty$;* 2) *from the sequence $(A_n^*)_{n\in\mathbb{N}}$ one can choose a subsequence which converges in the strong operator topology. Then A is invertible in the algebra $\mathcal{B}(G)$. Moreover, $A_n^{-1} \xrightarrow{s} A^{-1}$ and $\|A^{-1}\| \le \alpha$.*

Proof. Without loss of generality, we may assume that $A_n^* \xrightarrow{s} B \in \mathcal{B}(H)$. Then

$$(Af \mid g) = \lim_{n\to\infty}(A_n f \mid g) = \lim_{n\to\infty}(f \mid A_n^* g) = (f \mid Bg), \quad f, g \in H.$$

Hence $B = A^*$. For arbitrary $f \in G$ and $n \in \mathbb{N}$, $\|f\| \le \|A_n^{-1}\|\|A_n f\| \le \alpha\|A_n f\|$ and $\|f\| \le \|(A_n^*)^{-1}\|\|A_n^* f\| \le \alpha\|A_n^* f\|$. Thus, by passing to the limit, we obtain that

$$\inf\{\|Af\|, \|A^* f\|\} \ge \alpha^{-1}\|f\|, \quad f \in G,$$

so that the operators A and A^* are bounded below. Hence A has an inverse operator in $\mathcal{B}(G)$. By the relation $A_n^{-1} - A^{-1} = A_n^{-1}(A - A_n)A^{-1}$, we get that for an arbitrary $f \in G$

$$\|(A_n^{-1} - A^{-1})f\| \le \alpha\|(A - A_n)A^{-1}f\| \to 0 \quad \text{as} \quad n \to \infty,$$

and, therefore, $A_n^{-1} \xrightarrow{s} A^{-1}$. □

Lemma 20. *Let X be a Hausdorff topological space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Assume that:* 1) *from each subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$ one can choose a convergent subsequence;* 2) *all convergent subsequences of the sequence $(x_n)_{n\in\mathbb{N}}$ have the same limit $a \in X$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a.*

The proof of Lemma 20 is obvious.

Lemma 21. *If self-adjoint operators A and B are similar, then they are unitarily equivalent.*

Proof. Let self-adjoint operators A and B act in the Hilbert space G and

$$A = MBM^{-1}, \tag{30}$$

where an operator $M \in \mathcal{B}(G)$ is invertible in the algebra $\mathcal{B}(G)$. Let $M = UN$ be the polar decomposition of the operator M, i.e., $N = (M^*M)^{1/2}$ and $U = MN^{-1}$. Since M is invertible, the operator U is unitary. It follows from (30) that $MBM^{-1} = A = A^* = (M^*)^{-1}BM^*$. Therefore, $BM^*M = M^*MB$, i.e., the self-adjoint operator B commutes with the bounded positive operator M^*M. Thus B commutes with N too. Hence,

$$A = MBM^{-1} = UNBN^{-1}U^{-1} = UBU^{-1}.$$

□

Acknowledgements. The authors express their gratitude to Prof. Rostyslav Hryniv for stimulating discussions and helpful comments.

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Received 27.06.2023
Revised 11.09.2023