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ONE CLASS OF CONTINUOUS LOCALLY COMPLICATED FUNCTIONS RELATED TO INFINITE-SYMBOL Φ-REPRESENTATION OF NUMBERS

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In the paper, we introduce and study a massive class of continuous functions defined on the interval $(0;1)$ using a special encoding (representation) of the argument with an alphabet $\mathbb{Z}=\{0,\pm1,\pm2,...\}$ and base $\tau=\frac{\sqrt{5}-1}{2}$:

$$x=b_{\alpha_1}+\sum_{k=2}^{m}(b_{\alpha_k}\prod_{i=1}^{k-1}\Theta_{\alpha_i})\equiv\Delta^{\Phi}_{\alpha_1\alpha_2\ldots\alpha_m(\varnothing)},\quad x=b_{\alpha_1}+\sum_{k=2}^{\infty}(b_{\alpha_k}\prod_{i=1}^{k-1}\Theta_{\alpha_i})\equiv\Delta^{\Phi}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

where $\alpha_n\in\mathbb{Z}$, $\Theta_n=\Theta_{-n}=\tau^{3+|n|}$, $b_n=\sum\limits_{i=-\infty}^{n-1}\Theta_i=\begin{cases}\tau^{2-n}, & \text{if } n\le 0,\\ 1-\tau^{n+1}, & \text{if } n\ge 0.\end{cases}$

The function f, which is the main object of the study, is defined by equalities

$$\begin{cases} f(x=\Delta^{\Phi}_{i_1\ldots i_k\ldots})=\sigma_{i_1 1}+\sum\limits_{k=2}^{\infty}\sigma_{i_k k}\prod\limits_{j=1}^{k-1}p_{i_j j}\equiv\Delta_{i_1\ldots i_k\ldots},\\ f(x=\Delta^{\Phi}_{i_1\ldots i_m(\varnothing)})=\sigma_{i_1 1}+\sum\limits_{k=2}^{m}\sigma_{i_k k}\prod\limits_{j=1}^{k-1}p_{i_j j}\equiv\Delta_{i_1\ldots i_m(\varnothing)},\end{cases}$$

where an infinite matrix $||p_{ik}||$ $(i\in\mathbb{Z},\ k\in\mathbb{N})$ satisfies the conditions

1) $|p_{ik}|<1\ \forall i\in\mathbb{Z},\ \forall k\in\mathbb{N}$; 2) $\sum\limits_{i\in\mathbb{Z}}p_{ik}=1\ \forall k\in\mathbb{N}$;

3) $0<\sum\limits_{k=2}^{\infty}\prod\limits_{j=1}^{k-1}p_{i_j j}<\infty\ \ \forall(i_j)\in L$; 4) $0<\sigma_{ik}\equiv\sum\limits_{j=-\infty}^{i-1}p_{jk}<1\ \forall i\in\mathbb{Z},\forall k\in\mathbb{N}$.

This class of functions contains monotonic, non-monotonic, nowhere monotonic functions and functions without monotonicity intervals except for constancy intervals, Cantor-type and quasi-Cantor-type functions as well as functions of bounded and unbounded variation. The criteria for the function f to be monotonic and to be a function of the Cantor type as well as the criterion of nowhere monotonicity are proved. Expressions for the Lebesgue measure of the set of non-constancy of the function and for the variation of the function are found. Necessary and sufficient conditions for the function to be of unbounded variation are established.

1. Introduction. A class of continuous functions defined on an interval is surprisingly rich in cardinality as well as diversity of properties of the functions [13,14]. Functions with extremely inhomogeneous local behaviour are among them [9–12]. These are nowhere monotonic functions and functions without monotonicity intervals except for constancy intervals, functions whose sets of non-constancy (in particular, sets of growth for the monotonic functions) are

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of zero Lebesgue measure and fractional fractal Hausdorff-Besicovitch dimension [8] and the monotonic functions. Their monotonicity is interesting: for any arbitrary small interval from the domain of the functions, they have points where derivative is equal to zero, is equal to infinity as well as points, where derivative does not exists. This paper is devoted to such functions; and they require specific tools for their analytical definition. To define these functions we use a special infinite-symbol system of encoding (representation) of numbers generated by two two-sided sequences of real numbers such that one of them is one-parameter.

The peculiarity of this encoding for the numbers of interval $(0;1)$ consists in the fact that its alphabet is the set of all integer numbers and it has a single positive irrational base therewith. This is its principal difference from Q_∞-representation [2], q_0^∞-representation, representation of numbers by regular continued fractions, Lüroth, Engel, Sylvester, Ostrogradsky, Ostrogradsky-Sierpiński-Pierce series [3], etc. Some ideas on means for definition of locally complicated functions are borrowed from the previous papers [5,6], where the finite-symbol systems of number representation are considered.

2. Φ-representation of numbers. Let $A = \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ be an alphabet (set of digits), let $L = A \times A \times ...$ be the space of sequences of elements of the alphabet, and let $\tau = \frac{\sqrt{5}-1}{2} \approx 0.62$ be the golden ratio, i.e., it is a positive root of the equation $x^2 + x - 1 = 0$. Hence, $\tau = 1 - \tau^2 > 0, \quad \tau^2 = 1 - \tau, \quad \tau + 1 = \frac{1}{\tau}, \quad \tau = \frac{1}{\tau} - 1$.

It is possible to prove [4, 7] that for any number $x \in (0;1)$ there exists a unique finite tuple of integer numbers $(\alpha_1, \alpha_2, ..., \alpha_m)$ or a unique sequence $(\alpha_n) \in L$ such that

$$x = b_{\alpha_1} + \sum_{k=2}^{m} \left(b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \right) \equiv \Delta^{\Phi}_{\alpha_1\alpha_2...\alpha_m(\varnothing)} \ , \tag{1}$$

$$x = b_{\alpha_1} + \sum_{k=2}^{\infty} \left(b_{\alpha_k} \prod_{i=1}^{k-1} \Theta_{\alpha_i} \right) \equiv \Delta^{\Phi}_{\alpha_1\alpha_2...\alpha_n...} \ , \tag{2}$$

where

$$\Theta_n = \Theta_{-n} = \tau^{3+|n|}, \ b_n = \sum_{i=-\infty}^{n-1} \Theta_i = \begin{cases} \tau^{2-n}, & \text{if } n \le 0; \\ 1 - \tau^{n+1}, & \text{if } n \ge 0. \end{cases}$$

Let us remark that $\tau^{2k+1} = u_{2k+1}\tau - u_{2k}$, $\tau^{2k} = u_{2k-1} - u_{2k}\tau$, $k \in \mathbb{N}$, where u_n is the nth term of the classic Fibonacci sequence: $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$.

The expansion of a number x in sum (1) or series (2) is called its Φ-*expansion*, and symbolic notation $\Delta^{\Phi}_{\alpha_1\alpha_2...\alpha_m(\varnothing)}$ or $\Delta^{\Phi}_{\alpha_1\alpha_2...\alpha_n...}$ is called its Φ-*representation* (finite of infinite, respectively). In addition to that α_n is called the nth *digit of this* Φ-*representation.*

From the uniqueness of Φ-representation of a number it follows that digit $\alpha_n = \alpha_n(x)$ of Φ-representation of a number x is a well-defined function of a number that is represented.

The set $\Delta^{\Phi}_{c_1...c_m}$ of all numbers $x \in (0;1)$ that have finite or infinite Φ-representation with first m digits $c_1, c_2, ..., c_m$, respectively, i.e.

$$\Delta^{\Phi}_{c_1...c_m} = \{x : x = \Delta^{\Phi}_{c_1...c_m\alpha_{m+1}...\alpha_n(\varnothing)}, x = \Delta^{\Phi}_{c_1...c_m\beta_1\beta_2...}, (\beta_n) \in L\}$$

is called a *cylinder of rank* m *with base* $c_1c_2...c_m$.

The following properties of cylinders immediately follow from the definition:

0. $\Delta^{\Phi}_{c_1...c_m i} \subset \Delta^{\Phi}_{c_1...c_m}$; $\Delta^{\Phi}_{c_1...c_m} = \bigcup\limits_{i=-\infty}^{\infty} \Delta^{\Phi}_{c_1...c_m i}$.

1. An order of cylinders is defined by the equality $\sup \Delta^{\Phi}_{c_1...c_{m-1}c_m} = \min \Delta^{\Phi}_{c_1...c_{m-1}[c_m+1]}$.

2. The cylinder $\Delta^{\Phi}_{c_1\ldots c_m}$ is a half-segment $[a;d)$ with endpoints

$$a = b_{c_1} + \sum_{k=2}^{m} b_{c_k} \prod_{i=1}^{k-1} \Theta_{c_i} = \Delta^{\Phi}_{c_1\ldots c_m(\varnothing)} \quad \text{and} \quad d = a + \prod_{i=1}^{m} \Theta_{c_i} = \Delta^{\Phi}_{c_1\ldots c_{m-1}[c_m+1](\varnothing)}.$$

3. The length of cylinder: $|\Delta^{\Phi}_{c_1\ldots c_m}| = \prod\limits_{i=1}^{m} \Theta_{c_i}$; $|\Delta^{\Phi}_{c_1\ldots c_m i}| = \Theta_i |\Delta^{\Phi}_{c_1\ldots c_m}|$.

4. $\forall (c_m) \in L$: $\bigcap\limits_{m=1}^{\infty} \Delta^{\Phi}_{c_1\ldots c_m} = \Delta^{\Phi}_{c_1\ldots c_m\ldots}$.

Let us remark that the cylinders determine the system of reduced partitions of interval $(0;1)$ and describe the geometry of the Φ-representation of numbers.

3. The main object of the study. Let $||p_{ik}||$ be an infinite matrix $(i \in \mathbb{Z}, k \in \mathbb{N})$ whose elements are real numbers. Suppose that the following conditions are satisfied:

1) $|p_{ik}| < 1$ $(\forall i \in \mathbb{Z})$ $(\forall k \in \mathbb{N})$; 2) $\sum\limits_{i \in \mathbb{Z}} p_{ik} = 1$ $(\forall k \in \mathbb{N})$;

3) $0 < \sum\limits_{k=2}^{\infty} \prod\limits_{j=1}^{k-1} p_{i_j j} < \infty$ $(\forall (i_j) \in L)$; 4) $0 < \sigma_{ik} \equiv \sum\limits_{j=-\infty}^{i-1} p_{jk} < 1$ $(\forall i \in \mathbb{Z})(\forall k \in \mathbb{N})$.

Let us remark that the following statements follow from the previous conditions:

5) $p_{ik} \to \infty$ $(i \to \pm\infty)$; 6) $\sigma_{ik} = \sum\limits_{j=-\infty}^{i-1} p_{jk} \to 0$ $(i \to -\infty)$, $\sigma_{ik} \to 1$ $(i \to \infty)$.

Let us define a function f on the set $(0;1)$ by equalities

$$\begin{cases} f(x = \Delta^{\Phi}_{i_1\ldots i_k\ldots}) = \sigma_{i_1 1} + \sum\limits_{k=2}^{\infty} \sigma_{i_k k} \prod\limits_{j=1}^{k-1} p_{i_j j} \equiv \Delta_{i_1\ldots i_k\ldots}, \\ f(x = \Delta^{\Phi}_{i_1\ldots i_m(\varnothing)}) = \sigma_{i_1 1} + \sum\limits_{k=2}^{m} \sigma_{i_k k} \prod\limits_{j=1}^{k-1} p_{i_j j} \equiv \Delta_{i_1\ldots i_m(\varnothing)}. \end{cases} \tag{3}$$

The function is well defined by equalities (3) because of a uniqueness of the Φ-representation of numbers and conditions 1)–4) that provide the convergence of series (3).

Denote by $P(\Phi)$ the class of functions defined by equality (3).

Lemma 1. *For any sequence $(i_k) \in L$, the partial sums*

$$S_n = \sigma_{i_1 1} + \sum_{k=2}^{n} \sigma_{i_k k} \prod_{j=1}^{k-1} p_{i_j j}$$

of series (3) are positive and do not exceed 1.

Proof. The proof is by induction on n. $S_1 = \sigma_{\alpha_1 1} \in (0;1)$ by condition 4).

Consider $S_2 = \sigma_{\alpha_1 1} + \sigma_{\alpha_2 2} p_{\alpha_1 1}$. Since $0 < \sigma_{\alpha_2 2} < 1$, for positive $p_{\alpha_1 1}$, by condition 4), we have $0 < \sigma_{\alpha_1 1} < S_2 \le \sigma_{\alpha_1 1} + p_{\alpha_1 1} = \sigma_{[\alpha_1+1]1} < 1$, and for negative $p_{\alpha_1 1}$, we have $0 < \sigma_{[\alpha_1+1]1} = \sigma_{\alpha_1 1} + p_{\alpha_1 1} \le S_2 \le \sigma_{\alpha_1 1} < 1$.

Suppose that the proposition holds for $n = m$, i.e.

$$(\forall (\alpha_k) \in L)\colon \quad 0 < S_m = \sigma_{\alpha_1 1} + \sum_{k=2}^{m} \sigma_{\alpha_k k} \prod_{k=1}^{k-1} p_{\alpha_i i} < 1.$$

Consider $n = m+1$, i.e.

$$S_{m+1} = \sigma_{\alpha_1 1} + \sum_{k=2}^{m+1} \sigma_{\alpha_k k} \prod_{k=1}^{k-1} p_{\alpha_i i} = \sigma_{\alpha_1 1} + p_{\alpha_1 1} \left(\sigma_{\alpha_2 2} + \sum_{k=3}^{m+1} \sigma_{\alpha_k k} \prod_{i=2}^{k-1} p_{\alpha_i i} \right).$$

By the induction hypothesis, the expression in parentheses belongs to the interval $(0;1)$. Thus, for a positive $p_{\alpha_1 1}$, we have $0 < \sigma_{\alpha_1 1} < S_{m+1} \leq \sigma_{[\alpha_1+1]1} < 1$, and, for negative one, we have $0 < \sigma_{[\alpha_1+1]1} < S_{m+1} < \sigma_{\alpha_1 1} < 1$. This proves the lemma. □

Corollary 1. *The set D_f of values of the function f is a subset of the segment $[0,1]$.*

4. Continuity.

Theorem 1. *The function f is continuous at every point of the domain of the function.*

Proof. Since the function f is defined at every point of interval $(0;1)$, we see that its continuity at a point $x_0 \in (0;1)$ is equivalent to the equality $\lim\limits_{x\to x_0} |f(x) - f(x_0)| = 0$.

1. Let $x_0 = \Delta^{\Phi}_{c_1\ldots c_n\ldots}$ be a Φ-infinite point, $x_0 \neq x = \Delta^{\Phi}_{\alpha_1\ldots\alpha_n\ldots}$. Then there exists $m \in \mathbb{N}$ such that $\alpha_m \neq c_m$, but $\alpha_i = c_i$ for $i < m$. Consider the difference

$$|f(x) - f(x_0)| = \left|\prod_{i=1}^{m-1} p_{c_i i}\right| \left|\left(\sigma_{\alpha_m m} + \sum_{k=m+1}^{\infty} \sigma_{\alpha_k k} \prod_{j=m}^{k-1} p_{\alpha_j j} - \sigma_{c_m m} - \sum_{k=2}^{\infty} \sigma_{c_k k} \prod_{j=m}^{k-1} p_{\alpha_j j}\right)\right|.$$

Since the expression in the latter absolute value is a difference of two numbers belonging to the interval $(0;1)$, we see that the absolute value of this expression does not exceed 1.

Taking into account that from condition 3) for matrix $||p_{ik}||$ it follows that $\prod_{k=1}^{m-1} p_{i_k k} \to 0$ $(m \to \infty)$, we see that $|f(x) - f(x_0)| \to 0$ $(m \to \infty \Leftrightarrow x \to x_0)$. Hence, the function f is continuous at the point x_0.

2. Let $x_0 = \Delta^{\Phi}_{c_1\ldots c_m(\varnothing)}$ is a Φ-finite point, $x \to x_0 - 0$. For numbers $x < x_0$ that are close enough to the point x_0 we have $x = \Delta^{\Phi}_{c_1\ldots c_{m-1}[c_m-1]\alpha_{m+1}\alpha_{m+2}\ldots}$, and the condition $x \to x_0$ is equivalent to the condition $\alpha_{m+1} \to \infty$. Consider the difference $|f(x) - f(x_0)| = |\prod_{i=1}^{m-1} p_{c_i i}| \times B$, where

$$B = |\sigma_{[c_m-1]m} + p_{[c_m-1]m} \sum_{k=m+1}^{\infty} \sigma_{\alpha_k k} \prod_{i=m}^{k-1} p_{\alpha_i i} - \sigma_{c_m m}| = |\sigma_{[c_m-1]m} + p_{[c_m-1]m}\sigma_{\alpha_{m+1}[m+1]} +$$

$$+ p_{\alpha_m m} p_{\alpha_{m+1}[m+1]} \sum_{k=m+2}^{\infty} \sigma_{\alpha_k k} \prod_{i=m+1}^{k-1} p_{\alpha_i i} - \sigma_{c_m m}| \to 0$$

because of $\sigma_{\alpha_{m+1}[m+1]} \to 1$ as $\alpha_{m+1} \to \infty$.

3. Let $x \to x_0 + 0$, $x = \Delta^{\Phi}_{c_1\ldots c_m \alpha_{m+1}\alpha_{m+2}\ldots}$ and $\alpha_{m+1 \to -\infty}$. Consider the difference $|f(x) - f(x_0)| = \left|\prod_{i=1}^{m} p_{c_i i}\right| \times B$, where $B = \left|\sigma_{[c_m-1]m} + p_{[c_m-1]m} \sum\limits_{k=m+1}^{\infty} \sigma_{\alpha_k k} \prod\limits_{i=m}^{k-1} p_{\alpha_i i} - \sigma_{c_m m}\right| =$

$$= \left|\sigma_{[c_m-1]m} + p_{[c_m-1]m}\sigma_{\alpha_{m+1}[m+1]} + p_{\alpha_m m} p_{\alpha_{m+1}[m+1]} \sum_{k=m+2}^{\infty} \sigma_{\alpha_k k} \prod_{i=m+1}^{k-1} p_{\alpha_i i} - \sigma_{c_m m}\right| \to 0,$$

because of $\sigma_{\alpha_{m+1}[m+1]} \to 1$ as $\alpha_{m+1} \to \infty$.

Thus, f is continuous at every point $x_0 \in (0;1)$. □

5. The Cantor-type functions and distributions of their values. A defined and continuous on an interval function has the *Cantor type* if the total length of its constancy intervals is equal to the length of its domain of definition (interval). The classic Cantor function is the simplest example of such function. The distributions of the Cantor-type function values lead to an interesting class of purely discrete distributions [1].

We say that function has a *quasi-Cantor type* if its set of non-constancy (in particular, the set of points of growth for non-decreasing functions) has a positive Lebesgue measure, whereas it is of zero Lebesgue measure for functions of the Cantor type.

Lemma 2. *If $p_{cm} = 0$, then the function f is constant on every cylinder $\Delta^{\Phi}_{c_1c_2\ldots c_{m-1}c}$.*

Proof. Indeed, if $x \in \Delta^{\Phi}_{c_1\ldots c_{m-1}c} = [a, d)$, where

$$a = b_{c_1} + \sum_{k=2}^{m-1} b_{c_k} \prod_{i=1}^{k-1} \Theta_{c_i} + b_c \prod_{i=1}^{m-1} \Theta_{c_i}, \; d = a + \Theta_c \prod_{i=1}^{m-1} \Theta_i,$$

then $x = \Delta^{\Phi}_{c_1\ldots c_{m-1}c\alpha_1\alpha_2\ldots}$, $f(x) = f(a) + 0$, because of $p_{cm} \prod_{i=1}^{m-1} p_{c_i i} = 0$. □

Lemma 3. *If $p_{c_i i} \neq 0$, where $i = \overline{1, m}$, then the change in the function on the cylinder $\Delta^{\Phi}_{c_1\ldots c_m}$ is 1) positive, when $\prod_{i=1}^{m} p_{c_i i} > 0$, 2) or negative, when $\prod_{i=1}^{m} p_{c_i i} < 0$.*

Proof. Let $\Delta^{\Phi}_{c_1\ldots c_m} = [a; d)$, i.e.

$$a = b_{c_1} + \sum_{k=2}^{m} b_{c_k} \prod_{i=1}^{k-1} \Theta_{c_i} = \Delta^{\Phi}_{c_1\ldots c_m(\varnothing)}, \quad d = a + \prod_{i=1}^{m} \Theta_{c_i} = \Delta^{\Phi}_{c_1\ldots c_{m-1}[c_m+1](\varnothing)}.$$

Taking into account continuity of the function and $\sigma_{[c_m+1]m} - \sigma_{c_m m} = p_{c_m m}$, we have

$$f(d) - f(a) = \left(\prod_{i=1}^{m-1} p_{c_i i}\right)(\sigma_{[c_m+1]m} - \sigma_{c_m m}) = \prod_{i=1}^{m} p_{c_i i} \neq 0.$$

□

Corollary 2. *The change $\mu_f(\Delta^{\Phi}_{c_1\ldots c_m}) \equiv f(d) - f(a)$ in the function f on the cylinder $\Delta^{\Phi}_{c_1\ldots c_m} = [a; d)$ is given by the formula $\mu_f(\Delta^{\Phi}_{c_1\ldots c_m}) = \prod_{i=1}^{m} p_{c_i i}$.*

Corollary 3. *If $p_{ik} \geq 0$ for any $i \in \mathbb{Z}$, $k \in \mathbb{N}$ and the matrix $||p_{ik}||$ does not contain negative elements, then f is a probability distribution function on the segment $[0, 1]$.*

Corollary 4. *If matrix $||p_{ik}||$ does not contain zeroes and negative elements, then function f is a strictly increasing probability distribution function.*

Theorem 2. *The Lebesgue measure of the set of non-constancy S_f (i.e., complement of the union of constancy intervals) is given by the formula*

$$\lambda(S_f) = \prod_{k=1}^{\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})} = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}\right) = \prod_{k=1}^{\infty}(1 - W_k), \tag{4}$$

where $F_0 = [0, 1]$, F_k is a union of all Φ-cylinders of rank k that contain the cylinders of higher ranks with a nonzero changes in the function f, $\overline{F}_k \equiv F_{k-1} \setminus F_k$, $W_k = \sum_{i:p_{ik}=0} \Theta_{ik}$.

Proof. It is evident that $S_f \subset F_{k+1} \subset F_k \ \forall\ k \in \mathbb{N}$ and $S_f = \bigcap_{k=1}^{\infty} F_k = \lim\limits_{k\to\infty} F_k$. Because of measurability of S_f and continuity of the Lebesgue measure we have

$$\lambda(S_f) = \lim_{k\to\infty} \lambda(F_k) = \prod_{k=1}^{\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})}.$$

Since $F_k = F_{k-1} \setminus \overline{F}_k$, we obtain next to last of equalities (4). However $\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})} = W_k$, hence the last of equalities (4) holds. □

Corollary 5. $\lambda(S_f)=0 \iff \sum\limits_{k=1}^{\infty}\frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}=\sum\limits_{k=1}^{\infty}W_k=\infty.$

Theorem 3. *The function f is a singular function of the Cantor type if and only if*

$$\sum_{k\in\mathbb{Z}}W_k=\infty$$

Proof. This proposition follows from the definition of a function of the Cantor type, the previous proposition and known relation on convergence/divergence of infinite products and series. □

Theorem 4. *If f is a function of the Cantor type and X is an uniformly distributed on $[0,1]$ random variable, then the random variable $Y=f(X)$ has a pure discrete distribution whose atoms are points of the form*

$$y=f(\Delta^{\Phi}_{c_1\dots c_{m-1}i(\varnothing)}),\ \text{where } p_{c_kk}\neq 0,\ k=\overline{1,m-1},\quad p_{im}=0, \tag{5}$$

and mass of the atom is equal to the length of Φ-cylinder $|\Delta^{\Phi}_{c_1\dots c_{m-1}i}|=\Theta_i\prod_{j=1}^{m-1}\Theta_{c_j}$.

Proof. If $p_{c_kk}\neq 0$, when $k\in\{1,2,\dots,m-1\}$, and $p_{im}=0$, then, by Lemma 2, the function f is constant on cylinder $\Delta^{\Phi}_{c_1\dots c_{m-1}i}$. Since the random variable X has a uniform distribution, we have $P\{X\in\Delta^{\Phi}_{c_1\dots c_mi}\}=|\Delta^{\Phi}_{c_1\dots c_mi}|=\Theta_i\prod_{j=1}^{m-1}\Theta_{c_j}$. Then, by conditions of the theorem, $f(x=\Delta^{\Phi}_{c_1\dots c_{m-1}i(\varnothing)})=y_0$ for any $x\in\Delta^{\Phi}_{c_1\dots c_{m-1}i}$. Thus

$$P\{Y=y_0\}=P\{X\in\Delta^{\Phi}_{c_1\dots c_{m-1}i}\}=|\Delta^{\Phi}_{c_1\dots c_{m-1}i}|=\Theta_i\prod_{j=1}^{m-1}\Theta_{c_j}.$$

Hence, y_0 is an atom of distribution of random variable Y with a given mass. Taking into account that f is the Cantor-type function, i.e., total length of its constancy intervals is equal to 1, we conclude that the distribution of Y is pure discrete. □

Theorem 5. *If f is a function of the quasi-Cantor type and X is uniformly distributed on $[0,1]$ random variable, then the distribution of the random variable $Y=f(X)$ is a mixture of discrete and continuous distributions. Its point spectrum consists of all points of the form (5), and the Lebesgue structure of its probability distribution function is*

$$F_Y=\alpha F_d(x)+(1-\alpha)F_c(x),\ 0<\alpha<1.$$

Proof. The proof that the point spectrum of distribution Y consists of points of the form (5) is analogous to given considerations in the proof of the previous theorem. Since f is of quasi-Cantor type, we see that the sum of length of its constancy intervals, i.e., sum of masses of atoms is less then 1. Hence, the continuous spectrum of distribution of random variable Y is non empty and the distribution of Y is a mixture of discrete and continuous distributions.

Let $\{y_i\}$ be a point spectrum (set of atoms) with masses p_i (p_{y_i} respectively) of the distribution of random variable Y, and $y_i<y_{i+1}$. Let us denote $G(y):=\sum_{y_i<y}p_{y_i}$. Then we have $G(1)=\sum_{y_i<1}p_{y_i}$. Thus $F_d(x)=G(x)/G(1)$. □

6. The conditions of nowhere monotonicity of the function.

Theorem 6. *The function f is nowhere monotonic if and only if the matrix $||p_{ik}||$ does not contain zeroes and infinitely many columns of the matrix contain negative elements.*

Proof. First, prove that if the matrix does not contain zero elements and infinitely many columns of the matrix contain negative elements, then the function f is nowhere monotonic.

Since the matrix $||p_{ik}||$ does not contain zeroes, from Lemma 3 it follows that the function f does not have constancy intervals. It is enough to prove that the function does not have monotonicity intervals. To this end it is enough to show that this function is not monotonic on any Φ-cylinder.

Let $\Delta^{\Phi}_{c_1\ldots c_m}$ be any cylinder of rank m and $p_{ik} < 0$, where $k > m$.

The change in the function on the Φ-cylinder $\Delta^{\Phi}_{c_1\ldots c_m\ldots c_{k-1}c} = [a_1; d_1)$ is given by expression $f(d_1) - f(a_1) = p_{ck}\prod_{j=1}^{k-1} p_{c_jj}$, and the change in the function on the Φ-cylinder

$$\Delta^{\Phi}_{c_1\ldots c_m\ldots c_{k-1}cc_{k+1}\ldots c_{n-1}i} = [a_2; d_2),$$

where $p_{in} < 0$, and $p_{js} > 0$ $\forall j \in \mathbb{Z}$ i $k < s < n$ has the form

$$f(d_2) - f(a_2) = p_{ck}\left(\prod_{j=1}^{k-1} p_{c_jj}\right)\left(\prod_{j=k+1}^{n-1} p_{c_jj}\right)p_{in}.$$

Since $[a_2; d_2] \subset [a_1; d_1]$ and $\prod_{j=k+1}^{n-1} p_{c_jj} > 0$ (by the way, for $n = k+1$, this factor is just absent in the product), we see that the changes in the function on these cylinders that belongs to the cylinder $\Delta^{\Phi}_{c_1\ldots c_m}$ have different signs.

Thus, the function is not monotonic on the Φ-cylinder $\Delta^{\Phi}_{c_1\ldots c_m}$. Hence, it is nowhere monotonic because Φ-cylinder is arbitrary chosen.

Let f be a nowhere monotonic function. Assume that matrix $||p_{ik}||$ contains zero elements (let $p_{cs} = 0$). Then, by Lemma 2, the function has constancy intervals, and this contradicts its nowhere monotonicity. Hence, there are no zero elements in matrix $||p_{ik}||$.

Assume now that only finitely many columns of the matrix have negative elements and $p_{ik} > 0$ $\forall i \in \mathbb{Z}, k > k_0$. Then, by Corollary 3 from Lemma 3 the function f is monontonic on every cylinder of rank k. In this case it is piecewise monotonic, and it again contradicts its nowhere monotonicity. The obtained contradictions prove that the absence of zeros in the matrix $||p_{ik}||$ and the existence of infinitely many columns with negative elements is a necessary and sufficient condition for the function to be nowhere monotonic. $\square$

7. Variational properties of the function. In the class $P(\Phi)$, there exist functions of bounded and of unbounded variation depending on the properties of the matrix $||p_{ik}||$.

Lemma 4. *The function f takes maximal and minimal value on a Φ-cylinder at its endpoints.*

Proof. Consider an arbitrary Φ-cylinder

$$\Delta^{\Phi}_{c_1\ldots c_m} = [a; d), \quad a = \Delta^{\Phi}_{c_1\ldots c_m(\varnothing)},\ d = \Delta^{\Phi}_{c_1\ldots c_{m-1}[c_m+1](\varnothing)}$$

and two points $x_1 = \Delta^{\Phi}_{c_1\ldots c_m i(\varnothing)}$, $x_2 = \Delta^{\Phi}_{c_1\ldots c_m\ldots c_k\ldots}$ that belong to it. If $D \equiv \prod_{k=1}^{m-1} p_{c_kk}$, then $f(x_1) - f(a) = Dp_{c_mm}\cdot\sigma_{i[m+1]}$, $f(d) - f(x_1) = D(\sigma_{[c_m+1]m} - \sigma_{c_mm} - p_{c_mm}\sigma_{i[i+1]}) = Dp_{c_mm}(1 - \sigma_{i[m+1]})$. Since $\sigma_{i[m+1]} > 0$ and $1 - \sigma_{i[m+1]} > 0$, we see that values of expressions $f(x_1) - f(a)$ and $f(d) - f(x_1)$ have the same signs or are equal to 0 simultaneously. Then $f(a)$ is maximal and $f(d)$ is minimal, when $f(x_1) - f(a) < 0$, or vice versa otherwise.

Analogously,

$$f(x_2) - f(a) = Dp_{c_mm}\cdot\sum_{k=m+1}^{\infty}\sigma_{c_ki_k}\prod_{j=m+1}^{k-1} p_{c_jj}, \quad f(d) - f(x_2) =$$

$$= D\left(\sigma_{[c_m+1]m} - \sigma_{c_m m} - \sum_{k=m+1}^{\infty} \sigma_{c_k k} \prod_{j=m}^{k-1} p_{c_j j}\right) = D \cdot p_{c_m m}\left(1 - \sum_{k=m+1}^{\infty} \sigma_{c_k k} \prod_{j=m+1}^{k-1} p_{c_j j}\right).$$

Since

$$0 < 1 - \sum_{k=m+1}^{\infty} \sigma_{c_k k} \prod_{j=m+1}^{k-1} p_{c_j j} < 1,$$

we see that expressions $f(x_2) - f(a)$ and $f(d) - f(x_2)$ have the same signs or are equal to 0 simultaneously. So, we conclude the same.

Hence, the function f takes maximal and minimal values at the endpoints of considered Φ-cylinder. This proves the lemma. □

Theorem 7. *The variation $V_0^1(f)$ of the function f on the interval $(0;1)$ is calculated by formula*

$$V_1^0(f) = \prod_{i=m}^{\infty} V_m, \quad \text{where} \quad V_m = \sum_{i=-\infty}^{+\infty} |p_{im}|. \tag{6}$$

Proof. By Corollary 2 from Lemma 3, sum of changes in the function f on cylinders of the 1st rank is equal to $V_1 = \sum_{i=-\infty}^{+\infty} |p_{i1}|$. By conditions 1) and 2) from Section 3, we have $V_1 \geq 1$. The sum of changes of the function f on cylinders of 2nd rank that belong to cylinder $\Delta_{c_1}^{\Phi}$ is expressed in the form $|p_{c_1 1}| \sum_{i=-\infty}^{+\infty} |p_{i2}|$. Then the sum B_2 of changes in the function f on all cylinders of the 2nd rank is calculated by the formula $B_2 \equiv \sum_{c_1=-\infty}^{+\infty} (p_{c_1 1} \sum_{i=-\infty}^{+\infty} p_{i2}) = V_1 V_2$. By induction, we have the expression B_m of all changes in the function f on cylinders of rank m: $B_m = \prod_{k=1}^{m} V_k$. Since the function f takes maximal and minimal values at the endpoints of cylinder (see Lemma 4), we see that the variation of the function f on $(0;1)$ is equal to $V_0^1(f) = \lim\limits_{m\to\infty} B_m = \prod_{k=1}^{\infty} V_k$. □

Corollary 6. *The function f is of unbounded variation if and only if*

$$\sum_{k=1}^{\infty} (1 - V_k) = -\infty.$$

Indeed, since $V_k = 1 - (1 - V_k)$, we see that this proposition follows from the theorem on relation between convergence of infinite products and series.

Corollary 7. *If all columns of matrix $||p_{ik}||$ are the same and the function f is nowhere monotonic, then it is of unbounded variation.*

Proof. Indeed, if conditions of this proposition are satisfied, then $V_m = V_1$, $B_m = V_1^m$, $V_0^1(f) = \lim\limits_{m\to\infty} V_1^m$. However, from nowhere monotonicity of function f it follows that $V_1 > 1$. Hence, $V_0^1(f) = \infty$. □

Corollary 8. *If infinitely many columns of matrix $||p_{ik}||$ contain zero elements and*

$$\sum_{k=1}^{\infty} (1 - V_k) = \infty,$$

then f is a function of bounded variation that does not have monotonicity intervals except for constancy intervals.

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