ON ENTIRE DIRICHLET SERIES SIMILAR TO HADAMARD COMPOSITIONS


A function $F(s) = \sum_{n=1}^{\infty} a_n \exp{s\lambda_n}$ with $0 \leq \lambda_n \uparrow +\infty$ is called the Hadamard composition of the genus $m \geq 1$ of functions $F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp{s\lambda_n}$ if $a_n = P(a_{n,1}, \ldots, a_{n,p})$, where $P(x_1, \ldots, x_p) = \sum_{k_1 + \cdots + k_p = m} c_{k_1 \ldots k_p} x_1^{k_1} \cdots x_p^{k_p}$ is a homogeneous polynomial of degree $m \geq 1$. Let $M(\sigma, F) = \sup \{|F(\sigma + it)| : t \in \mathbb{R}\}$ and functions $\alpha, \beta$ be positive continuous and increasing to $+\infty$ on $[x_0, +\infty)$. To characterize the growth of the function $M(\sigma, F)$, we use the generalized order $\varrho_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}$, generalized type $T_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho_{\alpha, \beta}[F] \beta(\sigma))}$ and membership in the convergence class defined by the condition

$$
\int_{\sigma_0}^{\infty} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho_{\alpha, \beta}[F] \beta(\sigma))} d\sigma < +\infty.
$$

Assuming the functions $\alpha, \beta$ and $\alpha^{-1}(c \beta(\ln x))$ are slowly increasing for each $c \in (0, +\infty)$ and $\ln n = O(\lambda_n)$ as $n \to +\infty$, it is proved, for example, that if the functions $F_j$ have the same generalized order $\varrho_{\alpha, \beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T_{\alpha, \beta}[F_j] = T_j \in [0, +\infty)$, $c_{m_0 \ldots 0} = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to +\infty$ for $2 \leq j \leq p$, and $F$ is the Hadamard composition of genus $m \geq 1$ of the functions $F_j$ then $\varrho_{\alpha, \beta}[F] = \varrho$ and

$$
T_{\alpha, \beta}[F] \leq \sum_{k_1 + \cdots + k_p = m} (k_1 T_1 + \cdots + k_p T_p).
$$

It is proved also that $F$ belongs to the generalized convergence class if and only if all functions $F_j$ belong to the same convergence class.

1. Introduction. Let

$$
f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n, \quad 1 \leq j \leq p,
$$

be entire transcendental functions. As in [1], we say that the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is similar to the Hadamard composition of the functions $f_j$ if $a_n = w(a_{n,1}, \ldots, a_{n,p})$ for all $n$, where $w : \mathbb{C}^p \to \mathbb{C}$ is some function. Clearly, if $p = 2$ and $w(a_{n,1}, a_{n,2}) = a_{n,1} a_{n,2}$ then $f = (f_1 * f_2)$ is [2] the Hadamard composition (product) of the functions $f_1$ and $f_2$. Properties of this composition obtained by J. Hadamard find the applications [3, 4] in the theory of the analytic continuation of the functions represented by power series.

E. G. Calys [5] investigated the functions similar to Hadamard compositions of with $|w(x, y)| = \sqrt{|x y|}$ and proved in particular the following theorem.

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Theorem A ([5]). Let entire functions \( f_j, j = 1, 2, \) have the same order \( \rho[f_j] = \rho \in (0, +\infty) \) and types \( \sigma[f_j] = \sigma_j. \) Suppose that \( a_{n,1} \neq 0 \) and \( |a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|) \) for all \( n \geq n_0, \) where \( l \) is slowly varying function. If \( |a_n| = (1 + o(1))\sqrt{|a_{n,1}|/|a_{n,2}|} \) as \( n \to \infty, \) then the function \( f \) has order \( \rho[f] = \rho \) and type \( \sigma[f] \leq \sqrt{\sigma_1\sigma_2}. \)

In the paper [6] the results of E. G. Calys are generalized on the case of entire Dirichlet series of finite generalized orders, moreover instead of two entire functions \( m \geq 2 \) entire Dirichlet series were considered.

Let \( \Lambda = (\lambda_n) \) be an increasing to \(+\infty\) sequence of nonnegative number, \( S(\Lambda, A) \) be a class of Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it,
\]

(1)

with a given sequence \( (\lambda_n) \) of exponents and an abscissa of absolutely convergence \( \sigma_a[F] = A \in (-\infty, +\infty), \) and let \( M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\} \) for \( \sigma \in (-\infty, A). \)

As in [7], by \( L \) we denote the class of positive continuous functions \( \alpha \) on \((-\infty, +\infty)\) such that \( \alpha(x) = \alpha(x_0) \) for \( x \leq x_0 \) and \( 0 < \alpha(x) \uparrow +\infty \) as \( x_0 < x \uparrow +\infty. \) We say that \( \alpha \in L^0 \) if \( \alpha \in L \) and \( \alpha(1 + o(1))x = (1 + o(1))\alpha(x) \) as \( x \to +\infty. \) Finally, \( \alpha \in L_{si}, \) if \( \alpha \in L \) and \( \alpha(cx) = (1 + o(1))\alpha(x) \) as \( x \to +\infty \) for each \( c \in (0, +\infty), \) i. e. \( \alpha \) is a slowly increasing function. Clearly, \( L_{si} \subset L^0. \)

If \( \alpha \in L, \beta \in L \) and \( F \in S(\Lambda, +\infty), \) that is series (1) is entire, then the value

\[
\rho_{\alpha,\beta}[F] = \lim_{\sigma \to +\infty} \frac{\alpha(1 + o(1))\ln M(\sigma, F))}{\beta(\sigma)}
\]

is called the generalized order of \( F. \) If \( \rho_{\alpha,\beta}[F] \in (0, +\infty) \) the generalized type is defined as

\[
T_{\alpha,\beta}[F] = \lim_{\sigma \to +\infty} \frac{\ln M(\sigma, F))}{\rho_{\alpha,\beta}[F]^{(\beta(\sigma))}}.
\]

The following theorem is true.

Theorem B ([6]). Let the functions \( \alpha \in L_{si} \) and \( \beta \in L^0 \) be continuously differentiable, \( \frac{d\ln \alpha^{-1}(\rho(x))}{d \ln x} \rightarrow \rho \) and \( \alpha(x) = (1 + o(1))\ln x \) as \( x \to +\infty. \) Suppose that \( \ln n = o(\lambda_n) \) \( (n \to \infty) \) and Dirichlet series \( F_j \in S(\Lambda, +\infty) \) of form

\[
F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}, \quad 1 \leq j \leq p,
\]

(2)

have the same generalized order \( \rho_{\alpha,\beta}[F_j] = \rho \in (0, +\infty) \) and types \( T_{\alpha,\beta}[F_j] \in (0, +\infty). \) If \( a_{n,1} \neq 0 \) for all \( n \geq n_0 \) and \( \omega_j > 0 \) with \( \sum_{j=1}^{p} \omega_j = 1, \)

\[
\alpha^{-1}\left(\frac{\rho_{\alpha,\beta}}{\rho} \left(1 + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right) = (1 + o(1)) \prod_{j=1}^{p} \alpha^{-1}\left(\frac{\rho_{\alpha,\beta}}{\rho} \left(1 + \frac{1}{\lambda_n} \ln \frac{1}{|a_{n,j}|}\right)\right)^{\omega_j}, \quad n \to \infty,
\]

and

\[
\beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_{n,j}|}\right) \leq (1 + o(1))\beta\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_{n,1}|}\right), \quad n \to \infty,
\]

for all \( 2 \leq j \leq p, \) then Dirichlet series (1) has the generalized order \( \rho_{\alpha,\beta}[F] = \rho \) and the type

\[
T_{\alpha,\beta}[F] \leq \prod_{j=1}^{p} T_{\alpha,\beta}[F_j]^{\omega_j}.
\]
If \( T_{\alpha,\beta}[F] = 0 \) then for the characteristic of the growth of entire Dirichlet series (1) we define a generalized convergence class by the condition

\[
\int_{\sigma_0}^{\infty} \frac{\ln M(\sigma,F)}{\sigma^{\alpha-1}(\beta(\sigma))} d\sigma < +\infty, \quad \varrho = \varrho_{\alpha,\beta}[F].
\]  

Theorem C ([1]). Let \( \alpha \in L \) and \( \beta \in L \) be positive continuously differentiable functions such that \( \frac{d\ln \alpha^{-1}(\beta'(\sigma))}{\sigma} = O(1) \) as \( \sigma \to \infty \) for each \( \varrho \in (0, +\infty) \). Suppose that \( \ln n = O(\lambda_n) \) and \( |a_n| \times \prod_{j=1}^{p} |a_{n,j}|^{-\omega_j} \) as \( n \to \infty \) for some \( \omega_j > 0 \) such that \( \sum_{j=1}^{p} \omega_j = 1 \). If all functions (2) belong to the generalized convergence class then function (1) also belongs to this class. If, in addition, \( |a_{n,1}| > 0 \) for all \( n \geq 0 \) and \( |a_{n,j}| \asymp |a_{n,1}| \) as \( n \to \infty \) for all \( j = 2, \ldots, p \), then the belonging of function (1) to generalized convergence class implies the belonging of all functions (2) to this class.

Here we consider the case when \( w \) is a homogeneous polynomial.

2. Definition and convergence of Hadamard composition of the genus \( m \). Recall that a polynomial is called homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial \( P(x_1, \ldots, x_p) \) is homogeneous of degree \( m \) if and only if \( P(tx_1, \ldots, tx_p) = t^m P(x_1, \ldots, x_p) \) for all \( t \) from the field above which a polynomial is defined. Dirichlet series (1) is called the Hadamard composition of genus \( m \) of Dirichlet series (2) if

\[
P(x_1, \ldots, x_p) = \sum_{k_1+\cdots+k_p=m} c_{k_1 \ldots k_p} x_1^{k_1} \cdots x_p^{k_p}.
\]
is a homogeneous polynomial of degree \( m \geq 1 \). We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus \( m = 2 \).

Therefore, if the function \( F \) is the Hadamard composition of genus \( m \geq 1 \) of the functions \( F_j \) then

\[
|a_n| \leq \sum_{k_1+\cdots+k_p=m} |c_{k_1 \ldots k_p}| |a_{n,1}|^{k_1} \cdots |a_{n,p}|^{k_p}.
\]  

Denote

\[
\tau = \lim_{n \to \infty} \frac{\ln n}{\lambda_n}, \quad \alpha[F] = \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.
\]

Then \([8,9] \sigma_a[F] \leq \alpha[F] \leq \sigma_a[F] + \tau \). Hence it follows that if \( \tau < +\infty \) and either \( \sigma_a[F] = +\infty \) or \( \alpha[F] = +\infty \), then \( \sigma_a[F] = \alpha[F] \).

Therefore, if \( \tau < +\infty \) and all \( F_j \in S(\Lambda, +\infty) \), i.e. \( \alpha[F_j] = +\infty \), then for every \( a > 0 \) we have \( |a_{n,j}| \leq \exp\{-a\lambda_n\} \) for every \( a > 0 \) all \( j \) and all \( n \geq n_0(a) \). Therefore, (4) implies

\[
|a_n| \leq C \exp\{-a\lambda_n\}, \quad C = \sum_{k_1+\cdots+k_p=m} |c_{k_1 \ldots k_p}|,
\]
whence

\[
\alpha[F] = \lim_{n \to \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \geq a m,
\]
i.e. in view of the arbitrariness of \( a \) we get \( \alpha[F] = +\infty \), that is \( F \in S(\Lambda, +\infty) \).
3. Growth of entire Hadamard compositions of the genus \(m\). Since the polynomial \(P(x_1,\ldots,x_p)\) is homogeneous of the degree \(m \geq 1\), we have

\[
a_n e^{m \lambda_n} = \sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} (a_{n,1} e^{\lambda_n})^{k_1} \cdots (a_{n,p} e^{\lambda_n})^{k_p}.
\]

Let \(\mu(\sigma, F) = \max\{|a_n| \exp\{\sigma \lambda_n\}: n \geq 0\}\) be the maximal term of series (1). Since (5) implies

\[
|a_n| e^{m \sigma \lambda_n} \leq \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}|(|a_{n,1}| e^{\sigma \lambda_n})^{k_1} \cdots (|a_{n,p}| e^{\sigma \lambda_n})^{k_p},
\]

we have

\[
\mu(m\sigma, F) \leq \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}| \mu(\sigma, F_1)^{k_1} \cdots \mu(\sigma, F_p)^{k_p},
\]

whence for all \(\sigma\) large enough we get

\[
\ln \mu(m\sigma, F) \leq \sum_{k_1+\cdots+k_p=m} \ln \left(|c_{k_1\ldots k_p}| \mu(\sigma, F_1)^{k_1} \cdots \mu(\sigma, F_p)^{k_p}\right) + \ln (m+1) =
\]

\[
= \sum_{k_1+\cdots+k_p=m} (\ln \left(|c_{k_1\ldots k_p}|\right) + k_1 \ln \mu(\sigma, F_1) + \cdots + k_p \ln \mu(\sigma, F_p)) + \ln (m+1) =
\]

\[
= \sum_{k_1+\cdots+k_p=m} (k_1 \ln \mu(\sigma, F_1) + \cdots + k_p \ln \mu(\sigma, F_p)) + C_1,
\]

where \(C_1 = \sum_{k_1+\cdots+k_p=m} \ln^+ |c_{k_1\ldots k_p}| + \ln (m+1)\). In what follows, we will use the following lemma (see, for example, [8, p. 22] and [9, p. 184]).

**Lemma 1.** If \(\ln n = O(\lambda_n)\) as \(n \to \infty\) then \(\mu(\sigma, F) \leq M(\sigma, F) \leq \mu(\sigma + O(1), F)\) as \(\sigma \to +\infty\), and if \(\ln n = o(\lambda_n)\) as \(n \to \infty\) then \(\mu(\sigma, F) \leq M(\sigma, F) \leq \mu(\sigma + o(1), F)\) as \(\sigma \to +\infty\).

Hence it follows that if \(\alpha \in L\) and either \(\ln n = O(\lambda_n)\) as \(n \to \infty\) and \(\beta(\ln x) \in L_{si}\) or \(\ln n = o(\lambda_n)\) as \(n \to \infty\) and \(\beta(\ln x) \in L^0\) then

\[
\lim_{\sigma \to +\infty} \frac{\alpha(\ln \mu(\sigma, F))}{\beta(\sigma)} = \lim_{\sigma \to +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}.
\]

Suppose that the functions \(F_j\) have the same generalized order \(q_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)\). Then for every \(\varrho_1 > \varrho\) and all \(\sigma \geq \sigma_0\) we have \(\ln \mu(\sigma, F_j) \leq \alpha^{-1}(\varrho_1 \beta(\sigma))\) for \(1 \leq j \leq p\) and, thus, (6) implies

\[
\ln \mu(m\sigma, F) \leq \sum_{k_1+\cdots+k_p=m} ((k_1 + \cdots + k_p)\alpha^{-1}(\varrho_1 \beta(\sigma))) + C_1 = C_2 \alpha^{-1}(\varrho_1 \beta(\sigma)) + C_1.
\]

If \(\alpha \in L_{si}\) hence we obtain

\[
q_{\alpha,\beta}[F] = \lim_{\sigma \to +\infty} \frac{\alpha(\ln \mu(m\sigma, F))}{\beta(\sigma)} \leq \lim_{\sigma \to +\infty} \frac{\alpha(C_2 \alpha^{-1}(\varrho_1 \beta(\sigma)) + C_1)}{\beta(\sigma)} = \varrho_1.
\]

Thus, in view of the arbitrariness of \(\varrho_1\) the following statement is true.

**Proposition 1.** Let \(\alpha \in L_{si}\) and either \(\ln n = O(\lambda_n)\) as \(n \to \infty\) and \(\beta(\ln x) \in L_{si}\) or \(\ln n = o(\lambda_n)\) as \(n \to \infty\) and \(\beta(\ln x) \in L^0\). Suppose that all functions \(F_j\) have the same generalized order \(q_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)\) and the function \(F\) is Hadamard composition of the genus \(m \geq 1\) of the functions \(F_j\). Then \(q_{\alpha,\beta}[F] \leq \varrho\).
Suppose that the coefficient $|c_{m0...0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$. Put

$$\Sigma'_n = \sum_{k_1+\ldots+k_p = m, k_1 \neq m} c_{k_1\ldots k_p}(a_{n,1})^{k_1} \cdot \ldots \cdot (a_{n,p})^{k_p} = \sum_{k_1+\ldots+k_p = m} c_{k_1\ldots k_p}(a_{n,1})^{k_1} \cdot \ldots \cdot (a_{n,p})^{k_p} - c_{m0...0}(a_{n,1})^m.$$

Since for each monomial of the polynomial $\Sigma'_n$, the sum of the exponents is equal to $m$, we have

$$\frac{|a_{n,1}|^{k_1} \cdot \ldots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2} \cdot \ldots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \to 0, \quad n \to \infty$$

and, thus, $\Sigma'_n = o(|a_{n,1}|^m)$ as $n \to \infty$. Therefore,

$$|a_n| \geq |c|a_{n,1}|^m - |\Sigma'_n| = |c|a_{n,1}|^m - o(|a_{n,1}|^m) \geq c|a_{n,1}|^m/2, \quad n \geq n_0^*,$$

and, thus, $\ln |a_n| + m\lambda_n \sigma \geq m \ln |a_{n,1}| + m\lambda_n \sigma + \ln (c/2)$ for $n \geq n_0^*$. Since $|a_{n,j}| \leq |a_{n,1}|$ for all $n \geq n_0^*$ and $2 \leq j \leq p$, hence it follows that

$$\ln \mu(\sigma, F_j) \leq \ln \mu(\sigma, F_1) \leq \frac{1}{m} \ln \mu(m \sigma, F) + K \leq \ln \mu(\sigma, F) + K, \quad K = \text{const.} \quad (7)$$

Therefore, if $\alpha(\ln x) \in L_{si}$ and $\beta \in L_{si}$ then $\varrho_{\alpha,\beta}[F_j] \leq \varrho_{\alpha,\beta}[F]$ for all $1 \leq j \leq p$.

Thus, the following statement is true.

**Proposition 2.** Let $\alpha(\ln x) \in L_{si}$ and $\beta \in L_{si}$. If the function $F$ is the Hadamard composition of the genus $m \geq 1$ of the functions $F_j$, $|c_{m0...0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$ then $\varrho_{\alpha,\beta}[F_j] \leq \varrho_{\alpha,\beta}[F]$ for all $1 \leq j \leq p$.

Using Propositions 1 and 2 now prove the following theorem.

**Theorem 1.** Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and either $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L_{si}$ or $\ln n = o(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L^0$ for each $c \in (0, +\infty)$. Suppose that the functions $F_j \in S(\Lambda, +\infty)$ have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T_{\alpha,\beta}[F_j] = T_j \in [0, +\infty), |c_{m0...0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$. If the function $F$ is the Hadamard composition of the genus $m \geq 1$ of the functions $F_j$ then $\varrho_{\alpha,\beta}[F] = \varrho$ and $T_{\alpha,\beta}[F] \leq \sum_{k_1+\ldots+k_p = m} (k_1 T_1 + \ldots + k_p T_p)$.

**Proof.** Since the functions $F_j \in S(\Lambda, +\infty)$ have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho$, by Propositions 1 and 2, $\varrho_{\alpha,\beta}[F] = \varrho$. If $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L_{si}$ for each $c \in (0, +\infty)$ then by Lemma 1

$$\lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma, F)}{\alpha^{-1}(c\beta(\sigma))} \leq T_{\alpha,\beta}[F] \leq \lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma + O(1), F)}{\alpha^{-1}(c\beta(\sigma))} \leq \lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma, F)}{\alpha^{-1}(c\beta(\sigma))} = \lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma, F)}{\alpha^{-1}(c\beta(\sigma))},$$

i.e.

$$\lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma, F)}{\alpha^{-1}(c\beta(\sigma))} = T_{\alpha,\beta}[F].$$

According to Lemma 1, this equality is also valid if $\ln n = o(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L^0$ for each $c \in (0, +\infty)$.

Therefore, $\lim_{\sigma \to +\infty} \frac{\ln \mu(\sigma, F_j)}{\alpha^{-1}(c\beta(\sigma))} = T_j$ and $\ln \mu(\sigma, F_j) \leq (T_j + \varepsilon)\alpha^{-1}(c\beta(\sigma))$ for every $\varepsilon > 0$ and all $\sigma \geq \sigma_0(\varepsilon)$. Hence and from (6) we obtain
\[ \ln \mu(m\sigma, F) \leq \sum_{k_1 + \ldots + k_p = m} \{k_1(T_1 + \varepsilon) + \ldots + k_p(T_p + \varepsilon)\} \alpha^{-1}(\varrho\beta(\sigma)) + \text{const} \]

and, thus,
\[
T_{\alpha,\beta}[F] = \lim_{\sigma \to +\infty} \frac{\ln \mu(m\sigma, F)}{\sigma} \leq \lim_{\sigma \to +\infty} \frac{\ln \mu(m\sigma, F)}{\sigma} \leq \sum_{k_1 + \ldots + k_p = m} \{k_1(T_1 + \varepsilon) + \ldots + k_p(T_p + \varepsilon)\}.
\]

In view of the arbitrariness of \(\varepsilon\) Theorem 1 is proved. \(\square\)

If we choose \(\alpha(x) = \ln^+ x\) and \(\beta(x) = x^+\) then we obtain the definition of (the most commonly used characteristics of the growth of entire Dirichlet series) the \(R\)-order [10] \(\varrho_R[F] = \lim_{\sigma \to +\infty} \frac{\ln \ln M(\sigma, F)}{\sigma}\) and the \(R\)-type [11] \(T_R[F] = \lim_{\sigma \to +\infty} e^{-\varrho_R[F]} \ln M(\sigma, F)\). The functions \(\alpha(x) = \ln^+ x\) and \(\beta(x) = x^+\) satisfy the condition \(\alpha^{-1}(c\beta(\ln x)) \in L^0\) for each \(c \in (0, +\infty)\), but \(\beta \not\in L_{si}\). The condition \(\beta \in L_{si}\) is used in the proof of Proposition 2 to obtain from (7) the inequality \(\varrho_{\alpha,\beta}[F] \leq \varrho_{\alpha,\beta}[F]\). Clearly, this condition is not needed if \(m = 1\), that is \(a_n = c_1a_{n,1} + \cdots + c_pa_{n,p}\). Thus, the following statement is true.

**Proposition 4.** Let \(\ln n = o(\lambda_n)\) as \(n \to \infty\), the functions \(F_j \in S(\Lambda, +\infty)\) have the same \(R\)-order \(\varrho_R[F] = \varrho \in (0, +\infty)\) and the \(R\)-types \(T_R[F_j] = T_j \in [0, +\infty), |c_1| > 0\) and \(|a_{n,j}| = o(|a_{n,1}|)\) as \(n \to \infty\) for \(2 \leq j \leq p\). If the function \(F\) is Hadamard composition of the genus \(m = 1\) of the functions \(F_j\) then \(\varrho_R[F] = \varrho\) and \(T_R[F] \leq T_1 + \ldots + T_p\).

**4. Convergence classes of entire Hadamard compositions of the genus \(m\).** Let \(F \in S(\Lambda, +\infty)\), \(\ln n = \mathcal{O}(\lambda_n)\) as \(n \to \infty\) and \(\alpha^{-1}(c\beta(x)) \in L^0\) for each \(c \in (0, +\infty)\). In [12] it is proved that if \(h \in L^0\) then \(\lim_{x \to +\infty} h(Kx)/h(x) = B(K) < +\infty\) for \(K = \text{const} > 0\). Therefore, in view of Lemma 1 we have
\[
\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma} d\sigma \leq \ln M(\sigma, F) \leq \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma + C, F)}{\sigma} d\sigma = \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma + C, F)}{(\sigma + C)\alpha^{-1}(\varrho\beta(\sigma + C))} \frac{(\sigma + C)\alpha^{-1}(\varrho\beta(\sigma + C))}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \leq B \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma,
\]
where \(B = \text{const} > 0\), i.e. \(F\) belongs to the generalized convergence class if and only if
\[
\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma < +\infty.
\]

Therefore, if the function \(F\) is the Hadamard composition of the genus \(m \geq 1\) of the functions \(F_j\) and all functions \(F_j\) belong to the generalized convergence class then in view of (6)
\[
\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \leq \int_{\sigma_0}^{\infty} \frac{\ln \mu(m\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \leq \int_{\sigma_0}^{\infty} \frac{\ln \mu(m\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \leq \sum_{k_1 + \ldots + k_p = m} \left( k_1 \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma + \ldots + k_p \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \right) + \text{const} < +\infty,
\]
i.e. \(F\) belongs to the same convergence class.
On the other hand, if \( c_{m_0...0} = c \neq 0, |a_{n,1}| > 0 \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) then, as above, \( |a_n| \geq |c||a_{n,1}|^{m/2} \) for \( n \geq n_0 \), i.e. \( \ln |a_n| + m\sigma \lambda_n \geq m(\ln |a_{n,1}| + \sigma \lambda_n) + \ln (c/2) \) for \( n \geq n_0 \). Hence it follows that \( \ln \mu(\sigma, F_1) \leq \ln \mu(m\sigma, F)/m + \text{const} \) for \( \sigma \geq \sigma_0 \). Therefore, if \( \alpha^{-1}(c\beta(x)) \in L^0 \) for each \( c \in (0, +\infty) \) then

\[
\int_\sigma^\infty \frac{\ln \mu(\sigma, F_j)}{\sigma^{\alpha^{-1}(\beta(\sigma))}} d\sigma \leq \int_\sigma^\infty \frac{\ln \mu(\sigma, F_1)}{\sigma^{\alpha^{-1}(\beta(\sigma))}} d\sigma \leq \leq \frac{1}{m} \int_\sigma^\infty \frac{\ln \mu(m\sigma, F)}{m\sigma^{\alpha^{-1}(\beta(m\sigma))}} d\sigma + B_1 \leq B_2 \int_\sigma^\infty \frac{\ln \mu(\sigma, F)}{\sigma^{\alpha^{-1}(\beta(\sigma))}} d\sigma + B_1,
\]

where \( B_j = \text{const} \). Therefore, if \( F \) belongs to the generalized convergence class then all \( F_j \) belong to the same convergence class and, thus, the following theorem is true.

**Theorem 2.** Let \( \alpha \in L, \beta \in L \) and \( \alpha^{-1}(c\beta(x)) \in L^0 \) for each \( c \in (0, +\infty) \). Suppose that \( \ln n = O(\lambda_n) \) as \( n \to \infty \) and the function \( F \) is the Hadamard composition of genus \( m \geq 1 \) of the functions \( F_j \in S(\Lambda, +\infty) \). If all functions \( F_j \) belong to the generalized convergence class then \( F \) belongs to the same convergence class. If, in addition, \( c_{m_0...0} = c \neq 0, |a_{n,1}| > 0 \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) then the belonging of \( F \) to the generalized convergence class implies the belonging of all \( F_j \) to the same convergence class.

As in [13], let \( \Omega \) be a class of positive unbounded functions \( \Phi \) on \((-\infty, +\infty)\) such that the derivative \( \Phi' \) is positive continuously differentiable and increasing to \(+\infty\) on \((-\infty, +\infty)\). For \( \Phi \in \Omega \) let \( \varphi \) be the inverse function to \( \Phi' \) and \( \Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)} \) be the function associated with \( \Phi \) in the sense of Newton. Then [13] the function \( \Psi \) is continuously differentiable and increasing to \(+\infty\) on \((-\infty, +\infty)\) and the function \( \varphi \) is continuously differentiable and increasing to \(+\infty\) on \((x_0, +\infty)\). For entire Dirichlet series the convergence \( \Phi \)-class is defined in [14, p. 49] by the condition

\[
\int_\sigma^\infty \frac{\Phi'(\sigma) \ln M(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty.
\]

It is known [14, p. 57] that if \( \Phi \in \Omega \), the function \( \Phi'(\sigma)/\Phi(\sigma) \) is non-decreasing on \([\sigma_0, +\infty)\), \( \Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma)^2 \leq H < +\infty \) and

\[
\int_0^\infty \frac{\ln n(t)}{t\Phi'(\Psi(\varphi(t)))} dt < +\infty, \quad n(t) = \sum_{\lambda_n \leq t} 1 \tag{8}
\]

then \( F \) belongs to the convergence \( \Phi \)-class if and only if

\[
\int_\sigma^\infty \frac{\Phi'(\sigma) \ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty.
\]

Therefore, if the function \( F \) is the Hadamard composition of genus \( m \geq 1 \) of the functions \( F_j \) and all functions \( F_j \) belong to the convergence \( \Phi \)-class then in view of (6)

\[
\int_\sigma^\infty \frac{\Phi'(\sigma) \ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma \leq \int_\sigma^\infty \frac{\Phi'(\sigma) \ln \mu(m\sigma, F)}{\Phi^2(\sigma)} d\sigma \leq
\]
\[ \leq \sum_{k_1 + \cdots + k_p = m} \left( k_1 \int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, F_1)}{\Phi^2(\sigma)} d\sigma + \cdots + k_p \int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, F_p)}{\Phi^2(\sigma)} d\sigma \right) + \text{const} < +\infty, \]

i.e. \( F \) belongs to the same convergence class.

On the other hand, if \( c_{m0\ldots0} = c \neq 0, |a_{n,1}| > 0 \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) then, as above, we have \( \ln \mu(\sigma, F_1) \leq \ln \mu(\sigma F)/m + \text{const} \) for \( \sigma \geq \sigma_0 \). Therefore, assuming \( m = 1 \), we obtain

\[ \int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, F_1)}{\Phi^2(\sigma)} d\sigma \leq \int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma \leq \int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma) \ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma + \text{const}, \]

and the following theorem is true.

**Theorem 3.** Let \( \Phi \in \Omega \), the function \( \Phi'(\sigma)/\Phi(\sigma) \) be non-decreasing on \( [\sigma_0, +\infty) \), \( \Phi(\sigma) \Phi''(\sigma)/\Phi'(\sigma)^2 \leq H < +\infty \) and (8) holds. Suppose that the function \( F \) is the Hadamard composition of the genus \( m \geq 1 \) of the functions \( F_j \in S(\Lambda, +\infty) \). If all functions \( F_j \) belong to the convergence \( \Phi \)-class then \( F \) belongs to the same convergence class. If, in addition, \( m = 1 \), \( c_{m0\ldots0} = c \neq 0, |a_{n,1}| > 0 \) and \( |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) then the belonging of \( F \) to the convergence \( \Phi \)-class implies the belonging of all \( F_j \) to the same convergence class.

Studying the properties of entire functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) of the order \( \varrho \in (0, +\infty) \) G. Valiron [15, p 18] introduced the convergence class as

\[ \int_1^{ \infty } \ln \frac{M_f(r)}{r^{\varrho + 1}} \, dr < +\infty, \]

where \( M_f(r) = \max\{|f(z)|: |z| = r\} \). In the papers [16, 17] Valiron’s result is generalized to the case of entire Dirichlet series of \( R \)-order \( \varrho_R \in (0, +\infty) \) by introducing the convergence class as \( \int_{\sigma_0}^{\infty} \frac{\ln M(\sigma, F)}{\exp(\varrho_R \sigma)} d\sigma < +\infty \). From Theorem 3 we get the following statement.

**Corollary 1.** Let the function \( F \) be the Hadamard composition of the genus \( m = 1 \) of the functions \( F_j \in S(\Lambda, +\infty) \), \( c_{m0\ldots0} = c \neq 0, |a_{n,1}| > 0, |a_{n,j}| = o(|a_{n,1}|) \) as \( n \to \infty \) for \( 2 \leq j \leq p \) and \( \int_{t_0}^{\infty} \frac{\ln n(t)}{t^2} dt < +\infty \). In order that \( F \) belongs to the convergence class it is necessary and sufficient that all \( F_j \) belong to the convergence class.

Indeed, we choose \( \Phi(\sigma) = e^{\varrho_R \sigma} \). Then \( \Phi \) satisfies the assumptions of Theorem 3,

\[ \Phi'(\sigma) = \varrho_R e^{\varrho_R \sigma}, \quad \Psi(\sigma) = \sigma - \frac{1}{\varrho_R}, \quad \varphi(t) = \frac{1}{\varrho_R} \ln t, \quad t \Phi(\Psi(\varphi(t))) = \frac{t^2}{e^{\varrho_R}} \]

and, thus, conditions (8) and \( \int_{t_0}^{\infty} \frac{\ln n(t)}{t^2} dt < +\infty \) are equivalent. Therefore, Theorem 3 implies Corollary 1.

The logarithmic order of a series of Dirichlet is defined as the quantity

\[ \varrho_l[F] = \lim_{\sigma \to +\infty} \frac{\ln \ln M(\sigma, F)}{\ln \sigma}. \]

It is clear that \( \varrho_l[F] \geq 1 \). If \( \varrho_l \in (1, +\infty) \) then we say, as in [14, p. 20], that \( F \) belongs to the logarithmic convergence class if \( \int_{\sigma_0}^{\infty} \frac{\ln M(\sigma, F)}{\sigma^{\varrho_l + 1}} d\sigma < +\infty \). The function \( \Phi(\sigma) = \sigma^{\varrho_l} \) for \( \sigma \geq \sigma_0 \) does not satisfy the hypotheses of Theorem 3. But [14, p. 20-21], if \( \ln n = O(\lambda_n^{\varrho_l}/(\varrho_l - 1)) \) as \( n \to \infty \) then \( F \) belongs to the logarithmic convergence class if and only if \( \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma^{\varrho_l + 1}} d\sigma < +\infty \). Therefore, repeating the proof of Theorem 3, we arrive at the following assertion.
Proposition 5. Let the function $F$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, +\infty)$, $c_{m0\ldots0} = c \neq 0$, $|a_{n,1}| > 0$, $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$ and $\ln n = O(\lambda_n^{\rho}/(\rho - 1))$ as $n \to \infty$. In order that $F$ belongs to the logarithmic convergence class it is necessary and sufficient that all $F_j$ belong to the logarithmic convergence class.

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