УДК 517.55

Kh. Sukhorukova

SPACES OF NON-ADDITIVE MEASURES GENERATED BY TRIANGULAR NORMS

Kh. Sukhorukova. *Spaces of non-additive measures generated by triangular norms*, Mat. Stud. **59** (2023), 215–224.

We consider non-additive measures on the compact Hausdorff spaces, which are generalizations of the idempotent measures and max-min measures. These measures are related to the continuous triangular norms and they are defined as functionals on the spaces of continuous functions from a compact Hausdorff space into the unit segment.

The obtained space of measures (called $*$-measures, where $*$ is a triangular norm) are endowed with the weak* topology. This construction determines a functor in the category of compact Hausdorff spaces. It is proved, in particular, that the $*$-measures of finite support are dense in the spaces of $*$-measures. One of the main results of the paper provides an alternative description of $*$-measures on a compact Hausdorff space X, namely as hyperspaces of certain subsets in $X \times [0,1]$. This is an analog of a theorem for max-min measures proved by Brydun and Zarichnyi.

1. Introduction. Idempotent mathematics is a part of mathematics in which one of the ordinary arithmetic operations in $\mathbb{R}$ is replaced by an idempotent operation (e.g., maximum). The results and methods of idempotent mathematics find numerous applications in different parts of mathematics as well as in computer science and other disciplines. One can find a survey of some results of idempotent mathematics in [8].

The notion of probability measure has its counterparts in idempotent mathematics. The idempotent measures (also called Maslov measures) are introduced in [7]. The topological and categorical aspects of the theory of idempotent measures are considered in [13].

In this paper, to every triangular norm $*$ we assign a functor M^* acting in the category of compact Hausdorff spaces and continuous maps. This functor turns out to be normal in the sense of E. Shchepin [11]. The (modified) functor of idempotent measures as well as the functor of max-min measures are partial cases of our general construction.

The measure theories are often connected with the corresponding convexity theories. This is well known for the probability measures [12]. Also, the connection between the idempotent measures and the so-called max-plus convex sets defined in [5] is noticed in [13]. Note that the modified functor of idempotent measures, i.e., the functor of $\cdot$-measures (i.e., measures that correspond to the triangular norm of multiplication (see below)), corresponds to the theory of $\mathbb{B}$-convexity developed in [2]. In turn, the convexities are used in the equilibrium theory for games in measure-valued strategies, see [10].

2020 *Mathematics Subject Classification:* 54B20, 54B30.
Keywords: non-additive measure; triangular norm; max-min measure; compact Hausdorff space.
doi:10.30970/ms.59.2.215-224

2. Preliminaries. In the sequel, the proofs of some statements are (very) close to the proofs of the corresponding statements in [4]; we include them for the sake of reader's convenience. Unlikely to the case of spaces of idempotent measures and max-min measures we work with the spaces $C(X,[0,1])$. This allows to treat the known measures similarly (note that the coefficients are taken from $[-\infty,0]$ in the theory of idempotent measures, while they are taken from $[-\infty,\infty]$ in the theory of max-min measures).

2.1. Triangular norms. A triangular norm (t-norm) is a continuous, associative, commutative and monotonic binary operation on the unit segment $\mathbb{I}=[0,1]$ for which 1 is a unit.

Examples:

1. $a*b=ab$; 2. $a*b=\min\{a,b\}$; 3. $a*b=\max\{a+b-1,0\}$ (*Łukasiewicz t-norm*).

Note that $a*0=0$, for any t-norm $*$.

Given a countable family $*_i$ of t-norms and a countable disjoint family of subintervals (a_i,b_i) of $[0,1]$, we shrink $*_i$ into (a_i,b_i) and complete the rest with min. In this way we obtain a new triangular norm.

More formally, the ordinal sum of $(*_i,a_i,b_i)$ is given by the formula:

$$x*y=\begin{cases} a_i+(b_i-a_i)\left(\frac{x-a_i}{b_i-a_i}*_i,\frac{y-a_i}{b_i-a_i}\right) & \text{if } x,y\in[a_i,b_i];\\ \min\{x,y\}, & \text{otherwise.}\end{cases}$$

A t-norm $*$ is called Archimedean if for each $x,y\in(0,1)$ there is $n\in\mathbb{N}$ such that $x*\dots*x$ (n times) is less than or equal to y. By the Mostert–Shields theorem, every continuous t-norm is expressible as the ordinal sum of Archimedean continuous t-norms (see, e.g., [6]).

2.2. Hyperspaces. Let $\exp X$ denote the set of all nonempty compact subsets in a topological space X. Given $U_1,\dots,U_n\subset X$, let

$$\langle U_1,\dots,U_n\rangle=\Big\{A\in\exp X\mid A\subset\bigcup_{i=1}^n U_i,\ A\bigcap U_i\neq\varnothing,\ i=1,\dots,n\Big\}.$$

The family

$$\{\langle U_1,\dots,U_n\rangle\mid U_1,\dots,U_n \text{ are open},\ n\in\mathbb{N}\}$$

is known to be a base of the *Vietoris topology* on the set $\exp X$. The obtained topological space is called the *hyperspace* of X.

Given a family $\mathcal{U}$ of subsets of X, we say that $A,B\in\exp X$ are $\mathcal{U}$-close if $A\cap U\neq\varnothing\Leftrightarrow B\cap U\neq\varnothing$, for all $U\in\mathcal{U}$.

If $f\colon X\to Y$ is a map, then the map $\exp f\colon\exp X\to\exp Y$ is defined as follows: $\exp f(A)=f[A]$, $A\in\exp X$. Actually, exp is a functor in the category **Comp** of compact Hausdorff spaces and continuous maps.

3. Functionals. Let X be a compact Hausdorff space. Let $*$ be a t-norm. A functional $\mu\colon C(X,\mathbb{I})\to\mathbb{I}$ is a $*$-*measure* if

1. $\mu(c_X)=c$ (by c_X we denote the constant function on X whose value equals c);
2. $\mu(\lambda*\varphi)=\lambda*\mu(\varphi)$;
3. $\mu(\varphi\vee\psi)=\mu(\varphi)\vee\mu(\psi)$,

for all $c,\lambda\in\mathbb{I}$ and $\varphi,\psi\in C(X,\mathbb{I})$. (Hereafter $\vee$ denotes the maximum.)

Remark 1. The notion of $*$-measure resembles that of idempotent measure [13] as well as max-min measure [4].

Let $x \in X$. Denote by $\delta_x\colon C(X,\mathbb{I}) \to \mathbb{I}$ the Dirac measure, i.e. the map acting as follows: $\delta_x(\varphi) = \varphi(x)$. Clearly, δ_x is a $*$-measure.

One more example. For every $\varphi \in C(X,\mathbb{I})$, let $\omega_X(\varphi) = \sup\{\varphi(x) \mid x \in X\}$. It is easy to verify that ω_X is a $*$-measure.

Let $M^*(X)$ denote the set of all $*$-measures endowed with the weak* topology. A base of this topology can be described as follows. Given $\mu \in M^*(X)$, $\varphi_1, \dots, \varphi_n \in C(X,\mathbb{I})$ and $\varepsilon > 0$, we define the base neighborhood of μ as follows

$$O\langle \mu; \varphi_1, \dots, \varphi_n; \varepsilon\rangle = \{\nu \in M^*(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon,\ i = 1, \dots, n\}.$$

Let $\lambda_1, \dots, \lambda_n \in [0,1]$ be such that $\vee_{i=1}^n \lambda_i = 1$. Let also $\mu_1, \dots, \mu_n \in M^*(X)$. Define $\mu\colon C(X,\mathbb{I}) \to \mathbb{I}$ by the formula

$$\mu(\varphi) = \bigvee_{i=1}^n \lambda_i * \mu_i(\varphi). \tag{1}$$

Then it is clear than $\mu \in M^*(X)$. By $M^*_\omega(X)$ we denote the set of $*$-measures on X of form (1).

Remark 2. Hereafter, we assume that first $*$ is applied and then $\vee$ is applied.

Let X, Y be compact Hausdorff spaces,

$$\mu = \bigvee_{i=1}^n \alpha_i * \delta_{x_i} \in M^*(X),\ \nu = \bigvee_{j=1}^m \beta_i * \delta_{y_j} \in M^*(Y).$$

Define $\mu \otimes \nu = \bigvee_{i=1}^n \bigvee_{j=1}^m \alpha_i * \beta_j * \delta_{(x_i,y_j)}$. It is en easy exercise to prove that $\mu \otimes \nu \in M^*(X \times Y)$. Similarly, one can define $\mu_i \in M^*(X_i)$ like above, $i = 1, \dots, k$, one can define

$$\mu_1 \otimes \dots \otimes \mu_k = \otimes_{i=1}^k \mu_i \in M^*(X_1 \times \dots \times X_k).$$

Denote by $\iota\colon M^*(X) \to \prod_{\varphi \in C(X,\mathbb{I})} \mathbb{I}_\varphi$ (here $\mathbb{I}_\varphi$ is a copy of $\mathbb{I}$) a map defined as follows:

$$\iota(\mu) = (\mu(\varphi))_{\varphi \in C(X,\mathbb{I})},\ \mu \in M^*(X).$$

Proposition 1. *The map ι is an embedding and its image lies in the compact set*

$$\prod_{\varphi \in C(X,\mathbb{I})} \mathbb{I}_\varphi.$$

Proof. The fact that ι is an embedding immediately follows from the definition of the weak* topology. $\square$

In the sequel, we identify $M^*(X)$ with its image $\iota(M^*(X))$. Also, we regard every $x = (x_\varphi)_{\varphi \in C(X,\mathbb{I})}$ as a functional on $C(X,\mathbb{I})$, $x(\varphi) = x_\varphi$, $\varphi \in C(X,\mathbb{I})$.

Proposition 2. *The set $\iota(M^*(X))$ is a closed subset in $\prod_{\varphi \in C(X,\mathbb{I})} \mathbb{I}_\varphi$.*

Proof. Suppose that $\mu \in \left(\prod_{\varphi \in C(X,\mathbb{I})} \mathbb{I}_\varphi\right) \setminus M^*(X)$.

1) If there is $c \in \mathbb{I}$ such that $\mu(c) \neq c$, then $O(\mu; c_X; |c - \mu(c)|)$ is a neighborhood of μ that misses $M^*(X)$.

2) If $\mu(\varphi \vee \psi) \neq \mu(\varphi) \vee \mu(\psi)$, then

$$O\left(\mu; \varphi, \psi, \varphi \vee \psi; \frac{|\mu(\varphi \vee \psi) - (\mu(\varphi) \vee \mu(\psi))|}{2}\right)$$

is a neighborhood of μ that misses $M^*(X)$.

3) Suppose that $r = |\mu(c * \varphi) - c * \mu(\varphi)| > 0$. Since $*$ is continuous, there is $\varepsilon > 0$ such that $|c * a - c * \mu(\varphi)| < \frac{r}{3}$, whenever $|a - \mu(\varphi)| < \varepsilon$. Then $O\left(\mu; \varphi, c * \varphi; \min\{\varepsilon, r/3\}\right)$ is a neighborhood of μ that misses $M^*(X)$. □

Corollary 1. *For every compact Hausdorff space X the space $M^*(X)$ is compact.*

Proof. Indeed, by Propositions 1 and 2 the space $M^*(X)$ can be embedded as a closed subset in the compact Hausdorff space $\prod_{\varphi \in C(X,\mathbb{I})} \mathbb{I}_\varphi$ and therefore is compact Hausdorff as well. □

Let $f\colon X \to Y$ be a continuous map of compact Hausdorff spaces. For $\mu \in M^*(X)$, define $M^*(f)(\mu)\colon C(X,\mathbb{I}) \to \mathbb{I}$ by the formula $M^*(f)(\mu)(\varphi) = \mu(\varphi f)$, for every $\varphi \in C(Y,\mathbb{I})$.

Proposition 3. *We have then $M^*(f)(\mu) \in M^*(Y)$. The obtained map $M^*(f)\colon M^*(X) \to M^*(Y)$ is continuous.*

Proof. Let $\mu \in M^*(X)$ and $\varphi, \psi \in C(Y,\mathbb{I})$. Clearly, $M^*(f)(\mu)(c_X) = c,\ c \in \mathbb{I}$, and

$$M^*(f)(\mu)(\lambda * \varphi) = \mu((\lambda * \varphi)f) = \lambda * \mu(\varphi f) = \lambda * M^*(f)(\mu)(\varphi).$$

We have also

$$\begin{aligned} M^*(f)(\mu)(\varphi \vee \psi) =&\mu((\varphi \vee \psi)f) = \mu(\varphi f \vee \psi f) = \mu(\varphi f) \vee \mu(\psi f) = \\ =&M^*(f)(\mu)(\varphi) \vee M^*(f)(\mu)(\psi). \end{aligned}$$

Thus, $M^*(f)(\mu) \in M^*(Y)$. Therefore, we obtain a map $M^*(f)\colon M^*(X) \to M^*(Y)$.

Let $\mu \in M^*(X)$, $\psi_1, \dots, \psi_k \in C(Y,\mathbb{I})$, and $\varepsilon > 0$. Then the set $O(M^*(f)(\mu); \psi_1, \dots, \psi_n; \varepsilon)$ is a base neighborhood of $M^*(f)(\mu)$. Since

$$M^*(f)(\mu)(O(\mu; \psi_1 f, \dots, \psi_n f; \varepsilon)) \subset O(M^*(f)(\mu); \psi_1, \dots, \psi_n; \varepsilon),$$

we conclude that the map $M^*(f)$ is continuous.

□

Actually, M^* is a functor in the category **Comp**.

Proposition 4. *Let $\mu_1, \dots, \mu_n \in M^*(X)$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{I}$ be such that $\bigvee_{i=1}^n \lambda_i = 1$. Define $\mu = \bigvee_{i=1}^n \lambda_i * \mu_i\colon C(X,\mathbb{I}) \to \mathbb{I}$ as $\mu(\varphi) = \bigvee_{i=1}^n \lambda_i * \mu_i(\varphi)$. Then $\mu \in M^*(X)$.*

Proof. We verify the conditions from the definition of $*$-measure:

1) $\mu(c_X) = \bigvee_{i=1}^n \lambda_i * \mu_i(c_X) = \bigvee_{i=1}^n \lambda_i * c_X \leq 1 * c = c$;

2) $\mu(\lambda * \varphi) = \bigvee_{i=1}^n \lambda_i * \mu_i(\lambda * \varphi) = \bigvee_{i=1}^n \lambda_i * \lambda * \mu_i(\varphi) = \lambda * (\bigvee_{i=1}^n \lambda_i * \mu_i(\varphi)) = \lambda * \mu(\varphi)$;

3) $\mu(\varphi \vee \psi) = \bigvee_{i=1}^n \lambda_i * \mu_i(\varphi \vee \psi) = \bigvee_{i=1}^n \lambda_i * (\mu_i(\varphi) \vee \mu(\psi)) = \bigvee_{i=1}^n [(\lambda_i * \mu_i(\varphi)) \vee (\lambda_i * \mu_i(\psi))] =$
$= [\bigvee_{i=1}^n (\lambda_i * \mu_i(\varphi)] \vee [\bigvee_{i=1}^n (\lambda_i * \mu_i(\psi))] = \mu(\varphi) \vee \mu(\psi)$. □

Corollary 2. *If $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{I}, \vee_{i=1}^n \lambda_i = 1$, then*

$$\mu = \vee_{i=1}^n \lambda_i * \delta_{x_i} \in M^*(X).$$

Proposition 5. *Let $X = \{x_1, \dots, x_n\}$ and $\mu \in M^*$. Then there exist $\lambda_1, \dots, \lambda_n \in \mathbb{I}$ such that $\bigvee_{i=1}^n \lambda_i = 1$ and $\mu = \bigvee_{i=1}^n \lambda_i * \delta_{x_i}$.*

Proof. Let $\mu \in M^*(X)$ and $i \in \{1, \dots, n\}$. Consider the function $\varphi_i \colon X \to \mathbb{I}$ defined as follows: $\varphi_i(x_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$ Let $\lambda_j = \mu(\varphi_j)$. Then also

$$\left(\bigvee_{i=1}^n \lambda_i * \delta_{x_i}\right)(\varphi_j) = \bigvee_{i=1}^n \lambda_i * \varphi_j(x_i) = \lambda_j.$$

Now let $\varphi \colon X \to \mathbb{I}$ be an arbitrary function. Then $\varphi = \bigvee_{i=1}^n \varphi(x_i) * \varphi_i$ and therefore

$$\mu(\varphi) = \mu\Big(\bigvee_{i=1}^n \varphi(x_i) * \varphi_i\Big) = \bigvee_{i=1}^n \mu(\varphi(x_i) * \varphi_i) = \bigvee_{i=1}^n \varphi(x_i) * \mu(\varphi_i) = \bigvee_{i=1}^n \varphi(x_i) * \lambda_i.$$

□

We endow $C(X, \mathbb{I})$ with the uniform convergence topology.

Corollary 3. *Let X be a finite space and $\mu \in M^*(X)$. Then $\mu \colon C(X, \mathbb{I}) \to \mathbb{I}$ is a continuous map.*

Proposition 6. *Let X be a zero-dimensional space and $\mu \in M^*(X)$. Then $\mu \colon C(X, \mathbb{I}) \to \mathbb{I}$ is a continuous map.*

Proof. Let $\varphi \in C(X, \mathbb{I})$ and $\varepsilon > 0$. Since $*$ is (uniformly) continuous, there is $r > 0$ such that $|a - a'| < r$ and $|b - b'| < r$ together imply $|a * b - a' * b'| < \varepsilon$, for all $a, a', b, b' \in \mathbb{I}$.

There is a finite disjoint open cover $\mathcal{U}$ of X such that $\operatorname{diam}(\varphi(U)) < \varepsilon$ for every $U \in \mathcal{U}$. Let $Y = X/\mathcal{U}$ be the quotient space and $q \colon X \to Y$ the quotient map. There exist functions $\chi, \psi \colon Y \to \mathbb{I}$ such that $\|\chi - \psi\| < r$ and $\chi q < \varphi < \psi q$. The set $V = \{\varphi' \in C(X, \mathbb{I}) \mid \chi q < \varphi' < \psi q\}$ is a neighborhood of φ in the space $C(X, \mathbb{I})$. Then for every $\varphi' \in V$,

$$M^*(q)(\mu)(\chi) = \mu(\chi q) \leq \mu(\varphi') \leq \mu(\psi q) = M^*(q)(\mu)(\psi).$$

Suppose that $Y = \{y_1, \dots, y_n\}$ and $M^*(q)(\mu) = \vee_{i=1}^n \lambda_i * \delta_{y_i}$. Then

$$\mu(\chi q) = \bigvee_{i=1}^n \lambda_i * \chi(y_i) \leq \mu(\varphi') \leq \bigvee_{i=1}^n \lambda_i * \psi(y_i).$$

By the choice of r,

$$\Big|\bigvee_{i=1}^n \lambda_i * \chi(y_i) - \bigvee_{i=1}^n \lambda_i * \psi(y_i)\Big| \leq \bigvee_{i=1}^n |\lambda_i * \chi(y_i) - \lambda_i * \psi(y_i)| < \varepsilon.$$

We conclude that $|\mu(\varphi') - \mu(\varphi)| < \varepsilon$ and therefore, since $\varphi' \in V$ is arbitrary, the map μ is continuous at φ. □

Let $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ be an inverse system over a directed set $\mathcal{A}$. (See, e.g., [11] for the necessary information concerning inverse systems in the category **Comp**.) For any $\alpha \in \mathcal{A}$, let $p_\alpha\colon X = \varprojlim S \to X_\alpha$ denote the limit projection. By $M^*(\mathcal{S})$ we denote the inverse system $\{M^*(X_\alpha), M^*(p_{\alpha\beta}); \mathcal{A}\}$.

The following statement is a modification of that of Proposition 2.12 from [4]. Its proof is included for the sake of completeness.

Proposition 7. *Let X be a zero-dimensional compact space and $X = \varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$, for a directed set $\mathcal{A}$. Then the natural map*

$$h = (M^*(p_\alpha))_{\alpha\in\mathcal{A}}\colon M^*(X) \to \varprojlim M^*(\mathcal{S})$$

is a homeomorphism.

Proof. First, we are going to show that the map h is an embedding. Suppose the opposite and let $\mu, \nu \in M^*(X)$, $\mu \neq \nu$, be such that $h(\mu) = h(\nu)$. Since $\mu \neq \nu$, there exists $\varphi \in C(X, \mathbb{I})$ such that $\mu(\varphi) \neq \nu(\varphi)$.

Let $C' = \{\varphi p_\alpha \,|\, \varphi \in C(X_\alpha, \mathbb{I}),\ \alpha \in \mathcal{A}\}$. Note that the set C' is dense in $C(X, \mathbb{I})$. Since μ, ν are continuous, there is $\varphi' \in C'$ such that $\mu(\varphi') \neq \nu(\varphi')$. Then $\varphi' = \psi p_\alpha$, for some $\alpha \in \mathcal{A}$ and $\psi \in C(X_\alpha, \mathbb{I})$. Therefore,

$$M^*(p_\alpha)(\mu)(\psi) = \mu(\varphi') \neq \nu(\varphi') = M^*(p_\alpha)(\nu)(\psi)$$

and we obtain a contradiction.

Now, show that h is an onto map. Let $(\mu_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim M^*(S)$. We are going to show that there exists $\mu \in M^*(X)$ such that $M^*(p_\alpha)(\mu) = \mu_\alpha$, for any $\alpha \in \mathcal{A}$. Given $\varphi, \psi \in C'$, one can write $\varphi = \varphi' p_\alpha$, $\psi = \psi' p_\alpha$, for some $\alpha \in \mathcal{A}$, whence

$$\begin{aligned}\mu(\varphi \vee \psi) =&\mu((\varphi' p_\alpha) \vee (\psi' p_\alpha)) = M^*(p_\alpha)(\mu)(\varphi' \vee \psi') = \mu_\alpha(\varphi' \vee \psi') = \\ =&\mu_\alpha(\varphi') \vee \mu_\alpha(\psi') = \mu(\varphi' p_\alpha) \vee \mu(\psi' p_\alpha) = \mu(\varphi) \vee \mu(\psi).\end{aligned}$$

Since the set C' is dense in $C(X, \mathbb{I})$ and the operation $*$ is continuous, we conclude that $\mu(\varphi \vee \psi) = \mu(\varphi) \vee \mu(\psi)$, for all $\varphi, \psi \in C(X, \mathbb{I})$. Similarly, one can verify that $\mu(\lambda * \varphi) = \lambda * \mu(\varphi)$ for all $\varphi \in C(X, \mathbb{I})$ and $\lambda \in \mathbb{R}$. Thus, $\mu \in M^*(X)$ is as required. □

Proposition 8. *Let X be a zero-dimensional compact Hausdorff space. Then the set $M^*_\omega(X)$ is dense in $M^*(X)$.*

Proof. Let $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ be an inverse system such that all X_α are finite, $p_{\alpha\beta}$ are onto maps, and $X = \varprojlim \mathcal{S}$. Since $M^*(X) = \varprojlim \mathcal{M}^*(\mathcal{S})$ and all $M(p_\alpha)(M^*_\omega(X)) = M^*(X_\alpha)$, we conclude that $M^*_\omega(X)$ is dense in $M^*(X)$. □

Let $X = \prod_{\alpha\in A} X_\alpha$, where X_α is finite for every $\alpha \in A$. Let $\mu_\alpha \in M^*(X_\alpha)$, $\alpha \in A$. Similarly as in [4] using Proposition 7 one can define the tensor product $\otimes_{\alpha\in A}\mu_\alpha \in M^*\left(\prod_{\alpha\in A} X_\alpha\right)$ by the condition $M^*(\pi_{\alpha_1\dots\alpha_k})(\otimes_{\alpha\in A}\mu_\alpha) = \mu_{\alpha_1} \otimes \dots \otimes \mu_{\alpha_k}$, for every $\alpha_1, \dots, \alpha_k \in A$ (by $\pi_{\alpha_1\dots\alpha_k}$ the projection of X onto $X_{\alpha_1} \times \dots \times X_{\alpha_k}$ is denoted).

Remark 3. Let $X = \prod_{\alpha\in A} X_\alpha$, where X_α is finite for every $\alpha \in A$ and let $u_\alpha\colon X \to [0,1]$ be payoff a payoff function for each $\alpha \in A$. One can consider a game with $*$-measure strategies on X, where the payoff functions $u^*_\alpha\colon \prod_{\alpha\in A} M^*(X_\alpha) \to [0,1]$ are given by the formula

$$u^*_\alpha((\mu_\beta)_{\beta\in A}) = (\otimes_{\beta\in A}\mu_\beta)(u_\alpha).$$

Thus one can extend the considerations of Radul [10] onto the case of games with $*$-measure strategies on finite spaces.

4. $*$-Milyutin maps. To treat the general case we introduce a version of the notion of Milyutin map, which allows to reduce the general case to the zero-dimensional one.

First, given a compact Hausdorff space X and its closed subspace Y, we identify $M^*(Y)$ with the subspace $M^*(i)(M^*(Y))$ of $M^*(X)$, where $i\colon Y \to X$ stands for the inclusion map.

A map $f\colon X \to Y$ of compact metrizable spaces is called a *$*$-Milyutin map* if there is a map $s\colon Y \to M^*(X)$ such that $s(y) \in M^*(f^{-1}(y)) \subset M^*(X)$, for every $y \in Y$.

Theorem 1. *For every compact Hausdorff space X there exists a $*$-Milyutin map $f\colon Z \to X$, where Z is a zero-dimensional compact Hausdorff space.*

Proof. The proof consists in modifications of arguments from [4], which in turn relies on a construction from [1]. We provide the details for the sake of reader's convenience.

First, assume that X is metrizable and choose a compatible metric on X.

For every $n \in \mathbb{N}$, choose a finite family $\mathcal{A}_n$ of pairs of closed subsets of X with the properties:

1. $\bigcup\{B \mid (A,B) \in \mathcal{A}_n\} = X$;
2. $\mathrm{diam}(A) \le 1/n$, for every $(A,B) \in \mathcal{A}_n$;
3. $\mathrm{Int}(A) \supset B$, for every $(A,B) \in \mathcal{A}_n$.

Let $Z_n = \bigsqcup\{A \mid (A,B) \in \mathcal{A}_n\}$ (here $\bigsqcup$ denotes the disjoint union). Define $f_n\colon Z_n \to X$ by the condition: $f_n|A$ is the inclusion map $\iota_A\colon A \hookrightarrow X$. Then let

$$Z = \left\{ (z_n)_{n=1}^\infty \in \prod_{n=1}^\infty Z_n \mid f_i(z_i) = f_j(z_j),\ \text{for all } i,j \in \mathbb{N} \right\}.$$

For every $n \in \mathbb{N}$, let

$$Y_n = \left\{ (z_m)_{m=1}^n \in \prod_{m=1}^n Z_m \mid f_i(z_i) = f_j(z_j),\ \text{for all } i,j \le n \right\}.$$

For $n \ge k$, denote by $g_{nk}\colon Y_n \to Y_k$ the natural projection. Clearly, $Z = \varprojlim\{Y_n, g_{nk}; \mathbb{N}\}$. It is easy to verify that Z is a compact zero-dimensional space.

For any $(A,B) \in \mathcal{A}_n$, let $\alpha_{(A,B)}\colon X \to [0,1]$ be a continuous function such that $\alpha_{(A,B)}|B = 0$ and $\alpha_{(A,B)}|(X \setminus A) = 1$. Given $x \in X$, define

$$\mu_n(x) = \bigvee_{(A,B)\in\mathcal{A}_n} \alpha_{(A,B)}(x) * \delta_{\iota_A^{-1}(x)}.$$

Note that $\mu_n(x)$ is well-defined. We are going to show that the map $\mu_n\colon X \to M^*(Y_n)$ is continuous. Indeed, given $\varphi \in C(X)$, we see that the function

$$x \mapsto \mu_n(x)(\varphi) = \bigvee_{(A,B)\in\mathcal{A}_n} \alpha_{(A,B)}(x) * \varphi(x)\colon X \to \mathbb{R}$$

is continuous, and this implies the continuity of μ_n.

Define $f\colon Z \to X$ by the formula $f((z_n)_{n=1}^\infty) = f_1(z_1)$. For every $m \in \mathbb{N}$, let $h_m\colon Y_m \to X$ be defined by the formula $h_m((z_n)_{n=1}^m) = f_1(z_1)$.

For any $x \in X$, $f^{-1}(x) = \varprojlim\{h_m^{-1}(x), g_{mk}|h_m^{-1}(x); \mathbb{N}\}$. By Proposition 7, there exists $\mu(x) \in M^*(f^{-1}(x))$ such that $M^*(g_m)(\mu(x)) = \otimes_{i=1}^m \mu_i(x)$ (by $g_m\colon Z \to Y_m$ we denote the projection map).

Note that the continuity of the map map $x \mapsto \mu(x)$ is a consequence of the continuity of the maps μ_n, $n \in \mathbb{N}$.

Now, suppose that X is arbitrary. Then one may assume that $X \subset \prod_{\alpha\in T} X_\alpha$, for some family $\{X_\alpha \mid \alpha \in T\}$ of compact metrizable spaces. For every $\alpha \in T$ let $f_\alpha\colon Y_\alpha \to X_\alpha$ be a Milyutin map, where Y_α is a zero-dimensional compact Hausdorff space. Let $g = \prod_{\alpha\in T} f_\alpha\colon \prod_{\alpha\in T} Y_\alpha \to \prod_{\alpha\in T} X_\alpha$. Let $Z = g^{-1}(X)$ and let $f = g|Z\colon Z \to X$. Clearly Z is a zero-dimensional compact Hausdorff space.

We are going to show that f is a $*$-Milyutin map. For every $\alpha \in T$, let $s_\alpha\colon X_\alpha \to M^*(Y_\alpha)$ be a map such that $s_\alpha(x) \in M^*(f_\alpha^{-1}(x))$, for every $x \in X_\alpha$. Define $s((x_\alpha)_{\alpha\in T}) = \otimes_{\alpha\in T} s_\alpha(x_\alpha)$. Clearly, s is continuous and $s(x) \in M^*(s^{-1}(x))$, for every $x \in X$. □

Proposition 9. *Let X be a compact Hausdorff space. Then the set $M_\omega^*(X)$ is dense in $M^*(X)$.*

Proof. Let $f\colon Z \to X$ be a $*$-Milyutin map, where Z is a zero-dimensional compact Hausdorff space. Since $M_\omega^*(Z)$ is dense in $M^*(Z)$, the statement follows. □

5. Space $\bar{M}(X)$. Denote by $\bar{M}(X)$ the set of all $A \in \exp(X \times \mathbb{I})$ satisfying the following conditions:

1. $A \cap (X \times \{1\}) \neq \varnothing$;
2. $X \times \{0\} \subset A$;
3. A is saturated, i.e., if $(x,t) \in A$, then $(x,s) \in A$ for every $s \in [0,t]$.

The proof of the following statement can be obtained from the proof of Proposition 1.3 from [3] by replacing $[-\infty, 0]$ by $[0,1]$.

Proposition 10. *The set $\bar{M}(X)$ is closed in* $\exp(X \times \mathbb{I})$.

Let $f\colon X \to Y$ be a map. Define the map $\bar{M}(f)\colon \bar{M}(X) \to \bar{M}(Y)$ by the formula:

$$\bar{M}(f)(A) = \exp(f \times 1_{\mathbb{I}})(A) \cup (Y \times \{0\}).$$

Since the union map in the hyperspace is continuous (see, e.g., [9]), the map $\bar{M}(f)$ is continuous as well.

Actually, $\bar{M}$ is a functor in the category **Comp**.

Given $A \in \bar{M}^*(X)$, define the functional $h_A\colon C(X,\mathbb{I}) \to \mathbb{I}$ by the formula $h_A(\varphi) = \sup\{t * \varphi(x) \mid (x,t) \in A\}$.

Proposition 11. *If $A \in \bar{M}^*(X)$, then $h_A \in M^*(X)$.*

Proof. To prove this proposition we check the conditions from the definition of $*$-measure.

First, if $c \in \mathbb{I}$, then (by condition (2) of $\bar{M}^*(X)$) $h_A \leq 1 * \varphi(x) = 1 * c_X = c_X$. Clearly, $h_A \leq c_X$ and we are done.

Next,

$$h_A(\lambda * \varphi) = \sup\{t * (\lambda * \varphi)(x) \mid (x,t) \in A\} = \lambda * \sup\{t * \varphi(x) \mid (x,t) \in A\} = \lambda * h_A(\varphi).$$

Finally,

$$\begin{aligned} h_A(\varphi \vee \psi) =& \sup\{t * ((\varphi \vee \psi)(x)) \mid (x,t) \in A\} = \sup\{t * (\varphi(x) \vee \psi(x)) \mid (x,t) \in A\} = \\ =& \sup\{((t * \varphi)(x)) \vee ((t * \psi)(x)) \mid (x,t) \in A\} = \text{ (by monotonicity of } *) = \\ =& \sup\{t * \varphi(x) \mid (x,t) \in A\} \vee \sup\{t * \psi(x) \mid (x,t) \in A\} = h_A(\varphi) \vee h_A(\psi). \end{aligned}$$

□

Proposition 12. *The map $h\colon \bar{M}^*(X) \to M^*(X)$ is continuous.*

Proof. We choose a sequence (A_i) in $\bar{M}^*$ convergent to A in $\bar{M}^*$ and we need to show that $\lim_{i\to\infty} h(A_i) = h(A)$.

Let $\varphi \in C(X, \mathbb{I})$. We are going to show that $\lim_{i\to\infty} h(A_i)(\varphi) = h(A)(\varphi)$. Given $\varepsilon > 0$, find a finite open cover $\mathcal{U}$ of the space X such that oscillation of φ on every element of $\mathcal{U}$ is less that $\varepsilon/2$. Let $c < 1$ be such that $c < \inf\{\varphi(x) | x \in X\}$. Then consider a finite cover $\mathcal{V}$ of $[c, 1]$ such that the diameter of every element $\mathcal{V}$ is less than $\varepsilon/2$.

Consider the family $\mathcal{U} \times \mathcal{V} = \{U \times V | U \in \mathcal{U}, V \in \mathcal{V}\}$. There is $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$, the sets A_i and A are $(\mathcal{U} \times \mathcal{V})$-close.

There exists $(x,t) \in A$ such that $h(A)(\varphi) = t * \varphi(x)$. Then $(x,t) \in \bigcup(\mathcal{U} \times \mathcal{V})$, that is, there exists $U \times V \in \mathcal{U} \times \mathcal{V}$ such that $(x,t) \in U \times V$. Since A_i and A are $(\mathcal{U} \times \mathcal{V})$-close, there is $(x', t') \in A_i \cap (U \times V)$. Hence,

$$h(A_i)(\varphi) = t' * \varphi'(x) \geq h(A)(\varphi) = t * \varphi(x).$$

We conclude that $\lim_{i\to\infty} h(A_i)(\varphi) = h(A)(\varphi)$. Since φ is arbitrary, we conclude that $\lim_{i\to\infty} h(A_i) = h(A)$, by the definition of weak* topology. Thus, the map h is continuous. □

Proposition 13. *The map $h\colon \bar{M}^*(X) \to M^*(X)$ is an embedding.*

Proof. Suppose that $A, B \in \bar{M}$, $A \neq B$ and $A \setminus B \neq \varnothing$. Let $(x,t) \in A \setminus B$ and suppose that $(x, t') \notin A$, for all $t' > t$. Then there exists a neighborhood U of $x \in X$ and $r < t < 1$ such that $U \times (r, 1] \cap B = \varnothing$.

Let $\varphi \in C(X, [0,1])$ be a function such that:

(1) $\varphi(x) = \psi(x)$;

(2) $\varphi|(X \setminus U) < \psi(x)$.

Then, clearly, $h(A)(\varphi) \geq \psi(x)$ and $h(B)(\varphi) < \psi(x)$. Therefore $h(A) \neq h(B)$. □

Proposition 14. *The map h is onto.*

Proof. It is enough to show that $M^*_\omega(X)$ lies in the image of h. Given $\mu = \bigvee_{i=1}^n \alpha_i * \delta_{x_i} \in M^*_\omega(X)$, put

$$C = (X \times \{0\}) \cup \bigcup_{i=1}^n (\{x_i\} \times [0, \alpha_i]).$$

Then $h(C) = \mu$. □

Corollary 4. *The spaces $\bar{M}^*(X)$ and $M^*(X)$ are homeomorphic.*

REFERENCES

1. S. Ageev, E. D. Tymchatyn, *On exact atomless Milutin maps*, Topology and its Applications, **153** (2005), №2-3, 227–238. https://doi.org/10.1016/j.topol.2003.07.021
2. W. Briec, C. Horvath, $\mathbb{B}$-*convexity*, Optimization, **53** (2004), №2, 103–127. https://doi.org/10.1080/02331930410001695283
3. V. Brydun, A. Savchenko, M. Zarichnyi, *Fuzzy metrization of the spaces of idempotent measures*, European Journal of Mathematics, **6** (2020), №1, 98–109. https://doi.org/10.1007/s40879-019-00341-8
4. V. Brydun, M. Zarichnyi, *Spaces of max-min measures on compact Hausdorff spaces*, Fuzzy Sets and Systems, **396** (2020), 138–151. https://doi.org/10.1016/j.fss.2019.06.012
5. G. Cohen, S. Gaubert, J. Quadrat, I. Singer, *Max-plus convex sets and functions.* In: Litvinov, G.L., Maslov, V.P. (eds.): Idempotent Mathematics and Mathematical Physics. Contemporary Mathematics. American Mathematical Society, 2005, 105–129.
6. E. P. Klement, R. Mesiar, E. Pap, Triangular Norms. Dordrecht: Kluwer. 2000. ISBN 0-7923-6416-3.
7. V. Kolokoltsov, V. Maslov, Idempotent Analysis and Applications, Kluwer Acad. Publ., 1997.
8. G .L. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction*, J. Math. Sc. (New York), **140** (2007), №3, 426–444. https://doi.org/10.1007/s10958-007-0450-5
9. Sam B. Nadler, Hyperspaces of sets. Monographs and Textbooks on Pure and Applied Mathematics, №49, Marcel Dekker, Inc., New York, 1978, xvi + 707 pp.
10. T. Radul, *Equilibrium under uncertainty with fuzzy payoff*, Topol. Methods Nonlinear Anal., *59* (2002), №2B, 1029–1045. https://doi.org/10.12775/TMNA.2021.049
11. E.V. Shchepin, *Functors and uncountable powers of compacta*, Uspekhi Mat. Nauk, **31** (1981), 3–62.
12. T. Świrszcz, *Monadic functors and convexity*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **22** (1974), №1, 39–42.
13. M. Zarichnyi, Spaces and maps of idempotent measures, *Izvestia: Mathematics*, **74** (2010), №3, 481–499.

Ivan Franko National University of Lviv
Lviv, Ukraine
khrystyna.sukhorukova@lnu.edu.ua

Received 28.05.2023
Revised 19.06.2023