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A. O. Kuryliak

WIMAN'S TYPE INEQUALITY FOR ENTIRE MULTIPLE DIRICHLET SERIES WITH ARBITRARY COMPLEX EXPONENTS

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We prove analogs of the classical Wiman's inequality for the class $\mathcal{D}$ of absolutely convergents in the whole complex space $\mathbb{C}^p$ (entire) Dirichlet series of the form $F(z) = \sum_{\|n\|=0}^{+\infty} a_n e^{(z,\lambda_n)}$ with such a sequence of exponents (λ_n) that $\{\lambda_n\colon n\in\mathbb{Z}^p\}\subset\mathbb{C}^p$ and $\lambda_n\neq\lambda_m$ for all $n\neq m$. For $F\in\mathcal{D}$ and $z\in\mathbb{C}^p\setminus\{0\}$ we denote

$$\mathfrak{M}(z,F) := \sum_{\|n\|=0}^{+\infty} |a_n| e^{\mathrm{Re}(z,\lambda_n)}, \quad \mu(z,F) := \sup\{|a_n|e^{\mathrm{Re}(z,\lambda_n)}\colon n\in\mathbb{Z}^p_+\},$$

$(\mu_k)_{k\geq 0}$ is the sequence $(-\ln|a_n|)_{n\in\mathbb{Z}^p_+}$ arranged by non-decreasing. The main result of the paper: Let $F\in\mathcal{D}$. If

$$(\exists\alpha>0)\colon \int_{t_0}^{+\infty} t^{-2}(n_1(t))^\alpha dt < +\infty,\ n_1(t) \overset{def}{=} \sum_{\mu_n\leq t} 1, \quad t_0>0,$$

then there exists a set $E\subset\gamma_+(F)$ such that $\tau_{2p}(E\cap\gamma_+(F)) = \int_{E\cap\gamma_+(F)} |z|^{-2p}dxdy \leq C_p, z = x+iy\in\mathbb{C}^p$, and the relation $\mathfrak{M}(z,F) = o(\mu(z,F)\ln^{1/\alpha}\mu(z,F))$ holds as $z\to\infty$ $(z\in\gamma_R\setminus E)$ for each $R>0$, where

$$\gamma_R = \Big\{z\in\mathbb{C}^p\setminus\{0\}\colon\ K_F(z)\leq R\Big\}, \quad K_F(z) = \sup\Big\{\frac{1}{\Phi_z(t)}\int_0^t \frac{\Phi_z(u)}{u}du\colon\ t\geq t_0\Big\},$$

$$\gamma(F) = \{z\in\mathbb{C}\colon\ \lim_{t\to+\infty}\Phi_z(t) = +\infty\}, \quad \gamma_+(F) = \cup_{R>0}\gamma_R,$$

$\Phi_z(t) = \frac{1}{t}\ln\mu(tz,F)$. In general, under the specified conditions, the obtained inequality is exact.

Let $\mathbb{R}^p$ and $\mathbb{C}^p$ be real and complex vector spaces, respectively, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, $\mathbb{R}_+ = (0,+\infty)$, $p\in\mathbb{N}$. For $z=(z_1,\dots,z_p)\in\mathbb{C}^p$, $w=(w_1,\dots,w_p)\in\mathbb{C}^p$ denote $(z,w) = z_1w_1+\ldots+z_pw_p$, $\|z\| = z_1+\ldots+z_p$, $\mathrm{Re}\ z = (\mathrm{Re}\ z_1,\dots,\mathrm{Re}\ z_p)$, and for $R=(r_1,\dots,r_p)\in\mathbb{R}^p$ we write $\Pi_R = \{z\in\mathbb{C}^p\colon \mathrm{Re}\ z<R\}$.

By $\mathcal{D}$ denote the class of absolutely convergent in the whole complex space $\mathbb{C}^p$ (entire) Dirichlet series of the form

$$F(z) = \sum_{\|n\|=0}^{+\infty} a_n e^{(z,\lambda_n)} \tag{1}$$

with such a sequence of exponents (λ_n) that $\{\lambda_n\colon n\in\mathbb{Z}^p\}\subset\mathbb{C}^p$ and $\lambda_n\neq\lambda_m$ for all $n\neq m$. By $\mathcal{D}^+$ denote the class entire Dirichlet series with a sequence of exponents $\Lambda^p=(\lambda_n)$ such that $\lambda_n = (\lambda^{(1)}_{n_1},\dots,\lambda^{(p)}_{n_p})$, $n=(n_1,\dots,n_p)$ and $0=\lambda^{(j)}_0<\lambda^{(j)}_k\uparrow+\infty$ $(1\leq k\uparrow+\infty)$, $1\leq j\leq p$. For $F\in\mathcal{D}$ and $z\in\mathbb{C}^p$ we denote

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$$\mathfrak{M}(z,F) := \sum_{\|n\|=0}^{+\infty} |a_n| e^{\mathrm{Re}(z,\lambda_n)}, \quad \mu(z,F) := \sup\{|a_n| e^{\mathrm{Re}(z,\lambda_n)} \colon n \in \mathbb{Z}_+^p\},$$

and $\mathcal{N} := \bigcup_z \mathcal{N}(z)$, where $\mathcal{N}(z)$ is the set of such multi-indices $\nu = \nu(z,F) \in \mathbb{Z}_+^p$ that $|a_\nu| e^{\mathrm{Re}(z,\lambda_\nu)} = \mu(z,F)$ for a given z. Let us denote also the following function

$$\beta(z) := \sup\{\mathrm{Re}(z,\lambda_n) \colon\ n \in \mathbb{Z}_+\} \colon \mathbb{C}^p \to \mathbb{R}.$$

It is well known ([1–4, 6]) that for every nonconstant entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and every $\varepsilon > 0$ there exists an exceptional set $E = E(f,\varepsilon)$ of finite logarithmic measure, i.e. $\int_E \frac{dr}{r} < +\infty$, such that the inequality (*Wiman's inequality*) $M_f(r) \le \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$ holds for all $r \in [1,+\infty) \setminus E$, where

$$M_f(r) = \max\{|f(z)| \colon |z| = r\},\ \mu_f(r) = \max\{|a_n| r^n \colon n \ge 0\}.$$

Some analogs of the Wiman inequality for entire Dirichlet series $F \in \mathcal{D}^+$ with $p = 1$ were obtained in [5, 7].

Let $\mathcal{D}_1$ be the class of absolutely convergent for all Dirichlet series in $\mathbb{C}$ of form (1) with sequence of the exponents (λ_n) such that $\lambda_n \ge 0$ $(n \ge 0)$ and $\sup\{\lambda_n \colon\ n \ge 0\} = +\infty$. It should be noted that some asymptotic properties of functions $F \in \mathcal{D}_1$ were investigated in the papers [8–13].

For a function $F \in \mathcal{D}_1$ of form (1) denote by $(\mu_k)_{k\in\mathbb{Z}_+}$ the sequence $(-\ln|a_k|)_{k\in\mathbb{Z}_+}$ arranged by decreasing.

Let L be the class of positive continuous functions increasing to $+\infty$ on $[0;+\infty)$ and L_1 be the class of functions $\Phi \in L$ such that $\varphi(2t) = O(\varphi(t))$ $(t \to +\infty)$, where φ is an inverse function to Φ.

The following theorem was proved in paper [13].

Theorem 1 ([13], Ovchar, Skaskiv). *Let* $F \in \mathcal{D}_1$, $\Phi_1 \in L_1$, $\Phi_1(x) \overset{def}{=} \frac{1}{x} \ln \mu(x,F)$. *If*

$$(\exists \alpha > 0) \colon \quad \int_{t_0}^{+\infty} t^{-2} (n_1(t))^\alpha dt < +\infty, \quad n_1(t) \overset{def}{=} \sum_{\mu_n \le t} 1, \quad t_0 > 0,$$

then there exists a set $E \subset \mathbb{R}$ *such that* $\ln\text{-meas}\,(E) := \int_{E\cap[1,+\infty)} d\ln r < +\infty$ *and the relation* $\mathfrak{M}(x,F) = o(\mu(x,F) \ln^{1/\alpha} \mu(x,F))$ *holds as* $x \to +\infty$ $(x \notin E)$.

The following assertion shows that statement of Theorem 1 can not be improved in a general case.

Theorem 2 ([13], Kuryliak). *For every* $\alpha > 0$ *there exists a function* $F \in \mathcal{D}_1$ *such that the conditions* $\int_{t_0}^{+\infty} t^{-2}(n_1(t))^\alpha dt < +\infty$ *and* $(\forall \varepsilon > 0) \colon\ \int_{t_0}^{+\infty} t^{-2}(n_1(t))^{\alpha+\varepsilon} dt = +\infty$ *are satisfied, and*

$$\big(\forall\ \varepsilon \in (0;1/\alpha)\big) \colon\ \frac{F(x)}{\mu(x,F)(\ln \mu(x,F))^{1/\alpha-\varepsilon}} \to +\infty$$

holds as $x \to +\infty$.

In this paper we consider analogs of the Wiman inequality in the class $\mathcal{D}$ with arbitrary $p \ge 1$.

2. Auxiliary statements. For a Lebesgue measurable set (for example, for a Borel set) $E \subset \mathbb{C}^p$ and $\alpha > 0$ denote

$$\tau_\alpha(E) := \int_{E\cap\{z \colon |z|\ge 1\}} \frac{dxdy}{|z|^\alpha}, \quad z = x + iy \in \mathbb{C}^p.$$

For the ball $\mathbb{D}^p_R = \{z \in \mathbb{C}^p \colon |z| \le R\}$, $R > 0$, $\tau_{2p}(\mathbb{D}^p_R) = C_p \ln R$, $(R \ge 1)$, $\tau_{2p}(\mathbb{C}^p) = +\infty$, where C_p is the area of the unit sphere in $\mathbb{R}^{2p}$.

Let $\mathcal{L}$ be the class of positive continuos fucntions $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) \to +\infty$ $(t \to +\infty)$, and $\mathcal{L}_0$ be the class of the functions $\Phi \in \mathcal{L}$ such that $\int_{x_0}^x \frac{\Phi(t)}{t} dt = O(\Phi(x))$ $(x \to +\infty)$.

Denote by $\mathcal{D}_0$ the class of the functions $F \in \mathcal{D}$ such that $\mu(z, F) = 1$ $(z \in \mathbb{D}^p_1)$, where $\mathbb{D}^p_1 = \{z \in \mathbb{C}^p \colon |z| \le 1\}$. If for a function $F \in \mathcal{D}$ this condition is not fulfilled, then for $b := \max\{\mu(z, F) \colon z \in \mathbb{D}^p_1\}$ we define $F_2(z) := \frac{1}{2b}(F(z) - a_0 + 2b)$. Therefore, $\mu(z, F_2) = \mu(\mathbf{0}, F_2) = \max\{1, |a_n|/(2b) \colon n \ne \mathbf{0}\} = 1$ $(z \in \mathbb{D}^p_1)$, i.e. $F_2 \in \mathcal{D}_0$.

For the function $F \in \mathcal{D}_0$ and given $z \in \mathbb{C}^p$ we define the function

$$\Phi_z(t) = \frac{1}{t} \ln \mu(tz, F) \colon \; [0, +\infty) \to [0, +\infty).$$

For the function $F \in \mathcal{D}$, we define the following sets

$$\gamma(F) \overset{def}{=} \{z \in \mathbb{C} \colon \; \lim_{t \to +\infty} \Phi_z(t) = +\infty\}, \quad \gamma_+(F) \overset{def}{=} \{z \in \gamma(F) \colon \; \Phi_z \in \mathcal{L}_0\}.$$

From the convexity of the function $\ln \mu(tz, F)$ as function of $t \ge 0$ under the condition $\mu(\mathbf{0}, F) = 1$ for each fixed $z \in \gamma(F)$ we get that for $t_2 > t_1 \ge t_0$

$$\Phi_z(t_2) = \frac{\ln \mu(t_2 z, F) - \ln \mu(\mathbf{0}, F)}{(t_2 - 0)} \ge \frac{\ln \mu(t_1 z, F) - \ln \mu(\mathbf{0}, F)}{(t_1 - 0)} = \Phi_z(t_1) \ge 0, \qquad (2)$$

i.e. the function $\Phi_z(t)$ is continuous, non-negative on $[0, +\infty)$ and strictly increases on $[t_0, +\infty)$ for some $t_0 \ge 1$. Note that $t_0 = t_0(z) := \max\{t \in \mathbb{R} \colon \mu(tz, F) = 1\}$ has this property. Thus, we can define the inverse function $\varphi_z(u) \colon [0, +\infty) \to [t_0, +\infty)$ of the function $\Phi_z(t) \colon [t_0, +\infty) \to [\Phi_z(t_0), +\infty) = [0, +\infty)$ for fixed $z \in \gamma(F)$. Note that $\Phi_z(t) \equiv 0$ $(0 \le t \le t_0)$, $\varphi_z(0) = t_0 = t_0(z) = \max\{t \in \mathbb{R} \colon \mu(tz, F) = 1\}$.

The sets $\gamma(F)$, $\gamma_+(F)$ are real cones. It follows from the following elementary proposition.

Proposition 1. *For every function $F \in \mathcal{D}$*

$$z \in \gamma(F) \iff (\forall\, r > 0) \colon \; (rz) \in \gamma(F), \quad z \in \gamma_+(F) \iff (\forall\, r > 0) \colon \; (rz) \in \gamma_+(F).$$

The following statement (in the case $p = 1$, see indication in [15]) we will give a complete proof.

Proposition 2. *For every function $F \in \mathcal{D}$, $\gamma(F) = \{z \in \mathbb{C} \colon \; \beta(z) = +\infty\}$.*

Proof. Assume first that $\beta(z) < +\infty$. Then

$$\Phi_z(t) = \frac{\ln \mu(tz)}{t} = \frac{\ln |a_{\nu(tz)}|}{t} + \mathrm{Re}(z, \lambda_{\nu(tz)}) \le \frac{\ln |a_{\nu(tz)}|}{t} + \beta(z),$$

hence, $\overline{\lim\limits_{t \to +\infty}} \Phi_z(t) \le \beta(z)$, that is $z \notin \gamma(F)$. Therefore, $\gamma(F) \subset \{z \colon \beta(z) = +\infty\}$.

Suppose now that $z \notin \gamma(F)$. Then $(\forall t > 0) \colon\; \Phi_z(tz) \le C(z) < +\infty$, hence,

$$\frac{\ln |a_n|}{t} + \mathrm{Re}(\lambda_n, z) \le \frac{\ln \mu(tz, F)}{t} \le C(z) \quad (t > 0, \; n \in \mathbb{Z}^p_+).$$

Thus, for every fixed $n \in \mathbb{Z}^p_+$ we obtain

$$\mathrm{Re}(\lambda_n, z) \le \overline{\lim_{t \to +\infty}} \Big(C(z) - \frac{\ln |a_n|}{t} \Big) = C(z).$$

It follows that $\beta(z) = \sup\{\mathrm{Re}(\lambda_n, z) \colon n \ge 0\} \le C(z)$. Therefore, $\{z \colon \beta(z) = +\infty\} \subset \gamma(F)$ and finally $\gamma(F) = \{z \colon \beta(z) = +\infty\}$. □

Proposition 3. *Let $F \in \mathcal{D}$. For every cone K with the vertex at the point $z = \mathbf{0}$ such that $\overline{K} \setminus \{\mathbf{0}\} \subset \gamma(F)$ we have*

$$\min\{\|\nu(z,F)\|\colon \nu(z,F) \in \mathcal{N}(z)\} \to +\infty, \quad \frac{1}{|z|}\ln\mu(z,F) \to +\infty \quad (z \to \infty,\ z \in K).$$

Proof. Let us prove by contradiction that $\frac{1}{|z|}\ln\mu(z,F) \to +\infty$ $(z \to \infty,\ z \in K)$. Suppose that there exists a sequence (z_j), $z_j \in K$ $(j \geq 1)$ such that $z_j \to \infty$ $(j \to +\infty)$ and

$$\frac{1}{|z_j|}\ln\mu(z_j,F) \leq C < +\infty \quad (j \geq 1). \tag{3}$$

Denote $t_j = |z_j|$, $z_j^{(0)} = z_j/|z_j|$. Since $z_j^{(0)} \in K \cap \{z\colon\ |z| = 1\}$, the sequence $(z_j^{(0)})$ has an accumulation point $z_1 \in \overline{K} \cap \{z\colon\ |z| = 1\}$. Therefore, there exists a subsequence $(z_j^{(1)})$, $z_j^{(1)} = z_{k_j}^{(0)}$ $(j \geq 1)$ of the sequence $(z_j^{(0)})$ such that $\lim\limits_{j\to+\infty} z_j^{(1)} = z_1$. We put $z_j^* = z_j^{(1)} t_{k_j} = z_{k_j}$.

It follows from the condition $z_1 \in \gamma(F)$ that there exists $t_0 > 1$ such that

$$\Phi_{z_1}(t) = \frac{1}{t}\ln\mu(tz_1,F) \geq 2C \quad (t \geq t_0). \tag{4}$$

From (2) and (3) at $t = t_1 = t_0$, $t_2 = |z_j^*|$ we get

$$\Phi_{z_j^{(1)}}(t_0) = \frac{1}{t_0}\ln\mu(t_0 z_j^{(1)},F) \leq \Phi_{z_j^{(1)}}(t_2) = \frac{1}{|z_j^*|}\ln\mu(z_j^*,F) \leq C.$$

Passing here to the limit at $j \to +\infty$, together with (4) we obtain

$$2C \leq \Phi_{z_1}(t_0) = \lim_{j\to+\infty}\frac{1}{t_0}\ln\mu(t_0 z_j^{(1)},F) \leq C.$$

It is a contradiction.

Let us now prove that $\min\{\|\nu(z,F)\|\colon \nu(z,F) \in \mathcal{N}(z)\} \to +\infty$, $(z \to \infty,\ z \in K)$. Note that

$$\operatorname{Re}\Big(\frac{z}{|z|}, \lambda_{\nu(z)}\Big) = \frac{\ln\mu(z,F)}{|z|} - \frac{\ln|a_{\nu(z)}|}{|z|}, \quad \nu(z) = \nu(z,F) \in \mathcal{N}(z).$$

If we assume that there exists a sequence (z_j), $z_j \in K, z_j \to \infty (j \to +\infty)$ such that $(\forall j)\colon \|\nu(z_j,F)\| \leq C < +\infty$, then $\frac{\ln|a_{\nu(z_j)}|}{|z_j|} \to 0$ $(z_j \to \infty)$. Therefore,

$$\lim_{j\to+\infty}\operatorname{Re}\Big(\frac{z_j}{|z_j|}, \lambda_{\nu(z_j)}\Big) = \lim_{j\to+\infty}\frac{\ln\mu(z_j,F)}{|z_j|} = +\infty.$$

We again obtain a contradiction, because $\#\mathcal{N}(z_j) \leq (C+1)^p$ $(j \geq 1)$, and $\Big|\operatorname{Re}\frac{z_j}{|z_j|}\Big| \leq 1$. $\square$

For a fucntion $F \in \mathcal{D}$ and $z \in \gamma_F$ we put

$$K(z) = K_F(z) := \sup\Big\{\frac{1}{\Phi_z(t)}\int_0^t \frac{\Phi_z(u)}{u}du\colon\ t \geq t_0\Big\},$$

where $\Phi_z(t) = \frac{1}{t}\ln\mu(tz,F)$, and $t_0 = t_0(z) = \max\{t \in \mathbb{R}\colon \mu(tz,F) = 1\}$. It is clear that

$$\gamma_+(F) = \Big\{z \in \gamma(F)\colon\ K_F(z) < +\infty\Big\}.$$

For $R \in (0,+\infty)$ we also define

$$\gamma_R = \gamma_+(F,R) := \Big\{z\colon\ K_F(z) \leq R\Big\}.$$

It follows from Proposition 1 that for every $R>0$ the set of γ_R is also an unbounded real cone with the vertex at the point $\mathbf{0}$.

Since $a_n\to 0$ ($\|n\|\to+\infty$) for a function $F\in\mathcal{D}$ of form (1), the sequence $(a_n)_{n\in\mathbb{Z}_+^p}$ can be arranged by non-increasing. Denote by $(\mu_k)_{k\geq 0}$ the sequence $(-\ln|a_n|)_{n\in\mathbb{Z}_+^p}$ arranged by non-decreasing. It is clear that $\mu_k\nearrow+\infty$ ($k\to+\infty$). For each given $n\in\mathbb{Z}_+^p$ we put $k=k_n$ such that $\mu_{k_n}=-\ln|a_n|$, and for every given $k\in\mathbb{Z}_+$ we put $n=n(k)\in\mathbb{Z}_+^p$ such that $\mu_k=-\ln|a_{n(k)}|$.

Let us prove the following auxiliary general theorem containing the upper estimate of the general term of the series $F\in\mathcal{D}_0$ through its maximal term.

Theorem 3. *Let $F\in\mathcal{D}_0$, $v(u)\colon[0,+\infty)\to[0,+\infty)$ be a function such that $v(u)>0$ ($u\geq u_0$) and $\int_0^{+\infty}v(u)du<+\infty$. If $\ln k=o(\mu_k)$ ($k\to+\infty$), then there exist a function $c_1(u)\uparrow+\infty$ ($u\to+\infty$), $\int_0^{+\infty}c_1(u)v(4u)du<+\infty$, and a set $E\subset\gamma_+(F)$, $\tau_{2p}(E\cap\gamma_+(F))\leq C_p$, such that for each $R>0$, for all $n\geq 0$ and for all $t>0, tz\in\gamma_R\setminus E$*

$$|a_n|e^{t\operatorname{Re}(z,\lambda_n)}\leq\mu(tz,F)\exp\Big\{-t\int_{\mu_{k_\nu}}^{\mu_{k_n}}(\mu_{k_n}-u)\frac{c_z(u)}{\varphi_z^*(u)}v(4u)du\Big\},\tag{5}$$

where $\mu_{k_n}=-\ln|a_n|$, $c_z(u)=e^{-2K(z)}c_1(u)$, $\nu=\nu(tz,F)$:

$$\|\nu(tz)\|=\max\{\|n\|\colon|a_n|e^{t\operatorname{Re}(z,\lambda_n)}=\mu(tz,F)\}$$

is the central multi-index of series (1), and $\varphi_z^*(u)$ is the inverse function of the function $\Phi_z^*(t)=\ln\mu(tz,F)$.

Proof of Theorem 3. We fix $z\in\gamma_+(F),|z|=1$. Denote

$$l(x):=\int_x^{+\infty}v(u)du,\quad c_z^*(x):=e^{-2K(z)}c_1(x),\quad c_1(x):=(l(0)\cdot l(4x))^{-1/2},$$

where $K(z)=K_F(z)$ is the constant defined before the formulation of Theorem 3. Note that $l(x)\downarrow 0$, therefore $c_1(x)\uparrow+\infty$ ($x\to+\infty$), and

$$e^{2K(z)}\int_0^{+\infty}c_z^*(t)v(4t)dt\leq-\frac{1}{4}(l(0))^{-\frac12}\int_0^{+\infty}(l(x))^{-\frac12}dl(x)=\frac12.\tag{6}$$

For $t>0$ and $k\in\mathbb{Z}+$ we put

$$\alpha(t):=-\int_t^{+\infty}\frac{1}{\varphi_z^*(u)}c_z^*(u)v(4u)du,\ \alpha_k=\exp\Big\{-\int_0^{\mu_k}\alpha(t)dt\Big\},\ \tau_k=\alpha(\mu_k).$$

From (6) it follows

$$|\alpha(t)|\leq\frac{1}{\varphi_z^*(t)}\int_t^{+\infty}c_z^*(u)v(4u)du=o\left(\frac{1}{\varphi_z^*(t)}\right)\quad(t\to+\infty).\tag{7}$$

Therefore,

$$\int_0^{\mu_k}dt\int_t^{+\infty}\frac{c_z^*(u)}{\varphi_z^*(u)}v(4u)du=o(\mu_k)\quad(k\to+\infty).\tag{8}$$

We consider a Dirichlet series f of one variable $s\in\mathbb{C}$

$$f(s)=\sum_{k=0}^{+\infty}\frac{b_k}{\alpha_k}e^{s\mu_k},$$

where $b_k = e^{\mathrm{Re}(z,\lambda_n)}$, $n = n(k) \in \mathbb{Z}_+^p$ such that $\mu_k = -\ln|a_{n(k)}|$ $(k \in \mathbb{Z}_+)$.

Let us now prove that for every fixed $z \in \gamma_+(F)$ the Dirichlet series f is absolutely convergent in the half-plane $\{s = \sigma + it\colon \sigma < 0\}$, and also that the central index $\nu(x, f) \to +\infty$ $(x \to -0)$.

Indeed, the condition $F \in \mathcal{D}$ implies, that

$$\varliminf_{\|n\|\to+\infty} \frac{-\ln|a_n|}{\mathrm{Re}(z,\lambda_n)} = +\infty.$$

Thus, $\mathrm{Re}(z, \lambda_{n(k)}) = o(\mu_k)$ $(k \to +\infty)$. Hence and from (8) we obtain

$$\ln\frac{b_k}{\alpha_k} = \mathrm{Re}(z,\lambda_{n(k)}) - \int_0^{\mu_k} dt \int_t^{+\infty} \frac{c_z^*(u)}{\varphi_z^*(u)} v(4u)du = o(\mu_k) \quad (k \to +\infty).$$

Since $\ln k = o(\mu_k)$ $(k \to +\infty)$, by Valiron's theorem for abscissa of absolute convergence of Dirichlet series f we obtain

$$\sigma_a(f) = \varliminf_{k\to+\infty} \frac{-\ln(b_k/\alpha_k)}{\mu_k} = 0.$$

Therefore, the Dirichlet series f is absolute convergent in the half-plane $\{s = \sigma + it\colon \sigma < 0\}$ for every fixed $z \in \gamma_+(F)$.

Let us now prove that the cenral index $\nu(x, f) \to +\infty$ $(x \to -0)$. This follows from the relation $\mu(x, f) \to +\infty$ $(x \to -0)$ or, equivalently, from the condition

$$\sup\left\{\frac{b_k}{\alpha_k}:\ k \geq 0\right\} = +\infty. \tag{9}$$

Let us prove the last relation. We have, $0 \leq \ln\mu(tz, F) = \ln|a_\nu| + t\,\mathrm{Re}(z, \lambda_\nu)$ $(t \geq t_0(z))$, $\nu = \nu(tz, F)$. Hence, $-\ln|a_\nu| \leq t\,\mathrm{Re}(z, \lambda_\nu)$. Since $\Phi_z^*(t) = t\Phi_z(t) = \ln\mu(tz, F) \leq t\,\mathrm{Re}(z, \lambda_\nu)$ $(t \geq 0)$, $t \leq \varphi_z(\mathrm{Re}(z\lambda_\nu))$, where φ_z is the inverse function of the function Φ_z. Thus,

$$-\ln|a_\nu| \leq t\,\mathrm{Re}(z,\lambda_\nu) \leq \mathrm{Re}(z,\lambda_\nu)\varphi_z(\mathrm{Re}(z,\lambda_\nu)) \quad (t \geq t_0(z)), \tag{10}$$

where $\nu = \nu(tz, F)$. The function $u/\varphi_z^*(u)$ is the inverse function to the function $u\varphi_z(u)$, where the function φ_z is the inverse function to the function $\Phi_z(t) = \Phi_z^*(t)/t$, therefore,

$$\mathrm{Re}(z,\lambda_\nu) \geq \frac{u}{\varphi_z^*(u)}\bigg|_{-\ln|a_\nu|} = \frac{-\ln|a_\nu|}{\varphi_z^*(-\ln|a_\nu|)} \quad (t \geq 0) \quad \nu = \nu(tz, F). \tag{11}$$

Let $k_\nu \in \mathbb{Z}_+$ be such that $-\ln|a_\nu| = \mu_{k_\nu}$. Then, at $k = k_\nu$, $\nu = \nu(tz, F)$, $\ln\frac{b_k}{\alpha_k} \geq \frac{\mu_k}{\varphi_z^*(\mu_\nu)} - \int_0^{\mu_k}|\alpha(t)|dt$. The condition $z \in \gamma_+(F)$ implies that $t/\varphi_z^*(t) = \Phi_z(\varphi_z^*(t)) \to +\infty$ $(t \to +\infty)$, and

$$\int_0^x \frac{dt}{\varphi_z^*(t)} = \frac{x}{\varphi_z^*(x)} + \int_0^{\varphi_z^*(x)} \frac{\Phi_z^*(u)}{u^2}du + O(1) = \frac{x}{\varphi_z^*(x)} + O\left(\frac{\Phi_z^*(\varphi_z^*(x))}{\varphi_z^*(x)}\right) = O\left(\frac{x}{\varphi_z^*(x)}\right)$$

as $x \to +\infty$. Therefore, by (7) we get $\int_0^{\mu_k}|\alpha(u)|du = o\big(\mu_k/\varphi_z^*(\mu_k)\big)$ $(k \to +\infty)$, hence, finally, at $k = k_\nu$, $\ln\frac{b_k}{\alpha_k} \geq (1 + o(1))\frac{\mu_k}{\varphi_z^*(\mu_k)} \to +\infty$ $(t \to +\infty)$, $\nu = \nu(tz, F)$, that is (9) holds. Thus, $\nu(x, f) \to +\infty$ $(x \to -0)$.

Let (s_j) be the sequence of jump points of the central index $\nu(s, f)$, numbered in such a way that $\nu(s, f) = j$ for $s \in [s_j, s_{j+1})$ and, if $\nu(s_{j+1} - 0, f) = j$ and $\nu(s_{j+1}, f) = j + p$, then $s_{j+1} = s_{j+2} = \cdots = s_{j+p} < s_{j+p+1}$. It is clear that $s_j \to -0$ $(j \to +\infty)$.

If $x \in [s_k+\tau_k, s_{k+1}+\tau_k) \overset{def}{=} E_k^* \subset (-\infty; 0)$, then $\nu(x-\tau_k, f) = k$ and by definition $\mu(x-\tau_k, f)$ for all $m \geq 0$ we obtain $\frac{b_m}{\alpha_m}\mathrm{e}^{(x-\tau_k)\mu_m} \leq \mu(x-\tau_k, f)$. It follows from here for $\mu_m \neq \mu_k$

$$\frac{b_m}{b_k}\mathrm{e}^{x(\mu_m-\mu_k)} \leq \frac{\alpha_m}{\alpha_k}\mathrm{e}^{\tau_k(\mathrm{Re}(z,\lambda_{n(m)}-\lambda_{n(k)}))} = \exp\Big\{-\int_{\mu_k}^{\mu_m}(\alpha(u)-\alpha(\mu_k))du\Big\} < 1.$$

Substituting here $x = -\frac{1}{t},\ t > 0$, we obtain

$$\frac{|a_{n(m)}|\mathrm{e}^{t\,\mathrm{Re}(z,\lambda_{n(m)})}}{|a_{n(k)}|\mathrm{e}^{t\,\mathrm{Re}(z,\lambda_{n(k)})}} = \left(\frac{b_m\mathrm{e}^{x\mu_m}}{b_k\mathrm{e}^{x\mu_k}}\right)^t < 1 \quad (n \neq k), \tag{12}$$

i.e. $\nu(tz, F) = n(k)$ and $\mu(tz, F) = |a_{n(k)}|\mathrm{e}^{t\,\mathrm{Re}(z,\lambda_{n(k)})}$ for $t \in [-(s_k+\tau_k)^{-1}, -(s_{k+1}+\tau_k)^{-1})$.

Therefore, for every $t > 0$ тsuch that $x = -t^{-1} \in \bigcup_{k\in J} E_k^*$, where $J \subset \mathbb{N}\cup\{0\}$ is the range set of the cenral index $\nu(x, f)$, and for all $n \in \mathbb{Z}_+^p$ by inequality (12) we get

$$\frac{|a_{n(m)}|\mathrm{e}^{t\,\mathrm{Re}(z\lambda_{n(m)})}}{|a_{n(k)}|\mathrm{e}^{t\,\mathrm{Re}(z\lambda_{n(k)})}} = \left(\frac{b_m\mathrm{e}^{x\mu_m}}{b_k\mathrm{e}^{x\mu_k}}\right)^t \leq \exp\Big\{-t\int_{\mu_k}^{\mu_m}(\mu_m-u)\alpha'(u)du\Big\} \tag{13}$$

as $t = -\frac{1}{x} > 0$, where $n(m)$ such that $-\ln|a_{n(m)}| = \mu_m$. Therefore, for all $t \in \bigcup_{k\in J}\widetilde{E}_k \overset{def}{=} \widetilde{E}$ we obtain (5), where $\widetilde{E}_k \subset (0, +\infty)$ is the image of the set E_k^* by the mapping $t = -\frac{1}{x}$.

Estimate the logarithmic measure of the set

$$E_z^* = [-s_1^{-1}, +\infty)\setminus\widetilde{E} = \bigcup_{k=1}^{+\infty}[-(s_k+\tau_{k-1})^{-1}, -(s_k+\tau_k)^{-1}) = \bigcup_{k=1}^{+\infty} I_k^*.$$

Since $(\forall\ k \geq 0)(\forall\ t > 0)\colon\ -\mu_k + t\,\mathrm{Re}(z, \lambda_{n(k)}) \leq \ln\mu(tz, F)$, as $t = \varphi_z^*(\mu_k)$ we have

$$\mathrm{Re}(z, \lambda_{n(k)}) \leq \frac{\mu_k + \Phi_z^*(t)}{t} = \frac{2\mu_k}{\varphi_z^*(\mu_k)}. \tag{14}$$

For the constant $K = K_F(z) \in (0, +\infty)$ and fixed $z \in \gamma_+(F)$

$$2K\Phi_z(x\mathrm{e}^{-2K}) \leq \int_{x\mathrm{e}^{-2K}}^{x}\frac{\Phi_z(t)}{t}dt \leq \int_0^x\frac{\Phi_z(t)}{t}dt \leq K\Phi_z(x) \quad (x > 0),$$

that is $2\Phi_z\left(x\mathrm{e}^{-2K}\right) \leq \Phi_z(x)$ $(x > 0)$. Hence, $\varphi_z(2u) \leq e^{2K}\varphi_z(u)$ $(u \geq 0)$. Since, $t/\varphi_z^*(t) = \Phi_z(\varphi_z^*(t))$, the inequality $\varphi_z(\frac{2t}{\varphi_z^*(t)}) \leq c\varphi_z(\frac{t}{\varphi_z^*(t)}) = c\varphi_z^*(t)$ $(t \geq 0)$ holds with $c = \mathrm{e}^{2K}$, $K = K_F(z)$. By this inequality and (14) we get

$$t \leq \varphi_z(\mathrm{Re}(z, \lambda_{\nu(tz,F)})) \leq \varphi_z\left(\frac{2\mu_{k_\nu}}{\varphi_z^*(\mu_{k_\nu})}\right) \leq c\varphi_z^*(\mu_{k_\nu})\ (t \geq 0), \tag{15}$$

where $\nu = \nu(tz, F)$, k_ν such that $-\ln|a_\nu| = \mu_{k_\nu}$, and $c = \mathrm{e}^{2K}$, $K = K_F(z)$.

We now assume $k \in J$. Then $\nu\left(-\left(s_k+\tau_{k-1}-0\right)^{-1}z, F\right) = \nu(s_k+\tau_{k-1}-0, f) \leq k-1$. Hence and from the inequality (15) we have $|s_k+\tau_{k-1}|^{-1} = -(s_k+\tau_{k-1})^{-1} \leq c\varphi_z^*(\mu_{k-1})$. Therefore, by the definition of τ_k we obtain $|s_k+\tau_{k-1}|^{-1}(|\tau_{k-1}|-|\tau_k|) \leq c\int_{\mu_{k-1}}^{\mu_k}c_z^*(u)v(4u)du$.

Since, by inequality (6), $c\cdot\int_{\mu_{k-1}}^{\mu_k}c_z^*(u)v(4u)du \leq \frac{1}{2}$, for $I_k^* = [|s_k+\tau_{k-1}|^{-1}, |s_k+\tau_k|^{-1})$ one has

$$\ln\text{-}\mathrm{meas}(I_k^*) = \ln\text{-}\mathrm{meas}\left(([|s_k+\tau_{k-1}|^{-1}, |s_k+\tau_k|^{-1}))\right) = \ln\left|\frac{s_k+\tau_{k-1}}{s_k+\tau_k}\right| =$$

$$= \ln\left(1+\frac{|\tau_{k-1}|-|\tau_k|}{|s_k+\tau_k|}\right) \leq \frac{|\tau_{k-1}|-|\tau_k|}{|s_k+\tau_k|} = \frac{|\tau_{k-1}|-|\tau_k|}{|s_k+\tau_{k-1}|-(|\tau_{k-1}|-|\tau_k|)} \leq$$

$$\le c\int_{\mu_{k-1}}^{\mu_k} c_z^*(u)v(4u)du\Big(1-c\int_{\mu_{k-1}}^{\mu_k} c_z^*(u)v(4u)du\Big)^{-1}\le 2c\int_{\mu_{k-1}}^{\mu_k} c_z^*(u)v(4u)du.$$

Suppose now that $j\notin J$, $k,p\in J$ are such that $p<j<k$, $s_p<s_{p+1}=s_j=s_k<s_{k+1}$. Then

$$\bigcup_{j=p+1}^{k} I_j^*=\bigcup_{j=p+1}^{k}[|s_j+\tau_{j-1}|^{-1},|s_j+\tau_j|^{-1})=[|s_{p+1}+\tau_p|^{-1},|s_k+\tau_k|^{-1}).$$

By using the inequalities $|s_k+\tau_{k-1}|^{-1}=-(s_k+\tau_{k-1})^{-1}\le c\varphi_z^*(\mu_{k-1})$ and (6) we obtain

$$\ln\text{-}\operatorname{meas}\Big(\bigcup_{j=p+1}^{k} I_j^*\Big)\le \ln\frac{|s_{p+1}+\tau_p|}{|s_{p+1}+\tau_k|}\le\frac{|\tau_p|-|\tau_k|}{|s_{p+1}+\tau_p|-(|\tau_p|-|\tau_k|)}\le$$

$$\le c\int_{\mu_p}^{\mu_k} c_z^*(u)v(4u)du\Big(1-c\int_{\mu_p}^{\mu_k} c_z^*(u)v(4u)du\Big)^{-1}\le 2c\int_{\mu_p}^{\mu_k} c_z^*(u)v(4u)du.$$

Therefore, for the set $E_z^*=\bigcup\limits_{j=1}^{+\infty} I_j^*$ by inequality (6)

$$\ln\text{-}\operatorname{meas}(E_z^*)=\ln\text{-}\operatorname{meas}\Big(\bigcup_{j=1}^{+\infty} I_j^*\Big)\le 2c\int_0^\infty c_z^*(u)v(4u)du\le\frac12.$$

So, finally, for the set $E=\bigcup\limits_{z\in\gamma(F)\cap\{z:|z|=1\}} E_z$, where $E_z=\{tz\colon t\in E_z^*\}$, we get

$$\tau_{2p}(E)=\int_{z\in\gamma(F)\cap\{z:|z|=1\}}\Big(\int_{E_z}\frac{dt}{t}\Big)dS\le\frac12\cdot C_p,$$

where C_p is the area of the unit sphere in $\mathbb{C}^p$. □

3. Main result.

Theorem 4. *Let $F\in\mathcal{D}$. If*

$$(\exists\alpha>0)\colon\quad\int_{t_0}^{+\infty} t^{-2}(n_1(t))^\alpha dt<+\infty,\quad n_1(t)\stackrel{def}{=}\sum_{\mu_n\le t}1,\quad t_0>0,$$

then there exists a set $E\subset\gamma_+(F)$, such that $\tau_{2p}(E\cap\gamma_+(F))\le C_p$ and the relation

$$\mathfrak{M}(z,F)=o(\mu(z,F)\ln^{1/\alpha}\mu(z,F))$$

holds as $z\to\infty$ $(z\in\gamma_R\setminus E)$ for each $R>0$.

Proof of Theorem 4. Without a loss of generality we can assume that $F\in\mathcal{D}_0$, $\lambda_{\mathbf{0}}=\mathbf{0}$, $0=\mu_0\le\mu_m\nearrow+\infty$ $(1\le m\to+\infty)$. To prove Theorem 4, it is enough to use Theorem 3 and arguments according to the scheme of proving Theorem 1 in [13]. On the one hand, for a given $R>0$ and every fixed $z\in\gamma_R$ we will obtain that $\ln\mu(tz,F)\ge(n_1(3\mu_\nu))^\alpha c_1(\nu)$, $\nu=\nu(tz,F)$, holds for all $t>0$ such that $tz\notin E_1$, where the set $E_1\subset\gamma_+(F)$, by Theorem 3, such that $\tau_{2p}(E\cap\gamma_+(F))\le C_p/2$, and $c_1(\nu)\to+\infty$ as $tz\to\infty$ uniformly in K (by Proposition 3). On the other hand, we will obtain that

$$\sum_{\mu_k>3\mu_\nu}|a_n|\mathrm{e}^{t\operatorname{Re}(z,\lambda_n)}\le\mu(tz,F)/c_2(\nu),\quad\nu=\nu(tz,F),$$

is fulfilled for all $t>0$ such that $tz\notin E_2$, where again the set $E_2\subset\gamma_+(F)$, by Theorem 3, such that $\tau_{2p}(E_2\cap\gamma_+(F))\le C_p/2$, and $c_2(\nu)\to+\infty$ as $tz\to\infty$ uniformly in K (again by Proposition 3).

Remark, $\tau_{2p}((E_1\cup E_2)\cap\gamma_+(F))\le C_p$. Therefore, for all $tz\notin E=E_1\cup E_2$ we obtain

$$\mathfrak{M}(tz,F)\le\mu(tz,F)\Big(n(3\mu_\nu)+1/c_2(\nu)\Big)\le\mu(tz,F)\Big((\ln\mu(tz,F))^{1/\alpha}/c_1(\nu)+1/c_2(\nu)\Big).$$

Now, to complete the proof of Theorem 4, it remains to apply Proposition 3. In the case $F\in\mathcal{D}_0$, Theorem 4 is proved. The transition to a general case is obvious. □

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Ivan Franko National University of Lviv
andriykuryliak@gmail.com

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