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REAL UNIVARIATE POLYNOMIALS WITH GIVEN SIGNS OF COEFFICIENTS AND SIMPLE REAL ROOTS

V. P. Kostov. *Real univariate polynomials with given signs of coefficients and simple real roots*, Mat. Stud. **61** (2024), 22–34.

We continue the study of different aspects of Descartes' rule of signs and discuss the connectedness of the sets of real degree d univariate monic polynomials (i. e. with leading coefficient 1) with given numbers ℓ^+ and ℓ^- of positive and negative real roots and given signs of the coefficients; the real roots are supposed all simple and the coefficients all non-vanishing. That is, we consider the space $\mathcal{P}^d := \{P := x^d + a_1x^{d-1} + \cdots + a_d\}$, $a_j \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$, the corresponding *sign patterns* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d)$, where σ_j =sign(a_j), and the sets $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)} \subset \mathcal{P}^d$ of polynomials with given triples $(\sigma, (\ell^+, \ell^-))$. We prove that for degree $d \leq 5$, all such sets are connected or empty. Most of the connected sets are contractible, i. e. able to be reduced to one of their points by continuous deformation. Empty are exactly the sets with $d = 4$, $\sigma = (-,-,-,+)$, $\ell^+ = 0$, $\ell^- = 2$, with $d = 5$, $\sigma = (-,-,-,-,+)$, $\ell^+ = 0$, $\ell^- = 3$, and the ones obtained from them under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action defined on the set of degree d monic polynomials by its two generators which are two commuting involutions: $i_m\colon P(x) \mapsto (-1)^dP(-x)$ and $i_r\colon P(x) \mapsto x^dP(1/x)/P(0)$.

We show that for arbitrary d, the following two sets are contractible:

1) the set of degree d real monic polynomials having all coefficients positive and with exactly n complex conjugate pairs of roots ($2n \leq d$);

2) for $1 \leq s \leq d$, the set of real degree d monic polynomials with exactly n conjugate pairs ($2n \leq d$) whose first s coefficients are positive and the next $d + 1 - s$ ones are negative.

For any degree $d \geq 6$, we give an example of a set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ having $\Lambda(d)$ connected components, where $\Lambda(d) \to \infty$ as $d \to \infty$.

1. Introduction. We consider real univariate monic (i. e. with leading coefficient 1) polynomials. Their set $\mathcal{P}^d = \{x^d + a_1x^{d-1} + \cdots + a_d\}$ can be identified with $\mathbb{R}^d \cong Oa_1 \ldots a_d$. A well-known folklore observation about them claims that the set of all such polynomials of a given degree d having only simple real zeros is a union of $\left[\frac{d}{2}\right] + 1$ connected contractible components (i. e. open subsets of $\mathbb{R}^d$ which can be reduced to one of their points by continuous deformation) enumerated by the number of complex conjugate pairs. The observation is based on the fact that a polynomial with ℓ real roots and $\frac{d-\ell}{2}$ complex conjugate pairs where $\ell \equiv d$ mod 2 can be continuously deformed into $(x + 1)(x + 2) \ldots (x + \ell)(x^2 + 1)^{(d-\ell)/2}$.

However a similar question for monic polynomials with fixed signs of their coefficients (all assumed non-vanishing) becomes rather non-trivial. Below we present initial results in this direction. We need to introduce some relevant notation.

Given a *sign pattern* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d)$, where each $\sigma_j,\ j \in \{1, \ldots, d\}$, is either plus or minus, we denote by $\mathcal{P}^d_\sigma \subset \mathcal{P}^d$ the orthant of all polynomials in $\mathcal{P}^d$ such that the sign of a_1

2020 *Mathematics Subject Classification:* 26C10, 30C15.
Keywords: standard discriminant; Descartes' rule of signs; contractibility.
doi:10.30970/ms.61.1.22-34

is σ_1, the sign of a_2 is σ_2, ..., the sign of a_d is σ_d. In particular, all a_j's are non-vanishing. Further, if $\ell \le d$ is a non-negative integer satisfying $\ell \equiv d \mod 2$, denote by $\mathcal{P}^d_{\sigma,\ell} \subset \mathcal{P}^d_\sigma$ the subset of $\mathcal{P}^d_\sigma$ consisting of polynomials with ℓ simple real roots and no other real roots. One can easily notice that for each pair (σ, ℓ), the subset $\mathcal{P}^d_{\sigma,\ell}$, when non-empty, is an open subset of $\mathcal{P}^d$, i. e. if a given polynomial belongs to $\mathcal{P}^d_{\sigma,\ell}$, then all nearby polynomials of $\mathcal{P}^d$ are also in $\mathcal{P}^d_{\sigma,\ell}$.

Observe that for any sign pattern σ as above, the orthant $\mathcal{P}^d_\sigma$ contains no polynomials vanishing at 0 since the constant term of polynomials in $\mathcal{P}^d_\sigma$ must be non-vanishing. Therefore, in each connected component of $\mathcal{P}^d_{\sigma,\ell} \subset \mathcal{P}^d_\sigma$ the number of positive and negative roots stays the same.

Given a sign pattern σ as above, we call its *Descartes' pair* (p_σ, n_σ) the pair of non-negative integers counting sign changes and sign preservations in the *enlarged sign pattern* $\hat{\sigma} := (+, \sigma)$. The latter is obtained from σ by adding the positive sign of the leading coefficient in front of σ. By the famous Descartes' rule of signs (see [2], [4], [5], [6], [10], [11], [12], [17] or [18]), the Descartes' pair of σ gives the upper bound on the number of positive and negative roots of any (monic) polynomial of degree d whose signs of coefficients are given by σ, see below. One observes that for any σ, $p_\sigma + n_\sigma = d$.

Given σ as above, if a monic polynomial $Q(x)$ of degree d belongs to $\mathcal{P}^d_\sigma$ and

$$(\ell^+(Q), \ell^-(Q))$$

denotes its numbers of positive and negative roots, then these numbers satisfy the following simple-minded inequalities:

$$\ell^+(Q) \le p_\sigma, \ \ \ell^+(Q) \equiv p_\sigma \mod 2, \ \ \ell^-(Q) \le n_\sigma, \ \ \ell^-(Q) \equiv n_\sigma \mod 2. \tag{1}$$

In the particular case when all roots are real, i. e. when $\ell^+(Q) = p_\sigma$ and $\ell^-(Q) = n_\sigma$, the polynomial Q is called *hyperbolic.*

A pair (ℓ^+, ℓ^-) of non-negative integers is called *compatible with the sign pattern* σ (compatible in the sense of Descartes' rule of signs) if it satisfies the inequalities (1). Since the number of positive and negative roots cannot change in each connected component of $\mathcal{P}^d_{\sigma,\ell}$, we obtain that for any pair (σ, ℓ) as above,

$$\mathcal{P}^d_{\sigma,\ell} = \bigcup_{\ell^+ + \ell^- = \ell} \mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}. \tag{2}$$

Here for each pair (ℓ^+, ℓ^-) compatible with σ, $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ stands for the set of all polynomials in $\mathcal{P}^d$ whose sign pattern of coefficients is given by σ, and which have ℓ^+ positive and ℓ^- negative (simple) roots.

Notice that some of $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ might be empty, see examples below.

The problem of non-emptiness for the sets $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ has been discussed in some detail in [8, 15, 16] and a number of follow-up papers. The problem of disconnectedness of $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ has been mentioned in [15, Problem 2], where some special cases have been treated.

Below we will discuss the following question.

Problem 1. *Given a sign pattern $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_d)$ and a pair of nonnegative integers (ℓ^+, ℓ^-) compatible with σ, in which cases the set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ is connected/contractible?*

2. Preliminary and main results.

2.1. Preliminary results. The following group action on $\mathcal{P}^d$ preserves the properties of all sets under consideration to be (non)-empty, connected or contractible.

Definition 1. We define the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the triples of the form $(\hat{\sigma}, (\ell^+, \ell^-))$ by its two generators which are two commuting involutions i_r and i_m. As above, denote by σ_j the j-th component of the sign pattern σ. The first of the generators replaces the enlarged sign pattern $\hat{\sigma}$ by $\hat{\sigma}^r$, where $\hat{\sigma}^r$ stands for the reverted (i. e. read from the right) enlarged sign pattern multiplied by σ_d, and keeps the same pair (ℓ^+, ℓ^-). This generator corresponds to the fact that the polynomials $P(x)$ and $x^d P(1/x)/P(0)$ are both monic and have the same numbers of positive and negative roots. The second generator exchanges ℓ^+ with ℓ^- and changes the signs of σ corresponding to the monomials of odd (resp. even) powers if d is even (resp. odd); the rest of the signs are preserved. We denote the new enlarged sign pattern by $\hat{\sigma}^m$. This generator corresponds to the fact that the roots of the polynomials $P(x)$ and $(-1)^d P(-x)$ are mutually opposite, and if $\hat{\sigma}$ is the enlarged sign pattern of P, then $\hat{\sigma}^m$ is the one of $(-1)^d P(-x)$. The polynomials $P(x)$ and $x^d P(1/x)/P(0)$ have the same Descartes' pair (p_σ, n_σ) while the Descartes' pair of $(-1)^d P(-x)$ equals (n_σ, p_σ).

Example 2. For $d = 4$ and for the enlarged sign pattern $\hat{\sigma} := (+, -, +, +, +)$, the Descartes' pair equals $(2, 2)$ and the orbit under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action of the triple $T := (\hat{\sigma}, (2, 0))$ consists of the four triples T, $i_r(T) = ((+, +, +, -, +), (2, 0))$, $i_m(T) = ((+, +, +, -, +), (0, 2))$ and $i_m i_r(T) = ((+, -, +, +, +), (0, 2))$.

The question of non-emptiness of $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ initiated by an example of D. J. Grabiner [7] has been considered by a number of authors and seems to be quite difficult in general. At present the exhaustive answer is known up to degree 8 as well as several infinite series of non-realizable triples $(\hat{\sigma}, (\ell^+, \ell^-))$, see e.g. [3] and the references therein.

Although the question of non-emptiness is the most basic one and is still unanswered, some further information with regard to connectedness and contractibility is available about certain $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$, see [14, 15]. Namely, the following results were proven in the first of these papers.

Theorem 1 ([14], Theorem 2)**.** *For any sign pattern $\sigma = (\sigma_1, \ldots, \sigma_d)$, the orthant $\mathcal{P}^d_\sigma$ contains a nonempty and contractible set $\mathcal{P}^d_{\sigma,d}$. In other words, the intersection of the set of degree d polynomials with d real and simple zeros with each orthant $\mathcal{P}^d_\sigma$ is a nonempty contractible set.*

Theorem 2. (1) ([14, Part (1) of Theorem 1]) *For d even and any sign pattern $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_d = +$, the orthant $\mathcal{P}^d_\sigma$ contains a nonempty and contractible set $\mathcal{P}^d_{\sigma,0}$. In other words, the intersection of the set of (even) degree d polynomials with no real zeros with each orthant $\mathcal{P}^d_\sigma$ is a nonempty contractible set as soon as $\sigma_d = +$.*

(2) ([14, Part (2) of Theorem 1]) *For d odd and any sign pattern $\sigma = (\sigma_1, \ldots, \sigma_d)$, the orthant $\mathcal{P}^d_\sigma$ contains a nonempty and contractible set $\mathcal{P}^d_{\sigma,1}$. In other words, the intersection of the set of (odd) degree d polynomials with one real zero with each orthant $\mathcal{P}^d_\sigma$ is a nonempty contractible set.*

(3) ([14, Part (3) of Theorem 1]) *For d even and any sign pattern $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_d = -$, the orthant $\mathcal{P}^d_\sigma$ contains a nonempty and contractible set $\mathcal{P}^d_{\sigma,(1,1)}$. In other words, the intersection of the set of (even) degree d polynomials with one positive and one negative zeros with each orthant $\mathcal{P}^d_\sigma$ such that $\sigma_d = -$ is a nonempty contractible set.*

(4) ([14, Part (3) of Theorem 1]) *For d even and any sign pattern $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_d = +$, the set $\mathcal{P}^d_{\sigma,(2,0)}$ (resp. $\mathcal{P}^d_{\sigma,(0,2)}$) is either empty or contractible. It is empty exactly when σ_d is positive, and among the other even coefficients there is at least one which is negative, and all odd coefficients are positive (resp. negative).*

We denote by $\sigma_\bullet$ the sign pattern $(-,+,+,...,+,+,-,+)$ (the corresponding enlarged sign pattern has four sign changes).

Theorem 3 ([15], Theorem 1). (1) *For $d \geq 6$, the set $\mathcal{P}^d_{\sigma_\bullet,(2,d-4)}$ is non-empty and consists of more than one component. Hence the set $\mathcal{P}^d_{\sigma_\bullet,(2,d-4)}$ is not connected.*

(2) *For $d = 4$ and 5, the respective sets $\mathcal{P}^4_{\sigma_\bullet,(2,0)}$ and $\mathcal{P}^5_{\sigma_\bullet,(2,1)}$ are connected.*

Remark 3. It is clear that if one fixes just the total number ℓ of real roots (and not the pair (ℓ^+,ℓ^-)) and considers a somewhat non-trivial sign pattern σ, then one can have more than one connected component. A simple example of this kind is the sign pattern $\sigma = (-+++)$ and the set of polynomials having exactly one conjugate pair. This set consists of at least two components, one with two positive and one with two negative real roots. To pass from one to the other one must have a root at 0, i. e. the constant term must vanish.

2.2. New results. Here we continue the latter line of research and provide more information about the sets $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$. Our first result reads:

Theorem 4. (1) *Denote by $\mathcal{P}_{d,d-2n}$ the set of degree d real monic polynomials having all coefficients positive and with exactly n complex conjugate pairs of roots ($2n \leq d$). This set is contractible.*

(2) *Denote by $\mathcal{P}_{d,d-2n;s}$ the set of real degree d monic polynomials with exactly n conjugate pairs ($2n \leq d$) whose first s coefficients are positive and the next $d+1-s$ ones are negative, $1 \leq s \leq d$. This set is contractible.*

The theorem is proved in Section 3. When one applies the involution i_m (see Definition 1), the two sets mentioned in part (2) of Theorem 3 become $\mathcal{P}_{4,1}$ and $\mathcal{P}_{5,1;3}$ respectively.

Theorem 5. (1) *For $1 \leq d \leq 5$, all sets $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ are either connected or empty. For $1 \leq d \leq 4$, and for $d = 5$ in the cases described in Theorem 4 and in part (2) of Proposition 10, all non-empty components are contractible.*

(2) *For $1 \leq d \leq 5$, empty are exactly the sets $\mathcal{P}^4_{\sigma,(0,2)}$, (with $\sigma = (-,-,-,+)$), $\mathcal{P}^5_{\sigma,(0,3)}$ (with $\sigma = (-,-,-,-,+)$) and the ones equivalent to them in the sense of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, see Definition 1.*

The theorem is proved in Section 4.

Remark 6. Theorems 3 and 5 imply that 6 is the least degree in which one obtains a set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ consisting of more than one component. It would be interesting to know whether for $d = 6$, the example of a disconnected set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ given by Theorem 3 is the only one up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. In the case described by part (1) of Theorem 3 one can show that the set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ consists of at least two components. The following theorem formulates a much stronger result.

Theorem 7. *For any fixed number of complex conjugate pairs n, there exists a sequence of even and a sequence of odd degrees d, and for each d a sign pattern σ and a corresponding pair (ℓ^+,ℓ^-) with $n = (d - \ell^+ - \ell^-)/2$, such that the number of connected components of the set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ tends to infinity as d tends to infinity.*

The theorem is proved in Section 5.

3. Proof of Theorem 4.

Part (1). Indeed, suppose that P_1, $P_2 \in \mathcal{P}_{d,d-2n}$ and that the real roots of P_1 and P_2 are the same. Then the polynomials P_i are representable in the form

$$P_i = W_i \cdot \prod_{j=1}^{d-2n} (x + t_j) \ , \quad W_i := \prod_{\nu=1}^{n} (x^2 + u_{\nu,i} x + v_{\nu,i}) \ , \quad i \in \{1, \ 2\},$$

where $-t_j$ are the real roots of P_1 and P_2 (hence $t_j > 0$), $v_{\nu,i} > 0$ and $u_{\nu,i}^2 - 4v_{\nu,i} < 0$. The quantities $u_{\nu,1}$ can be positive, zero or negative; we suppose that $u_{\nu,2} > 0$, $\nu \in \{1, \ldots, n\}$. Then every polynomial in the family $\tau P_1 + (1-\tau)P_2$, $\tau \in [0,1]$, belongs to the set $\mathcal{P}_{d,d-2n}$. Indeed, for $\tau \in [0,1]$, the degree $2n$ polynomial $\tau W_1 + (1-\tau)W_2$ is monic and takes only positive values, so it has no real roots. Thus the set $\mathcal{P}_{d,d-2n}$ can be retracted onto its part $\mathcal{P}^+_{d,d-2n}$ consisting of polynomials with negative real parts of their complex roots. The latter set is contractible, that is each polynomial of this set can be continuously deformed (without leaving the set $\mathcal{P}^+_{d,d-2n}$) into $\prod_{j=1}^{d-2n}(x+j) \prod_{\nu=1}^{n}((x+\nu)^2+1)$.

Part (2). It is clear from Descartes' rule of signs that such a polynomial Q has a single positive root h (which is simple). So it can be represented in the form $Q = (x-h)R$, where the polynomial R has n complex conjugate pairs and $d - 2n - 1$ negative roots. A priori there is an even number of sign changes in the sequence of coefficients of R. We prove first that all coefficients of R are positive.

Indeed, set $Q := \sum_{j=0}^{d} a_j x^{d-j}$, $a_0 = 1$, and $R := \sum_{j=0}^{d-1} b_j x^{d-1-j}$, $b_0 = 1$. Suppose that there exist indices $1 \leq \mu < \nu$ such that $b_{\mu-1} > 0$, $b_\mu < 0$, $b_{\nu-1} < 0$ and $b_\nu > 0$. As $a_j = b_j - hb_{j-1}$, one has $a_0 > 0$, $a_\mu < 0$, $a_\nu > 0$ and $a_d < 0$, i.e. there are at least three sign changes in the sequence of coefficients of Q – a contradiction. Hence $R \in \mathcal{P}_{d-1,d-1-2n}$.

Consider for each fixed $R \in \mathcal{P}_{d-1,d-1-2n}$ the polynomial $Q = (x-h)R$ as a one-parameter family with parameter $h > 0$. For h small enough, the first d coefficients of the polynomial Q are positive and the last one is negative. There exists a strictly decreasing sequence h_1, $\ldots$, h_{d-1} of positive numbers such that for $h = h_j$, one has $a_j = 0$. Indeed, for each s fixed, the coefficient a_s is an affine decreasing function in h, so the values h_i exist. The inequality $h_s < h_{s+1}$ is impossible, because then for $h \in (h_s, h_{s+1})$, the polynomial Q has at least three sign changes. The equality $h_s = h_{s+1}$ is also impossible, because the set $\mathcal{P}_{d-1,d-1-2n}$ to which the polynomial R belongs is open, so for certain nearby polynomials R one has $h_s < h_{s+1}$.

Thus the set $\mathcal{P}_{d,d-2n;s}$ of polynomials Q is fibered over the set $\mathcal{P}_{d-1,d-1-2n}$ the fibre being the interval (h_{s+1}, h_s). The quantities h_i depend continuously on $R \in \mathcal{P}_{d-1,d-1-2n}$. In part (1) of this theorem we proved that the latter set is contractible. Therefore the set $\mathcal{P}_{d,d-2n;s}$ is also contractible.

4. Proof of Theorem 5.

4.1. Plan of the proof.

Notation 8. We denote by $\Sigma_{m,r}$ the enlarged sign pattern consisting of m pluses (including the sign of the leading monomial) followed by r minuses (so $m + r = d + 1$). We denote by $\Sigma_{m,r,q}$ the enlarged sign pattern consisting of m pluses followed by r minuses followed by q pluses (so $m + r + q = d + 1$).

Some of the statements of Theorem 5 follow from already known results. Thus part (2) can be deduced from the results in [1]. The proof of part (1) for $1 \leq d \leq 4$ and when the polynomial is not hyperbolic follows directly from Theorem 2. Indeed, in this case there are not more than 2 real roots. For hyperbolic polynomials the proof of part (1) follows from [14, Theorem 2], see Theorem 1.

Suppose that $d = 5$. Then with the help of the involution i_m (see Definition 1) one can change an enlarged sign pattern with 3, 4 or 5 sign changes into a sign pattern with 2, 1 or 0 sign changes respectively.

In the cases of 0 or 1 sign changes and

– when the polynomial has exactly one real root the proof of part (1) of Theorem 5 follows from part (2) of Theorem 2;

– when the polynomial has exactly three real roots the proof of part (1) of Theorem 5 results from Theorem 4;

– when the polynomial has five real roots part (1) of Theorem 5 is a consequence of [14, Theorem 2], see Theorem 1.

For $d = 5$ and up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, there are 6 enlarged sign patterns with two sign changes:

$$\Sigma_{4,1,1}\ , \quad \Sigma_{3,1,2}\ , \quad \Sigma_{3,2,1}\ , \quad \Sigma_{2,2,2}\ , \quad \Sigma_{2,3,1} \quad \text{and} \quad \Sigma_{1,4,1}.$$

By part (2) of Theorem 5, all of them are realizable with the admissible pair $(2, 1)$.

It remains to prove part (1) of Theorem 5 for these 6 enlarged sign patterns. For a given enlarged sign pattern $\Sigma_{m,d-m,1}$ $(1 \leq m \leq d-1)$, we denote by σ its corresponding sign pattern. The proof of part (1) for $d = 5$ and enlarged sign patterns with two sign changes results from the following two propositions (proved respectively in Subsections 4.2 and 4.3).

Proposition 9. (1) *For $d = 5$ and for the enlarged sign pattern $\Sigma_{m,5-m,1}$, $1 \leq m \leq 4$, the set $\mathcal{P}^5_{\sigma,(2,1)}$ is connected.*

(2) *For $d = 5$ and for the enlarged sign pattern $\Sigma_{3,1,2}$, the set $\mathcal{P}^5_{\sigma,(2,1)}$ is connected.*

(3) *For $d = 5$ and for the enlarged sign pattern $\Sigma_{2,2,2}$, the set $\mathcal{P}^5_{\sigma,(2,1)}$ is connected.*

Proposition 10. (1) *For the enlarged sign patterns $\Sigma_{1,1,4}$, $\Sigma_{1,2,3}$, $\Sigma_{2,1,3}$ and $\Sigma_{2,2,2}$, the corresponding sets $\mathcal{P}^5_{\sigma,(0,3)}$ are connected.*

(2) *For the enlarged sign pattern $\Sigma_{1,3,2}$, the corresponding set $\mathcal{P}^5_{\sigma,(0,3)}$ is contractible.*

And we remind that for the enlarged sign pattern $\Sigma_{1,4,1}$, the corresponding set $\mathcal{P}^5_{\sigma,(0,3)}$ is empty.

4.2. Proof of Proposition 9.

Part (1). For a monic degree 5 polynomial $g \in \mathcal{P}^5_{\sigma,(2,1)}$ defining the enlarged sign pattern $\Sigma_{m,5-m,1}$, the polynomial $f := g'/5$ is monic and defines the enlarged sign pattern $\Sigma_{m,5-m}$. It has one positive and one or three negative roots (counted with multiplicity) in which cases we denote the sets of such degree 4 polynomials f by K_1 and K_3 respectively. For $f \in K_3$, f may have a multiple negative root.

The set K_1 is contractible, see part (3) of Theorem 2. As $g = a_5 + 5\int_0^x f(t)dt$, the set of polynomials g obtained by integrating polynomials $f \in K_1$ is contractible. Indeed for each $f \in K_1$, the constant $a_5 > 0$ takes its values from the open interval $(0, a_5^*)$, where for $a_5 = a_5^*$, the polynomial g has a double positive root.

Suppose that $f \in K_3$ and $m \leq 3$. Denote by $-\xi$ the negative root of the polynomial g. Consider the one-parameter family of polynomials $g_t := g - tx(x + \xi)$, $t \geq 0$. For $t \geq 0$, this polynomial defines the enlarged sign pattern $\Sigma_{m,5-m,1}$ and has a single negative root at $-\xi$. As t increases, its smaller positive root decreases without reaching 0 and its larger positive root increases. For $t > 0$ large enough, the polynomial g_t has no inflection points for $x < 0$. Hence its derivative has only one negative root.

Thus the polynomial g can be connected by a continuous path with a polynomial g_t for which $(g_t)' \in K_1$. So the set $\mathcal{P}^5_{\sigma,(2,1)}$ is connected.

Suppose that $m = 4$. As above, the set of polynomials g obtained by integrating polynomials $f \in K_1$ is contractible. Suppose that $f \in K_3$. The enlarged sign pattern of f is $(+,+,+,+,-)$ and f has four real roots. We denote them by $-\eta_3 < -\eta_2 < -\eta_1 < 0 < \zeta$ and the roots of g by $-\xi < 0 < \rho_1 < \rho_2$. By [13, part (1) of Theorem 1], $\zeta < \eta_1$. Hence $\rho_1 < \xi$, because $\rho_1 < \zeta < \rho_2$ and $-\xi < -\eta_1$, i. e. $\eta_1 < \xi$, so $\rho_1 < \zeta < \eta_1 < \xi$.

We consider the one-parameter family $g^*_t := g + tx(x^2 - \xi^2)$, $t \geq 0$. It defines the same enlarged sign pattern for all $t \geq 0$ and $-\xi$ is the only negative root of g^*_t. For $x = 0$, one has $g^*_t = a_5 > 0$. For $x > \rho_1$ and close to ρ_1, one has $g^*_t(x) < 0$, so g^*_t has two positive roots. Observe that $(g^*_t)''(0) > 0$ and $(x(x^2 - \xi^2))'' < 0$ for $x < 0$. As $(x(x^2 - \xi^2))''' = 6 > 0$, for $t > 0$ large enough, the polynomial $(g^*_t)'''$ is positive-valued, so the polynomial g^*_t has a single inflection point for $x < 0$ and $(g^*_t)' \in K_1$. As above we conclude that the set $\mathcal{P}^5_{\sigma,(2,1)}$ is connected.

Part (2). Given a polynomial h defining the enlarged sign pattern $\Sigma_{3,1,2}$ and the pair $(2,1)$ and having a single negative root $-\xi$, we consider the one-parameter family $h_t := h + tx^2(x + \xi)$, $t \geq 0$. For some $t = t_0 > 0$, the polynomial h_{t_0} has a double positive root. Hence its coefficient of x^2 is still negative, otherwise all coefficients of h_{t_0} are positive and it can have no positive roots. Thus the set $\mathcal{P}^5_{\sigma,(2,1)}$ can be retracted to the set of polynomials of the form

$$F := (x-1)^2(x+a)(x^2 + Bx + C)\ , \quad a > 0\ , \quad C > B^2/4$$

(we normalize the double positive root by a linear change of the variable x). The coefficients of $F = x^5 + F_4x^4 + \cdots + F_1x + F_0$ of the respective powers of x equal

$$F_4 = -2 + a + B, \quad F_3 = 1 - 2a + (-2 + a)B + C, \quad F_2 = a + (1 - 2a)B + (-2 + a)C,$$
$$F_1 = aB + (1 - 2a)C, \quad F_0 = aC.$$

Hence the condition $F_0 > 0$ is fulfilled. We suppose first that $0 < a \leq 1/2$. The inequality $F_4 > 0$ implies $B > 2 - a > 0$. As $F_1 > 0$, the triple of quantities (a, B, C) belongs to the domain

$$\Delta_1 := \big\{0 < a \leq 1/2,\ a + B > 2,\ C > \max(B^2/4,\ 2a - 1 + (2 - a)B,$$
$$(a + (1 - 2a)B)/(2 - a)\big\}.$$

It is clear that the domain Δ_1 is connected — it consists of all the points above the graph of a continuous function in the variables (a, B) (the maximum of three such functions) defined over the contractible domain $\{0 < a \leq 1/2,\ a + B > 2\}$.

If $1/2 < a < 2$, then the triple (a, B, C) belongs to the domain

$$\Delta_2 := \{1/2 < a < 2,\ a + B > 2,$$
$$\max(B^2/4,\ 2a - 1 + (2 - a)B,\ (a + (1 - 2a)B)/(2 - a)) < C < aB/(2a - 1)\}.$$

In the plane of the variables (B, C) the straight lines $L_3\colon C = 2a - 1 + (2 - a)B$ and $L_1\colon C = aB/(2a - 1)$ intersect for $a \neq 1$; for $a = 1$, they are parallel. For $a \neq 1$, the slope of L_3 is smaller than the slope of L_1, because

$$(2 - a) - a/(2a - 1) = -2(a - 1)^2/(2a - 1) < 0.$$

Their intersection point is

$$L_3 \cap L_1 = \big\{\big((2a - 1)^2/2(a - 1)^2,\ a(2a - 1)/2(a - 1)^2\big)\big\}. \qquad (3)$$

The quantity $C - B^2/4$ computed for the intersection point (3) is

$$-(2a - 1)(4a^2 - 2a - 1)/16(a - 1)^4.$$

It is negative for $a > a_0 := 0.8090169944\ldots$ and positive for $1/2 < a < a_0$. (The roots of the quadratic factor are a_0 and $a^* < 0$.) The domain Δ_2 consists of points of the sector which is below the line L_1 and above the line L_3. For $a \geq a_0$, this (open) sector is entirely below the parabola $C = B^2/4$. Hence the domain Δ_2 contains no points with $a > a_0$. This conclusion is valid without the restriction $a < 2$, because in the reasoning we did not use the straight line $L_2\colon C = (a + (1-2a)B)/(2-a)$, but only L_1 and L_3.

For $1/2 < a < a_0$, the domain Δ_2 consists of all points which are below the line L_1, above the parabola and the lines L_3 and L_2 and to the right of the line $L_0\colon B = 2-a$. This is the intersection of three convex domains. We show that it is non-empty. One has

$$L_1 \cap \{C = B^2/4\} = \{(0,0), (B_0, C_0)\},\ (B_0, C_0) := (4a/(2a-1), 4a^2/(2a-1)^2).$$

The point (B_0, C_0) lies above the lines L_2 and L_3:

$$C_0 - (a + (1-2a)B_0)/(2-a) = a(8a^2 - 4a + 3)/((2-a)(2a-1)^2) > 0,$$
$$C_0 - (2a - 1 + (2-a)B_0) = -(4a^2 - 2a - 1)/(2a-1)^2 > 0.$$

It lies to the right of the line L_0:

$$B_0 - (2-a) = 4a/(2a-1) - (2-a) = (2a^2 - a + 2)/(2a-1) > 0.$$

Hence for $1/2 < a < a_0$ fixed, in the space $OaBC$ and close to the point (B_0, C_0), there are points of the set Δ_2.

The union of the domains Δ_1 and Δ_2 is a connected set. To prove this one can consider a point T of the set Δ_1 belonging to the plane (in the space $OaBC$) $a = 1/2$. One can choose T for $(B, C) = (2, 4)$ in which case

$$F = x^5 + x^4/2 + x^3 - 11x^2/2 + x + 2.$$

The conditions defining the set of polynomials F are strict polynomial inequalities which are satisfied by all points from some neighbourhood of the point T hence by points from Δ_1 and Δ_2.

Part (3). In the case of the sign pattern $\Sigma_{2,2,2}$ one obtains the following system of inequalities:

$$F_4 = a + B - 2 > 0, \quad F_3 = C + (a-2)B - 2a + 1 < 0,$$
$$F_2 = (a-2)C + (1-2a)B + a < 0\ , \quad F_1 = (1-2a)C + aB > 0, \quad F_0 = aC > 0.$$

Suppose that $0 < a \leq 1/2$. Then one has $F_1 > 0$ and

$$S := (a + (1-2a)B)/(2-a) < C < (2-a)B + 2a - 1 =: U.$$

The condition $S < U$ is fulfilled for $0 < a \leq 1/2$, $B > 2-a$. Indeed, this condition is tantamount to $B > 2(a-1)^2/(a^2 - 2a + 3)$ and the right hand-side is $< 2-a$.

We denote by L_i the lines $F_i = 0$. For a fixed, we denote by $F_{3,i}$ and $F_{2,i}$, $i = 1$, 2, the B-coordinates of the intersection points of the lines L_3 and L_2 with the parabola $C = B^2/4$, $F_{j,1} < F_{j,2}$. One checks directly that for $0 < a \leq 1/2$, it is true that

$$F_{2,1} < F_{3,1} < F_{2,2} < 2-a < F_{3,2}.$$

(The quantities $F_{j,i}$ as functions in a are expressed by explicit formulas involving only rational functions and square roots of quadratic polynomials.) Thus for $0 < a \leq 1/2$, the set of polynomials F is fibered over its projection on the plane OaB. The projection is defined by the conditions $2-a < B < F_{3,2}$ and the fibre is of the form (W, U), where W is a point of the parabola and U a point of the line L_3. Clearly this set (denoted by Δ_3) is contractible.

Suppose that $1/2 < a < 2$. We denote by Δ_4 the corresponding set of polynomials F. Then

$$F_{2,1} < F_{3,1} < F_{1,1} = 0 < F_{2,2}, \quad 2-a < F_{1,2}, \quad F_{3,2} \quad \max(B^2/4, F_2) < C < \min(F_1, F_3)$$

(the relative orders of $F_{2,2}$ and $2-a$, and of $F_{1,2}$ and $F_{3,2}$, are not discussed). The slope of the line L_2 is negative, the slopes of L_1 and L_3 are positive, therefore Δ_4 is not empty. It is the intersection of two convex sets, so it is convex hence contractible.

For $a = 2$, the set Δ_5 of polynomials F is defined by the conditions

$$B > 2/3, \quad B^2/4 < C < \min(2B/3, 3),$$

hence it is non-empty and contractible. (One observes that dim $\Delta_5 = 2$ whereas dim $\Delta_3 =$ dim $\Delta_4 = 3$.)

Suppose that $a > 2$. We denote by Δ_6 the corresponding set of polynomials F (dim $\Delta_6 = 3$). Then

$$B^2/4 < C < \min((2-a)B + 2a - 1,\ ((2a-1)B - a)/(a-2),\ aB/(2a-1)).$$

Using the same notation $F_{i,j}$ as above (with $F_{1,1} = 0$ and $F_{1,2} = 4a/(2a-1)$), one checks that $F_{3,1} < 0 < F_{2,1} < F_{1,2} < F_{3,2} < F_{2,2}$. Hence the set Δ_6 comprises the points satisfying the conditions

$$2 < a, \quad F_{2,1} < B < F_{1,2}, \quad B^2/4 < C < \min(F_1, F_2, F_3).$$

(One observes that the inequality $B > 2 - a$ is less restrictive than $B > F_{2,1}$.) The set Δ_6 is not empty. Indeed, for $B \in (F_{2,1}, F_{1,2})$ and for $a > 2$ fixed, the line L_3 is above the line L_1, so one has in fact $B^2/4 < C < \min(F_1, F_2)$. The slope of the line L_1 is smaller than the slope of L_2. Denote by (B^0, C^0) the coordinates of the intersection point of the lines L_1 and L_2. Hence for $(B^*, C^*) \in \Delta_6$,

$$(B^*)^2/4 < C^* < F_2(a, B^*, C^*), \quad \text{if} \quad B \in (F_{2,1}, B^0) \quad \text{and}$$
$$(B^*)^2/4 < C^* < F_1(a, B^*, C^*), \quad \text{if} \quad B \in (B^0, F_{1,2}).$$

It is clear that Δ_6 is contractible.

The fact that the sets Δ_3, Δ_4, Δ_5 and Δ_6 are parts of a connected open set can be proved in the same way as this was done for Δ_1 and Δ_2 in the proof of part (2). Namely, we consider the points $T_1 \in \Delta_3|_{a=1/2}$ and $T_2 \in \Delta_5$ corresponding respectively to the triples $(a, B, C) = (1/2, 2, 2)$ and $(2, 1, 1/2)$, i. e. to the polynomials F of the form

$$x^5 + x^4/2 - x^3 - 5x^2/2 + x + 1 \quad \text{and} \quad x^5 + x^4 - 5x^3/2 - x^2 + x/2 + 1.$$

There exist neighbourhoods $\tilde{V}_i \subset OaBC$ of T_i such that all polynomials from these neighbourhoods define the enlarged sign pattern $\Sigma_{2,2,2}$. Thus there exists a continuous path connecting the sets Δ_3 and Δ_4 and passing through $\tilde{V}_1$ and such a path connecting the sets Δ_4, Δ_5 and Δ_6 and passing through $\tilde{V}_2$, so the four sets Δ_3, Δ_4, Δ_5 and Δ_6 are parts of one and the same connected set.

4.3. Proof of Proposition 10. Part (1). In the case of $\Sigma_{1,1,4}$ we use the fact that the last four coefficients are positive. In the other cases this is not true and therefore we point out the technical differences in the proof.

For a not necessarily monic polynomial h defining the enlarged sign pattern $\Sigma_{1,1,4}$, we denote by $-\eta_3 < -\eta_2 < -\eta_1 < 0$ its three negative roots (h is supposed to have also a complex conjugate pair). We denote the set of polynomials h by H. Clearly for $h \in H$, one has $h/h(0) \in \mathcal{P}^5_{\sigma,(0,3)}$.

The homotopy $\tilde{h}(t) := th + (1-t)(x+\eta_1)(x+\eta_2)(x+\eta_3)$ connects h with the degree 3 polynomial $w := (x+\eta_1)(x+\eta_2)(x+\eta_3)$. The set W of polynomials w is connected (this is the set of triples $\{(\eta_1, \eta_2, \eta_3)|0 < \eta_1 < \eta_2 < \eta_3\}$). For any two polynomials h_1, $h_2 \in H$, one can find their corresponding polynomials w_1, $w_2 \in W$ and a path $\gamma \subset W$ connecting w_1 with w_2. Thus one can connect h_1 with h_2 by a path belonging to $H \cup W$. We parametrize the path γ by $s \in [0, 1]$ so that $\gamma(0) = w_1$ and $\gamma(1) = w_2$. There exists $t_0 > 0$ small enough such that for any $(s, t) \in [0, 1] \times (0, t_0]$, one has

$$h^{\dagger}_{s,t}(x) := t(sh_2 + (1-s)h_1) + (1-t)w(s) \in H.$$

Indeed, there exists $\alpha > 0$ such that along the homotopy $w(s)$ one has $|\eta_i(s) - \eta_{i+1}(s)| \geq \alpha$, $i \in \{1, 2\}$, and the values of any derivative of the quantity $sh_2 + (1-s)h_1$ are uniformly bounded. This means that for $t > 0$ small enough, the polynomial $h^{\dagger}_{s,t}(.)$ is of degree 5 and has three negative simple roots. By Descartes' rule of signs, it has no other negative roots.

For $x > 0$, the values of h_1, h_2 and w are positive, so $h^{\dagger}_{s,t}$ has no positive roots. Hence $h^{\dagger}_{s,t} \in H$.

Thus one connects by paths h_1 with $\tilde{h}_1(t_0)$, $\tilde{h}_1(t_0)$ with $\tilde{h}_2(t_0)$ and $\tilde{h}_2(t_0)$ with h_2 hence h_1 with h_2 by a path belonging to the set H, i. e. the set H is connected and $\mathcal{P}^5_{\sigma,(0,3)}$ as well.

Suppose that the polynomial h defines the enlarged sign pattern $\Sigma_{2,2,2}$. We define the polynomial w in a different way. Namely, we look for a polynomial of the form $w = x^5 + Ax^4 + Bx + C$, A, B, $C > 0$. We denote by $\mathrm{sym}(k_1, k_2, k_3)$ the symmetric polynomial of (η_1, η_2, η_3) containing exactly all monomials $\eta^{k_1}_{i_1}\eta^{k_2}_{i_2}\eta^{k_3}_{i_3}$, where (i_1, i_2, i_3) is a permutation of $(1, 2, 3)$. The conditions $w(-\eta_i) = 0$, $i \in \{1, 2, 3\}$, yield

$$A = (\mathrm{sym}(3,0,0) + \mathrm{sym}(2,1,0) + \eta_1\eta_2\eta_3)/S,\ \ B = (\mathrm{sym}(3,3,0) + \mathrm{sym}(3,2,1) + \eta_1^2\eta_2^2\eta_3^2))/S,$$
$$C = \eta_1\eta_2\eta_3(\mathrm{sym}(2,2,0) + \mathrm{sym}(2,1,1))/S,$$

where $S := \mathrm{sym}(2,0,0) + \mathrm{sym}(1,1,0)$. (One needs to use computer algebra here.) Hence $A > 0$, $B > 0$, $C > 0$, and except the real roots $-\eta_i$ the polynomial w has a complex conjugate pair. (By Descartes' rule of signs it has no positive roots and not more than 3 negative roots.) As in the case of $\Sigma_{1,1,4}$, we conclude that the set H is connected.

Suppose that h defines the enlarged sign pattern $\Sigma_{1,2,3}$ or $\Sigma_{2,1,3}$. Then we look for a polynomial w of the form $w = x^5 + Dx^2 + Ex + F$, where

$$D = \mathrm{sym}(3,0,0) + \mathrm{sym}(2,1,0) + \eta_1\eta_2\eta_3, \quad E = \mathrm{sym}(3,1,0) + \mathrm{sym}(2,2,0) + 2\,\mathrm{sym}(2,1,1),$$
$$F = \mathrm{sym}(3,1,1) + \mathrm{sym}(2,2,1),$$

so D, E, $F > 0$. We conclude that the set H is connected as in the previously considered cases.

Part (2). We prove contractibility of the set $H^* := \mathcal{P}^5_{\sigma,(0,3)}$ for the enlarged sign pattern $\Sigma_{1,3,2}$. For each polynomial $h \in H^*$ with negative roots $-\eta_3 < -\eta_2 < -\eta_1$, there exists a positive number t_0 such that for $t = t_0$, the polynomial $h_t := h - t(x + \eta_1)(x + \eta_2)(x + \eta_3)$ is hyperbolic and has a double positive root. Indeed, in the family h_t one cannot obtain a negative coefficient of the monomial x, because the set $\mathcal{P}^5_{\sigma,(0,3)}$ is empty for the enlarged sign pattern $\Sigma_{1,4,1}$.

One cannot obtain at the same time a double positive root and a zero coefficient of x. Indeed, in this case one can add to h_t a polynomial $\varepsilon x(x - \eta)$, where $\eta > 0$ is smaller than the (double) positive root of h_t and $\varepsilon > 0$ is so small that the signs of the other coefficients of h_t do not change, the simple negative roots of h_t remain such, no new positive roots appear and the coefficient of x becomes $-\varepsilon\eta$. Thus the modified polynomial h_t must belong to the empty set $\mathcal{P}^5_{\sigma,(0,3)}$ defined for $\Sigma_{1,4,1}$ which is a contradiction.

We denote the set of polynomials h_{t_0} by H^*_0. Observe that in the family of polynomials h_t one can obtain polynomials from H^* also for some negative values of the parameter t. Thus the set H^* is fibered over the set H^*_0 the fibres being intervals of the form $t_{\dagger} < t < t_0$, where for $t = t_{\dagger}$, one of the coefficients of x^4, x^3 or x^2 of h_t vanishes. So now we concentrate on proving contractibility of H^*_0. One can transform the positive root of h_{t_0} into 1 by a linear change $x \mapsto ax$, $a > 0$, so we set

$$h_{t_0} := (x-1)^2(x^2 + Sx + T)(x + w) = x^5 - Ax^4 - Bx^3 - Cx^2 + Dx + E,$$
$$w = \eta_3 > 0, \quad S = \eta_1 + \eta_2 > 0\ , \quad T = \eta_1\eta_2 > 0,$$

where the condition $E = Tw > 0$ is automatic. The following system of equalities and inequalities must hold true

$$A = 2 - S - w > 0, \quad B = (2 - S)w + 2S - T - 1 > 0,$$
$$C = (2S - T - 1)w - (S - 2T) > 0, \quad D = (S - 2T)w + T > 0.$$

The couple (S, T) satisfies the conditions $0 < S < 2$ and $0 < T < S^2/4$. They define a domain in the space OST which is entirely below the straight line $T = S/2$, so the condition $D > 0$ is fulfilled. If $2S - T - 1 \leq 0$, then $C < 0$, so one must have $2S - T - 1 > 0$. Then the condition $B > 0$ takes the form

$$w > -(2S - T - 1)/(2 - S),$$

which is automatically fulfilled, because the right hand-side is negative while $w > 0$. Thus w must satisfy the inequalities

$$(S - 2T)/(2S - T - 1) < w < 2 - S.$$

So w takes the values of an open interval when the couple (S, T) satisfies the condition

$$(S - 2T) < (2S - T - 1)(2 - S), \quad \text{i.e.} \quad T > y(S) := 2(S - 1)^2/S.$$

For $S > 0$, the graph of the function y has an ordinary tangency with the S-axis for $S = 1$; it lies above this axis for $S \in (0, 1) \cup (1, \infty)$. This graph intersects the parabola $T = S^2/4$ for $S = 2$ and $S = 3 \pm \sqrt{5}$. Thus the set H_0^* can be retracted to the domain $\{3 - \sqrt{5} < S < 2, y(S) < T < S^2/4\} \subset OST$ which is contractible.

5. Proof of Theorem 7. A polynomial from a given set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$ is said to realize the sign pattern σ and the pair (ℓ^+, ℓ^-). In the proof we make use of the following concatenation lemma, see [8, Lemma 14].

Lemma 11. *We represent the enlarged sign patterns of the monic polynomials P_1 and P_2 of degrees d_1 and d_2 in the form $(+, \sigma^1)$ and $(+, \sigma^2)$ respectively, where σ^i are the respective sign patterns. The polynomials are supposed to realize the pairs (ℓ_1^+, ℓ_1^-) and (ℓ_2^+, ℓ_2^-). Then:*
(1) if the last position of σ^1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the enlarged sign pattern $(+, \sigma^1, \sigma^2)$ and the pair $(\ell_1^+ + \ell_2^+, \ell_1^- + \ell_2^-)$;
(2) if the last position of σ^1 is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the enlarged sign pattern $(+, \sigma^1, -\sigma^2)$ and the pair $(\ell_1^+ + \ell_2^+, \ell_1^- + \ell_2^-)$. Here $-\sigma^2$ is obtained from σ^2 by changing each $+$ by $-$ and vice versa.

It is clear that for ε small enough, the roots of the polynomial $P_2(x/\varepsilon)$ are much smaller in modulus than the roots of the polynomial $P_1(x)$. In particular, when both polynomials P_i are without vanishing coefficients, one knows the order of the moduli of the real roots and of the moduli of the real parts of complex roots of the product $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ on the real positive half-line.

We apply the concatenation lemma. Suppose that d is even. We concatenate polynomials f_1, f_2, $\ldots$ with double positive roots and polynomials g_1, g_2, $\ldots$ with double negative roots. The enlarged sign patterns of the polynomials f_i and g_j equal $(+, -, +)$ and $(+, +, +)$ respectively.

The concatenation is of the form $P := M_1 f_1 M_2 f_2 \ldots M_{s-1} f_{s-1} M_s$, where each M_k stands for the concatenation of $m_k \geq 1$ polynomials g_j and $\deg P = d = 2(m_1 + \cdots + m_s + s - 1)$. The sign pattern of such a concatenation contains minus-signs separated by at least three plus-signs from each side. Minus-signs can occur only at odd degree monomials. The moduli of the double roots of the concatenated polynomials form a decreasing sequence.

One can perturb the roots of the constructed polynomial so that the signs of the coefficients do not change. One can find perturbations in which each concatenation factor independently gives rise either to two simple roots (positive for the factors f_i and negative for the factors g_j) or to a complex conjugate pair. We consider only perturbations in which all factors f_i are perturbed so as to give rise to simple positive roots. We assume that the perturbation is so small that the order of the moduli of the real parts of the roots remains the same throughout the perturbation.

We consider perturbations in which exactly n factors g_j give rise to complex conjugate pairs. For a given number n, there are different possible choices of these n polynomials g_j. We call these polynomials *marked*. Moreover, the distributions of the *quantities* of marked polynomials g_i among the intervals between two consecutive polynomials f_i (i. e. among the products M_k) could be different.

Example 12. Suppose that $s = 3$, $m_1 = m_2 = m_3 = 2$ (so $P = M_1 f_1 M_2 f_2 M_3$, $M_k = g_{2k-1}g_{2k}$) and $n = 2$. Then there are exactly six possible distributions of the marked polynomials g_j. Namely, one can choose both polynomials g_j from the same concatenation product M_1, M_2 or M_3 (this gives three possibilities), or from two different such products which gives another three possibilities.

Denote by P_1 and P_2 two degree d polynomials P corresponding to two different choices of the marked factors g_j, in which the distributions of the numbers of marked polynomials g_j are different. If P_1 and P_2 belong to one and the same component C of the set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$, then one can connect them by a path $\gamma \subset C$ in which there is at least one polynomial P_* having a positive and a negative root of the same modulus. Indeed, for each of the polynomials P_1 and P_2 one can consider the moduli of their positive and negative roots on the real positive half-axis $\mathbb{R}^*_+$. Moduli of positive (resp. negative) roots are coloured in red (resp. blue). The distributions of the numbers of marked polynomials g_j being different for P_1 and P_2, the orders on $\mathbb{R}^*_+$ of the coloured moduli of their real roots are also different. As these moduli change continuously along γ, there has to be a point of γ for which a red modulus is equal to a blue one.

This however is impossible. Indeed, one can assume that these two roots equal ± 1 (which can be achieved by a linear change of the variable x). Then $P_*(1) = P_*(-1) = 0$, so $(P_*(1) + P_*(-1))/2 = 0$. But this is exactly the sum of the even coefficients of P_* which are all positive, so this is impossible. Hence for the different distributions of the quantities of marked polynomials g_j among the intervals between two consecutive polynomials f_i one obtains different components of the set $\mathcal{P}^d_{\sigma,(\ell^+,\ell^-)}$.

When d tends to infinity, one can construct polynomials P whose numbers s also tend to infinity. In this case the number of the aforementioned distributions tends to infinity which proves the theorem for even degrees d. To prove it for odd degrees d one can consider in the same way instead of P polynomials of the form $LM_1f_1M_2f_2\ldots$, where L is a degree 1 polynomial with a negative root.

Acknowledgment. The author is grateful to Boris Shapiro from the University of Stockholm for the fruitful and stimulating discussions of this subject.

REFERENCES

1. A. Albouy, Y. Fu, *Some remarks about Descartes' rule of signs*, Elemente der Mathematik, **69** (2014), 186–194.
2. F. Cajori, *A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity*, Colo. Coll. Publ. Sci. Ser. **12–7** (1910), 171–215.
3. H. Cheriha, Y. Gati, V.P. Kostov, *A non-realization theorem in the context of Descartes' rule of signs*, arXiv:1911.12255.

4. D.R. Curtiss, *Recent extensions of Descartes' rule of signs*, Annals of Mathematics, **19** (1918), №4, 251–278.
5. J.-P. de Gua de Malves, *Démonstrations de la Règle de Descartes*, Pour connoître le nombre des Racines positives & négatives dans les Équations qui n'ont point de Racines imaginaires, Memoires de Mathématique et de Physique tirés des registres de l'Académie Royale des Sciences (1741), 72–96.
6. The Geometry of René Descartes with a facsimile of the first edition, translated by D. E. Smith and M.L. Latham, New York, Dover Publications, 1954.
7. D.J. Grabiner, *Descartes' rule of signs: Another construction*, Amer. Math. Mon., **106** (1999), 854–856.
8. J. Forsgård, V. Kostov, B. Shapiro, *Could René Descartes have known this?*, Exp. Math., **24** (2015), №4, 438–448.
9. J. Forsgård, V. Kostov, B. Shapiro, *Corrigendum: "Could René Descrates have known this?"*, Exp. Math., **28** (2019), №2, 255–256.
10. J. Fourier, *Sur l'usage du théorème de Descartes dans la recherche des limites des racines*, Bulletin des sciences par la Société philomatique de Paris, (1820), 156–165, 181–187; œuvres 2, 291–309, Gauthier-Villars, 1890.
11. C.F. Gauss, *Beweis eines algebraischen Lehrsatzes*, J. Reine Angew. Math. **3** (1828), №1–4; Werke 3, 67–70, Göttingen, 1866.
12. J.L.W. Jensen, *Recherches sur la théorie des équations*, Acta Math., **36** (1913), 181–195.
13. V.P. Kostov, *Descartes' rule of signs and moduli of roots*, Publicationes Mathematicae Debrecen, **96** (2020), №1–2, 161–184.
14. V.P. Kostov, *Univariate polynomials and the contractibility of certain sets*, Annual of Sofia University St. Kliment Ohridski, Faculty of Mathematics and Informatics, **107** (2020), 75–99.
15. V.P. Kostov, *The disconnectedness of certain sets defined after uni-variate polynomials*, Constr. Math. Anal., **5** (2022), №3, 119–133.
16. V. Kostov, B. Shapiro, *New aspects of Descartes' rule of signs*, in book "Polynomials — Theory and Applications", DOI: 10.5772/intechopen.82040.
17. E. Laguerre, *Sur la théorie des équations numériques*, Journal de Mathématiques Pures et Appliquées, s.3, **9** (1883), 99–146; œuvres 1, Paris, 1898, Chelsea, New-York, 1972, 3–47.
18. B.E. Meserve, Fundamental Concepts of Algebra, New York, Dover Publications, 1982.

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Received 16.06.2023
Revised 26.01.2024