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O. M. Mulyava, M. M. Sheremeta

ON ENTIRE DIRICHLET SERIES SIMILAR TO HADAMARD COMPOSITIONS

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A function $F(s)=\sum_{n=1}^{\infty}a_n\exp\{s\lambda_n\}$ with $0\le\lambda_n\uparrow+\infty$ is called the Hadamard composition of the genus $m\ge1$ of functions $F_j(s)=\sum_{n=1}^{\infty}a_{n,j}\exp\{s\lambda_n\}$ if $a_n=P(a_{n,1},...,a_{n,p})$, where $P(x_1,...,x_p)=\sum\limits_{k_1+\dots+k_p=m}c_{k_1...k_p}x_1^{k_1}\cdot...\cdot x_p^{k_p}$ is a homogeneous polynomial of degree $m\ge1$. Let $M(\sigma,F)=\sup\{|F(\sigma+it)|:\,t\in\mathbb{R}\}$ and functions α, β be positive continuous and increasing to $+\infty$ on $[x_0,+\infty)$. To characterize the growth of the function $M(\sigma,F)$, we use generalized order $\varrho_{\alpha,\beta}[F]=\varlimsup\limits_{\sigma\to+\infty}\dfrac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)}$, generalized type $T_{\alpha,\beta}[F]=\varlimsup\limits_{\sigma\to+\infty}\dfrac{\ln M(\sigma,F)}{\alpha^{-1}(\varrho_{\alpha,\beta}[F]\beta(\sigma))}$ and membership in the convergence class defined by the condition
$$\int_{\sigma_0}^{\infty}\frac{\ln M(\sigma,F)}{\sigma\alpha^{-1}(\varrho_{\alpha,\beta}[F]\beta(\sigma))}d\sigma<+\infty.$$

Assuming the functions α,β and $\alpha^{-1}(c\beta(\ln x))$ are slowly increasing for each $c\in(0,+\infty)$ and $\ln n=O(\lambda_n)$ as $n\to\infty$, it is proved, for example, that if the functions F_j have the same generalized order $\varrho_{\alpha,\beta}[F_j]=\varrho\in(0,+\infty)$ and the types $T_{\alpha,\beta}[F_j]=T_j\in[0,+\infty)$, $c_{m0...0}=c\ne0$, $|a_{n,1}|>0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\le j\le p$, and F is the Hadamard composition of genus $m\ge1$ of the functions F_j then $\varrho_{\alpha,\beta}[F]=\varrho$ and
$$T_{\alpha,\beta}[F]\le\sum_{k_1+\dots+k_p=m}(k_1T_1+...+k_pT_p).$$
It is proved also that F belongs to the generalized convergence class if and only if all functions F_j belong to the same convergence class.

1. Introduction. Let
$$f_j(z)=\sum_{n=0}^{\infty}a_{n,j}z^n,\ 1\le j\le p,$$
be entire transcedental functions. As in [1], we say that the function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is similar to the Hadamard composition of the functions f_j if $a_n=w(a_{n,1},...,a_{n,p})$ for all n, where $w\colon\mathbb{C}^p\to\mathbb{C}$ is some function. Clearly, if $p=2$ and $w(a_{n,1},\,a_{n,2})=a_{n,1}a_{n,2}$ then $f=(f_1*f_2)$ is [2] the Hadamard composition (product) of the functions f_1 and f_2. Properties of this composition obtained by J. Hadamard find the applications [3, 4] in the theory of the analytic continuation of the functions represented by power series.

E. G. Calys [5] investigated the functions similar to Hadamard compositions of with $|w(x,y)|=\sqrt{|xy|}$ and proved in particular the following theorem.

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Theorem A ([5]). *Let entire functions f_j, $j = 1, 2$, have the same order $\varrho[f_j] = = \varrho \in (0, +\infty)$ and types $\sigma[f_j] = \sigma_j$. Suppose that $a_{n,1} \neq 0$ and $|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \geq n_0$, where l is slowly varying function. If $|a_n| = (1 + o(1))\sqrt{|a_{n,1}||a_{n,2}|}$ as $n \to \infty$, then the function f has order $\varrho[f] = \varrho$ and type $\sigma[f] \leq \sqrt{\sigma_1\sigma_2}$.*

In the paper [6] the results of E. G. Calys are generalized on the case of entire Dirichlet series of finite generalized orders, moreover instead of two entire functions $m \geq 2$ entire Dirichlet series were considered.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative number, $S(\Lambda, A)$ be a class of Diriclet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with a given sequence (λ_n) of exponents and an abscissa of absolutely convergence $\sigma_a[F] = A \in (-\infty, +\infty)$, and let $M(\sigma, F) = \sup\{|F(\sigma + it)|\colon\ t \in \mathbb{R}\}$ for $\sigma \in (-\infty, A)$.

As in [7], by L we denote the class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, +\infty)$, that is series (1) is entire, then the value

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}$$

is called the generalized order of F. If $\varrho_{\alpha,\beta}[F] \in (0, +\infty)$ the generalized type is defined as

$$T_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\ln M(\sigma, F)}{\alpha^{-1}(\varrho_{\alpha,\beta}[F]\beta(\sigma))}.$$

The following theorem is true.

Theorem B ([6]). *Let the functions $\alpha \in L_{si}$ and $\beta \in L^0$ be continuously differentiable, $\frac{d\ln\alpha^{-1}(\varrho\beta(x))}{d\ln x} \to \varrho$ and $\alpha(x) = (1+o(1))\ln x$ as $x \to +\infty$. Suppose that $\ln n = o(\lambda_n)$ $(n \to \infty)$ and Dirichlet series $F_j \in S(\Lambda, +\infty)$ of form*

$$F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}, \quad 1 \leq j \leq p, \tag{2}$$

have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and types $T_{\alpha,\beta}[F_j] \in (0, +\infty)$. If $a_{n,1} \neq 0$ for all $n \geq n_0$ and $\omega_j > 0$ with $\sum_{j=1}^{p} \omega_j = 1$,

$$\alpha^{-1}\left(\varrho\beta\left(\frac{1}{\varrho} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)\right) = (1 + o(1))\prod_{j=1}^{p}\alpha^{-1}\left(\varrho\beta\left(\frac{1}{\varrho} + \frac{1}{\lambda_n}\ln\frac{1}{|a_{n,j}|}\right)\right)^{\omega_j}, \quad n \to \infty,$$

and

$$\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_{n,j}|}\right) \leq (1 + o(1))\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_{n,1}|}\right), \quad n \to \infty,$$

for all $2 \leq j \leq p$, then Dirichlet series (1) has the generalized order $\varrho_{\alpha,\beta}[F] = \varrho$ and the type

$$T_{\alpha,\beta}[F] \leq \prod_{j=1}^{p} T_{\alpha,\beta}[F_j]^{\omega_j}.$$

If $T_{\alpha,\beta}[F]=0$ then for the characteristic of the growth of entire Dirichlet series (1) we define a generalized convergence class by the condition

$$\int\limits_{\sigma_0}^{\infty}\frac{\ln M(\sigma,F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}d\sigma<+\infty,\quad \varrho=\varrho_{\alpha,\beta}[F]. \tag{3}$$

Theorem C ([1]). *Let $\alpha\in L$ and $\beta\in L$ be positive continuously differentiable functions such that $\frac{d\ln\alpha^{-1}(\varrho\beta(\sigma))}{d\sigma}=O(1)$ as $\sigma\to\infty$ for each $\varrho\in(0,+\infty)$. Suppose that $\ln n=O(\lambda_n)$ and $|a_n|\asymp\prod_{j=1}^{p}|a_{n,j}|^{\omega_j}$ as $n\to\infty$ for some $\omega_j>0$ such that $\sum_{j=1}^{p}\omega_j=1$. If all functions (2) belong to the generalized convergence class then function (1) also belongs to this class. If, in addition, $|a_{n,1}|>0$ for all $n\ge 0$ and $|a_{n,j}|\asymp|a_{n,1}|$ as $n\to\infty$ for all $j=2,\dots,p$, then the belonging of function (1) to generalized convergence class implies the belonging of all functions (2) to this class.*

Here we consider the case when w is a homogeneous polynomial.

2. Definition and convergence of Hadamard composition of the genus m. Recall that a polynomial is called homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial $P(x_1,...,x_p)$ is homogeneous of degree m if and only if $P(tx_1,...,tx_p)=t^mP(x_1,...,x_p)$ for all t from the field above which a polynomial is defined. Dirichlet series (1) is called the Hadamard composition of genus m of Dirichlet series (2) if $a_n=P(a_{n,1},...,a_{n,p})$, where

$$P(x_1,...,x_p)=\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}x_1^{k_1}\cdot\ldots\cdot x_p^{k_p}.$$

is a homogeneous polynomial of degree $m\ge 1$. We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m=2$.

Therefore, if the function F is the Hadamard composition of genus $m\ge 1$ of the functions F_j then

$$|a_n|\le\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}||a_{n,1}|^{k_1}\cdot\ldots\cdot|a_{n,p}|^{k_p}. \tag{4}$$

Denote

$$\tau=\varliminf_{n\to\infty}\frac{\ln n}{\lambda_n},\ \ \alpha[F]=\varliminf_{n\to\infty}\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}.$$

Then [8, 9] $\sigma_a[F]\le\alpha[F]\le\sigma_a[F]+\tau$. Hence it follows that if $\tau<+\infty$ and either $\sigma_a[F]=+\infty$ or $\alpha[F]=+\infty$, then $\sigma_a[F]=\alpha[F]$.

Therefore, if $\tau<+\infty$ and all $F_j\in S(\Lambda,+\infty)$, i. e. $\alpha[F_j]=+\infty$, then for every $a>0$ we have $|a_{n,j}|\le\exp\{-a\lambda_n\}$ for every $a>0$ all j and all $n\ge n_0(a)$. Therefore, (4) implies

$$|a_n|\le C\exp\{-am\lambda_n\},\quad C=\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|,$$

whence

$$\alpha[F]=\varliminf_{n\to\infty}\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\ge am,$$

i.e. in view of the arbitrariness of a we get $\alpha[F]=+\infty$, that is $F\in S(\Lambda,+\infty)$.

3. Growth of entire Hadamard compositions of the genus m. Since the polynomial $P(x_1,...,x_p)$ is homogeneous of the degree $m\geq 1$, we have

$$a_n e^{ms\lambda_n}=\sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}(a_{n,1}e^{s\lambda_n})^{k_1}\cdot\ldots\cdot(a_{n,p}e^{s\lambda_n})^{k_p}. \tag{5}$$

Let $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}\colon\ n\geq 0\}$ be the maximal term of series (1). Since (5) implies

$$|a_n|e^{m\sigma\lambda_n}\leq\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|(|a_{n,1}|e^{\sigma\lambda_n})^{k_1}\cdot\ldots\cdot(|a_{n,p}|e^{\sigma\lambda_n})^{k_p},$$

we have

$$\mu(m\sigma,F)\leq\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|\mu(\sigma,F_1)^{k_1}\cdot\ldots\cdot\mu(\sigma,F_p)^{k_p},$$

whence for all σ large enough we get

$$\ln\,\mu(m\sigma,F)\leq\sum_{k_1+\dots+k_p=m}\ln\left(|c_{k_1\dots k_p}|\mu(\sigma,F_1)^{k_1}\cdot\ldots\cdot\mu(\sigma,F_p)^{k_p}\right)+\ln\left(m+1\right)=$$

$$=\sum_{k_1+\dots+k_p=m}(\ln\left(|c_{k_1\dots k_p}|+k_1\ln\,\mu(\sigma,F_1)+\ldots+k_p\ln\,\mu(\sigma,F_p)\right)+\ln\left(m+1\right)=$$

$$=\sum_{k_1+\dots+k_p=m}(k_1\ln\,\mu(\sigma,F_1)+\ldots+k_p\ln\,\mu(\sigma,F_p))+C_1, \tag{6}$$

where $C_1=\sum_{k_1+\dots+k_p=m}\ln^+|c_{k_1\dots k_p}|+\ln\left(m+1\right)$. In what follows, we will use the following lemma (see, for example, [8, p. 22] and [9, p. 184]).

Lemma 1. *If* $\ln\,n=O(\lambda_n)$ *as* $n\to\infty$ *then* $\mu(\sigma,F)\leq M(\sigma,F)\leq\mu(\sigma{+}O(1),F)$ *as* $\sigma\to+\infty$, *and if* $\ln\,n=o(\lambda_n)$ *as* $n\to\infty$ *then* $\mu(\sigma,F)\leq M(\sigma,F)\leq\mu(\sigma+o(1),F)$ *as* $\sigma\to+\infty$.

Hence it follows that if $\alpha\in L$ and either $\ln\,n=O(\lambda_n)$ as $n\to\infty$ and $\beta(\ln\,x)\in L_{si}$ or $\ln\,n=o(\lambda_n)$ as $n\to\infty$ and $\beta(\ln\,x)\in L^0$ then

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln\,\mu(\sigma,F))}{\beta(\sigma)}=\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln\,M(\sigma,F))}{\beta(\sigma)}.$$

Suppose that the functions F_j have the same generalized order $\varrho_{\alpha,\beta}[F_j]=\varrho\in(0,+\infty)$. Then for every $\varrho_1>\varrho$ and all $\sigma\geq\sigma_0$ we have $\ln\,\mu(\sigma,F_j)\leq\alpha^{-1}(\varrho_1\beta(\sigma))$ for $1\leq j\leq p$ and, thus, (6) implies

$$\ln\,\mu(m\sigma,F)\leq\sum_{k_1+\dots+k_p=m}((k_1+\ldots+k_p)\alpha^{-1}(\varrho_1\beta(\sigma))+C_1=C_2\alpha^{-1}(\varrho_1\beta(\sigma))+C_1.$$

If $\alpha\in L_{si}$ hence we obtain

$$\varrho_{\alpha\beta}[F]=\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln\,\mu(m\sigma,F))}{\beta(m\sigma)}\leq\varlimsup_{\sigma\to+\infty}\frac{\alpha(C_2\alpha^{-1}(\varrho_1\beta(\sigma))+C_1)}{\beta(\sigma)}=\varrho_1.$$

Thus, in view of the arbitrariness of ϱ_1 the following statement is true.

Proposition 1. *Let* $\alpha\in L_{si}$ *and either* $\ln\,n=O(\lambda_n)$ *as* $n\to\infty$ *and* $\beta(\ln\,x)\in L_{si}$ *or* $\ln\,n=o(\lambda_n)$ *as* $n\to\infty$ *and* $\beta(\ln\,x)\in L^0$. *Suppose that all functions* F_j *have the same generalized order* $\varrho_{\alpha,\beta}[F_j]=\varrho\in(0,+\infty)$ *and the function* F *is Hadamard composition of the genus* $m\geq 1$ *of the functions* F_j. *Then* $\varrho_{\alpha,\beta}[F]\leq\varrho$.

Suppose that the coefficient $|c_{m0\dots 0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$. Put

$$\Sigma_n' = \sum_{k_1+\dots+k_p=m,\, k_1\neq m} c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\ldots\cdot(a_{n,p})^{k_p} =$$
$$= \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\ldots\cdot(a_{n,p})^{k_p} - c_{m0\dots 0}(a_{n,1})^m.$$

Since for each monomial of the polynomial Σ_n' the sum of the exponents is equal to m, we have

$$\frac{|a_{n,1}|^{k_1}\cdot\ldots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2}\cdot\ldots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \to 0, \quad n\to\infty$$

and, thus, $\Sigma_n' = o(|a_{n,1}|^m)$ as $n \to \infty$. Therefore,

$$|a_n| \ge c|a_{n,1}|^m - |\Sigma_n'| = c|a_{n,1}|^m - o(|a_{n,1|}^m) \ge c|a_{n,1}|^m/2, \quad n \ge n_0^*,$$

and, thus, $\ln|a_n| + m\lambda_n\sigma \ge m\ln|a_{n,1}| + m\lambda_n\sigma + \ln(c/2)$ for $n \ge n_0^*$. Since $|a_{n,j}| \le |a_{n,1}|$ for all $n \ge n_0^{**}$ and $2 \le j \le p$, hence it follows that

$$\ln\mu(\sigma, F_j) \le \ln\mu(\sigma, F_1) \le \frac{1}{m}\ln\mu(m\sigma, F) + K \le \ln\mu(m\sigma, F) + K, \quad K = \text{const}. \tag{7}$$

Therefore, if $\alpha(\ln x) \in L_{si}$ and $\beta \in L_{si}$ then $\varrho_{\alpha,\beta}[F_j] \le \varrho_{\alpha,\beta}[F]$ for all $1 \le j \le p$.

Thus, the following statement is true.

Proposition 2. *Let $\alpha(\ln x) \in L_{si}$ and $\beta \in L_{si}$. If the function F is the Hadamard composition of the genus $m \ge 1$ of the functions F_j, $|c_{m0\dots0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$ then $\varrho_{\alpha,\beta}[F_j] \le \varrho_{\alpha,\beta}[F]$ for all $1 \le j \le p$.*

Using Propositions 1 and 2 now prove the following theorem.

Theorem 1. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and either $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L_{si}$ or $\ln n = o(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L^0$ for each $c \in (0, +\infty)$. Suppose that the functions $F_j \in S(\Lambda, +\infty)$ have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T_{\alpha,\beta}[F_j] = T_j \in [0, +\infty)$, $|c_{m0\dots0}| = c \neq 0$, $|a_{n,1}| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$. If the function F is the Hadamard composition of the genus $m \ge 1$ of the functions F_j then $\varrho_{\alpha,\beta}[F] = \varrho$ and $T_{\alpha,\beta}[F] \le \sum_{k_1+\dots+k_p=m}(k_1T_1 + \ldots + k_pT_p)$.*

Proof. Since the functions $F_j \in S(\Lambda, +\infty)$ have the same generalized order $\varrho_{\alpha,\beta}[F_j] = \varrho$, by Propositions 1 and 2, $\varrho_{\alpha,\beta}[F] = \varrho$. If $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L_{si}$ for each $c \in (0, +\infty)$ then by Lemma 1

$$\varlimsup_{\sigma\to+\infty}\frac{\ln\mu(\sigma,F)}{\alpha^{-1}(\varrho\beta(\sigma))} \le T_{\alpha,\beta}[F] \le \varlimsup_{\sigma\to+\infty}\frac{\ln\mu(\sigma+O(1),F)}{\alpha^{-1}(\varrho\beta(\sigma))} \le$$
$$\le \varlimsup_{\sigma\to+\infty}\frac{\ln\mu(\sigma,F)}{\alpha^{-1}(\varrho\beta(\sigma))}\varlimsup_{\sigma\to+\infty}\frac{\alpha^{-1}(\varrho\beta(\sigma+O(1)))}{\alpha^{-1}(\varrho\beta(\sigma))} = \varlimsup_{\sigma\to+\infty}\frac{\ln\mu(\sigma,F)}{\alpha^{-1}(\varrho\beta(\sigma))},$$

i.e. $\varlimsup\limits_{\sigma\to+\infty}\dfrac{\ln\mu(\sigma,F)}{\alpha^{-1}(\varrho\beta(\sigma))} = T_{\alpha,\beta}[F]$. According to Lemma 1, this equality is also valid if $\ln n = o(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(\ln x)) \in L^0$ for each $c \in (0, +\infty)$.

Therefore, $\varlimsup\limits_{\sigma\to+\infty}\frac{\ln\mu(\sigma,F_j)}{\alpha^{-1}(\varrho\beta(\sigma))} = T_j$ and $\ln\mu(\sigma, F_j) \le (T_j+\varepsilon)\alpha^{-1}(\varrho\beta(\sigma))$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma_0(\varepsilon)$. Hence and from (6) we obtain

$$\ln \mu(m\sigma, F) \le \sum_{k_1+\dots+k_p=m} \{k_1(T_1+\varepsilon)+...+k_p(T_p+\varepsilon)\}\alpha^{-1}(\varrho\beta(\sigma)) + \text{const}$$

and, thus,

$$T_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \frac{\ln \mu(m\sigma, F)}{\alpha^{-1}(\varrho\beta(m\sigma))} \le \varlimsup_{\sigma\to+\infty} \frac{\ln \mu(m\sigma, F)}{\alpha^{-1}(\varrho\beta(\sigma))} \le \sum_{k_1+\dots+k_p=m} \{k_1(T_1+\varepsilon)+...+k_p(T_p+\varepsilon)\}.$$

In view of the arbitrariness of ε Theorem 1 is proved. □

If we choose $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ then we obtain the definition of (the most commonly used characteristics of the growth of entire Dirichlet series) the R-order [10] $\varrho_R[F] = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln M(\sigma,F)}{\sigma}$ and the R-type [11] $T_R[F] = \varlimsup_{\sigma\to+\infty} e^{-\varrho_R[F]\sigma} \ln M(\sigma, F)$. The functions $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ satisfy the condition $\alpha^{-1}(c\beta(\ln x)) \in L^0$ for each $c \in (0, +\infty)$, but $\beta \notin L_{si}$. The condition $\beta \in L_{si}$ is used in the proof of Proposition 2 to obtain from (7) the inequality $\varrho_{\alpha,\beta}[F_j] \le \varrho_{\alpha,\beta}[F]$. Clearly, this condition is not needed if $m = 1$, that is $a_n = c_1 a_{n,1} + \dots + c_p a_{n,p}$. Thus, the following statement is true.

Proposition 4. *Let $\ln n = o(\lambda_n)$ as $n \to \infty$, the functions $F_j \in S(\Lambda, +\infty)$ have the same R-order $\varrho_R[F_j] = \varrho \in (0, +\infty)$ and the R-types $T_R[F_j] = T_j \in [0, +\infty)$, $|c_1| > 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$. If the function F is Hadamard composition of the genus $m = 1$ of the functions F_j then $\varrho_R[F] = \varrho$ and $T_R[F] \le T_1 + ... + T_p$.*

4. Convergence classes of entire Hadamard compositions of the genus m. Let $F \in S(\Lambda, +\infty)$, $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\alpha^{-1}(c\beta(x)) \in L^0$ for each $c \in (0, +\infty)$. In [12] it is proved that if $h \in L^0$ then $\varlimsup_{x\to+\infty} h(Kx)/h(x) = B(K) < +\infty$ for $K = \text{const} > 0$. Therefore, in view of Lemma 1 we have

$$\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \le \int_{\sigma_0}^{\infty} \frac{\ln M(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \le \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma+C, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma =$$

$$= \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma+C, F)}{(\sigma+C)\alpha^{-1}(\varrho\beta(\sigma+C))} \frac{(\sigma+C)\alpha^{-1}(\varrho\beta(\sigma+C))}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \le B \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma,$$

where $B = \text{const} > 0$, i.e. F belongs to the generalized convergence class if and only if

$$\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma < +\infty.$$

Therefore, if the function F is the Hadamard composition of the genus $m \ge 1$ of the functions F_j and all functions F_j belong to the generalized convergence class then in view of (6)

$$\int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \le \int_{\sigma_0}^{\infty} \frac{\ln \mu(m\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \le$$

$$\le \sum_{k_1+\dots+k_p=m} \left(k_1 \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma + ... + k_p \int_{\sigma_0}^{\infty} \frac{\ln \mu(\sigma, F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))} d\sigma \right) + \text{const} < +\infty,$$

i.e. F belongs to the same convergence class.

On the other hand, if $c_{m0\dots0}=c\neq0$, $|a_{n,1}|>0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\le j\le p$ then, as above, $|a_n|\ge|c||a_{n,1}|^m/2$ for $n\ge n_0$, i. e. $\ln|a_n|+m\sigma\lambda_n\ge m(\ln|a_{n,1}|+\sigma\lambda_n)+\ln(c/2)$ for $n\ge n_0$. Hence it follows that $\ln\mu(\sigma,F_1)\le\ln\mu(m\sigma,F)/m+\text{const}$ for $\sigma\ge\sigma_0$. Therefore, if $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$ then

$$\int\limits_{\sigma_0}^{\infty}\frac{\ln\mu(\sigma,F_j)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}d\sigma\le\int\limits_{\sigma_0}^{\infty}\frac{\ln\mu(\sigma,F_1)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}d\sigma\le$$

$$\le\frac{1}{m}\int\limits_{\sigma_0}^{\infty}\frac{\ln\mu(m\sigma,F)}{m\sigma\alpha^{-1}(\varrho\beta(m\sigma))}\frac{\sigma\alpha^{-1}(\varrho\beta(m\sigma))}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}dm\sigma+B_1\le B_2\int\limits_{\sigma_0}^{\infty}\frac{\ln\mu(\sigma,F)}{\sigma\alpha^{-1}(\varrho\beta(\sigma))}d\sigma+B_1,$$

where $B_j=\text{const}$. Therefore, if F belongs to the generalized convergence class then all F_j belong to the same convergence class and, thus, the following theorem is true.

Theorem 2. *Let $\alpha\in L$, $\beta\in L$ and $\alpha^{-1}(c\beta(x))\in L^0$ for each $c\in(0,+\infty)$. Suppose that $\ln n=O(\lambda_n)$ as $n\to\infty$ and the function F is the Hadamard composition of genus $m\ge1$ of the functions $F_j\in S(\Lambda,+\infty)$. If all functions F_j belong to the generalized convergence class then F belongs to the same convergence class. If, in addition, $c_{m0\dots0}=c\neq0$, $|a_{n,1}|>0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\le j\le p$ then the belonging of F to the generalized convergence class implies the belonging of all F_j to the same convergence class.*

As in [13], let Ω be a class of positive unbounded functions Φ on $(-\infty,+\infty)$ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$. For $\Phi\in\Omega$ let φ be the inverse function to Φ' and $\Psi(\sigma)=\sigma-\frac{\Phi(\sigma)}{\Phi'(\sigma)}$ be the function associated with Φ in the sense of Newton. Then [13] the function Ψ is continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$ and the function φ is continuously differentiable and increasing to $+\infty$ on $(x_0,+\infty)$. For entire Dirichlet series the convergence Φ-class is defined in [14, p. 49] by the condition

$$\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln M(\sigma,F)}{\Phi^2(\sigma)}d\sigma<+\infty.$$

It is known [14, p. 57] that if $\Phi\in\Omega$, the function $\Phi'(\sigma)/\Phi(\sigma)$ is non-decreasing on $[\sigma_0,+\infty)$, $\Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma)^2\le H<+\infty$ and

$$\int\limits_{t_0}^{\infty}\frac{\ln n(t)}{t\Phi(\Psi(\varphi(t)))}dt<+\infty,\quad n(t)=\sum_{\lambda_n\le t}1\tag{8}$$

then F belongs to the convergence Φ-class if and only if

$$\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\mu(\sigma,F)}{\Phi^2(\sigma)}d\sigma<+\infty.$$

Therefore, if the function F is the Hadamard composition of genus $m\ge1$ of the functions F_j and all functions F_j belong to the convergence Φ-class then in view of (6)

$$\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\mu(\sigma,F)}{\Phi^2(\sigma)}d\sigma\le\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\mu(m\sigma,F)}{\Phi^2(\sigma)}d\sigma\le$$

$$\leq \sum_{k_1+\dots+k_p=m}\left(k_1\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\,\mu(\sigma,F_1)}{\Phi^2(\sigma)}d\sigma+\ldots+k_p\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\,\mu(\sigma,F_p)}{\Phi^2(\sigma)}d\sigma\right)+\text{const}<+\infty,$$

i.e. F belongs to the same convergence class.

On the other hand, if $c_{m0\dots0}=c\neq 0$, $|a_{n,1}|>0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\leq j\leq p$ then, as above, we have $\ln\,\mu(\sigma,F_1)\leq\ln\,\mu(m\sigma,F)/m+\text{const}$ for $\sigma\geq\sigma_0$. Therefore, assuming $m=1$, we obtain

$$\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\,\mu(\sigma,F_1)}{\Phi^2(\sigma)}d\sigma\leq\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\,\mu(\sigma,F_1)}{\Phi^2(\sigma)}d\sigma\leq\int\limits_{\sigma_0}^{\infty}\frac{\Phi'(\sigma)\ln\,\mu(\sigma,F)}{\Phi^2(\sigma)}d\sigma+\text{const},$$

and the following theorem is true.

Theorem 3. *Let $\Phi\in\Omega$, the function $\Phi'(\sigma)/\Phi(\sigma)$ be non-decreasing on $[\sigma_0,+\infty)$, $\Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma)^2\leq H<+\infty$ and (8) holds. Suppose that the function F is the Hadamard composition of the genus $m\geq 1$ of the functions $F_j\in S(\Lambda,+\infty)$. If all functions F_j belong to the convergence Φ-class then F belongs to the same convergence class. If, in addition, $m=1$, $c_{m0\dots0}=c\neq 0$, $|a_{n,1}|>0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\leq j\leq p$ then the belonging of F to the convergence Φ-class implies the belonging of all F_j to the same convergence class.*

Studying the properties of entire functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ of the order $\varrho\in(0,+\infty)$ G. Valiron [15, p 18] introduced the convergence class as

$$\int\limits_{1}^{\infty}\frac{\ln\,M_f(r)}{r^{\varrho+1}}dr<+\infty,$$

where $M_f(r)=\max\{|f(z)|\colon|z|=r\}$. In the papers [16, 17] Valiron's result is generalized to the case of entire Dirichlet series of R-order $\varrho_R\in(0,+\infty)$ by introducing the convergence class as $\int_{\sigma_0}^{\infty}\frac{\ln\,M(\sigma,F)}{\exp\{\varrho_R\sigma\}}d\sigma<+\infty$. From Theorem 3 we get the following statement.

Corollary 1. *Let the function F be the Hadamard composition of the genus $m=1$ of the functions $F_j\in S(\Lambda,+\infty)$, $c_{m0\dots0}=c\neq 0$, $|a_{n,1}|>0$, $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\leq j\leq p$ and $\int_{t_0}^{\infty}\frac{\ln\,n(t)}{t^2}dt<+\infty$. In order that F belongs to the convergence class it is necessary and sufficient that all F_j belong to the convergence class.*

Indeed, we choose $\Phi(\sigma)=e^{\varrho_R\sigma}$. Then Φ satisfies the assumptions of Theorem 3,

$$\Phi'(\sigma)=\varrho_Re^{\varrho_R\sigma},\ \Psi(\sigma)=\sigma-\frac{1}{\varrho_R},\ \varphi(t)=\frac{1}{\varrho_R}\ln\,\frac{t}{\varrho_R},\ t\Phi(\Psi(\varphi(t)))=\frac{t^2}{e\varrho_R}$$

and, thus, conditions (8) and $\int_{t_0}^{\infty}\frac{\ln\,n(t)}{t^2}dt<+\infty$ are equivalent. Therefore, Theorem 3 implies Corollary 1.

The logarithmic order of a series of Dirichlet is defined as the quantity

$$\varrho_l[F]=\varlimsup_{\sigma\to+\infty}\frac{\ln\,\ln\,M(\sigma,F)}{\ln\,\sigma}.$$

It is clear that $\varrho_l[F]\geq 1$. If $\varrho_l\in(1,+\infty)$ then we say, as in [14, p. 20], that F belongs to the logarithmic convergence class if $\int_{\sigma_0}^{\infty}\frac{\ln\,M(\sigma,F)}{\sigma^{\varrho_l+1}}d\sigma<+\infty$. The function $\Phi(\sigma)=\sigma^{\varrho_l}$ for $\sigma\geq\sigma_0$ does not satisfy the hipotheses of Theorem 3. But [14, p. 20-21], if $\ln\,n=O(\lambda_n^{\varrho_l/(\varrho_l-1)})$ as $n\to\infty$ then F belongs to the logarithmic convergence class if and only if $\int_{\sigma_0}^{\infty}\frac{\ln\,\mu(\sigma,F)}{\sigma^{\varrho_l+1}}d\sigma<+\infty$. Therefore, repeating the proof of Theorem 3, we arrive at the following assertion.

Proposition 5. *Let the function F be the Hadamard composition of genus $m\ge 1$ of the functions $F_j\in S(\Lambda,+\infty)$, $c_{m0\dots 0}=c\not=0$, $|a_{n,1}|>0$, $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\le j\le p$ and $\ln\, n=O(\lambda_n^{\varrho_l/(\varrho_l-1)})$ as $n\to\infty$. In order that F belongs to the logarithmic convergence class it is necessary and sufficient that all F_j belong to the logarithmic convergence class.*

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Kyiv National University of Food Technologies
Kyiv, Ukraine
oksana.m@bigmir.net

Ivan Franko National University of Lviv
Lviv, Ukraine
m.m.sheremeta@gmail.com

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