УДК 517.54

I. V. Denega, Ya. V. Zabolotnyi

APPLICATION OF UPPER ESTIMATES FOR PRODUCTS OF INNER RADII TO DISTORTION THEOREMS FOR UNIVALENT FUNCTIONS

I. V. Denega, Ya. V. Zabolotnyi. *Application of upper estimates for products of inner radii to distortion theorems for univalent functions*, Mat. Stud. **60** (2023), 138–144.

In 1934 Lavrentiev solved the problem of maximum of product of conformal radii of two non-overlapping simply connected domains. In the case of three or more points, many authors considered estimates of a more general Mobius invariant of the form

$$T_n := \prod_{k=1}^{n} r(B_k, a_k)\left(\prod_{1\leqslant k<p\leqslant n} |a_k - a_p|\right)^{-\frac{2}{n-1}},$$

where $r(B,a)$ denotes the inner radius of the domain B with respect to the point a (for an infinitely distant point under the corresponding factor we understand the unit). In 1951 Goluzin for $n=3$ obtained an accurate evaluation for T_3. In 1980 Kuzmina showed that the problem of the evaluation of T_4 is reduced to the smallest capacity problem in the certain continuum family and obtained the exact inequality for T_4. No other ultimate results in this problem for $n \geqslant 5$ are known at present. In 2021 [4,15] effective upper estimates are obtained for T_n, $n \geqslant 2$. Among the possible applications of the obtained results in other tasks of the function theory are the so-called distortion theorems. In the paper we consider an application of upper estimates for products of inner radii to distortion theorems for univalent functions in disk U, which map it onto a star-shaped domains relative to the origin.

1. Preliminaries. An important place in geometric function theory of complex variable is occupied by the direction associated with the study of extremal decomposition of the complex plane. The first significant result on this direction was the M. A. Lavrentiev theorem ([1]) where the problem on the product of conformal radii of two non-overlapping simply connected domains relative to the two predetermined fixed points was set and solved. For three fixed points and, accordingly, three non-overlapping domains, the similar problem was solved by G. M. Goluzin ([2]). Also in the monograph [2] the similar problem for n non-overlapping domains that contain, respectively, n given points, for an arbitrary natural $n \geqslant 2$, was considered and the following problem was posed.

Problem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, and $\{a_k \colon k \in \{1, 2, \dots, n\}\}$ be any set of fixed points of the complex plane. Find the maximum of the product*

$$\prod_{k=1}^{n} |f_k'(0)|\,, \tag{1}$$

2020 *Mathematics Subject Classification:* 30C75.
Keywords: the Green function; inner radius of the domain; star-shaped domain.
doi:10.30970/ms.60.2.138-144

where f_k $\big(k \in \{1, 2, \ldots, n\}\big)$ *are the functions regular in the disk* $|z| < 1$, *for which* $f_k(0) = a_k$, $f_i(z_1) \neq f_j(z_2)$, *for an arbitrary integer positive numbers* i, j *such that* $1 \leqslant i, j \leqslant n$, $i \neq j$, *and arbitrary different* $z_1, z_2 \in U$.

In the case of $n = 4$ the problem (1) was reduced to the smallest capacity problem in the certain continuum family and was solved in [3]. Currently for the case of $n \geqslant 5$ complete solution of this problem is unknown. Some estimates for the expression (1) for an arbitrary natural $n \geqslant 2$ was obtained in [4].

Among the possible applications of the obtained results in other tasks of the function theory are the so-called distortion theorems. The essence of these theorems lies in the study of restrictions imposed by the univalence of the mapping on some elements related to this mapping, in particular, on the quantity of the modulus and argument of the derivative of a given function, that is, on the quantity of distortion that makes this function at the selected points of the domain. One of the first results on this topic was obtained by T. H. Gronwall ([5]) so-called "square theorem". This result is the following: if $F(\zeta) = \zeta + \sum_{k=1}^{\infty} \frac{a_k}{\zeta^k}$ is a regular function, except the point $\zeta = \infty$ and univalent in the domain $|\zeta| > 1$, then the inequality $\sum_{k=1}^{\infty} n|a_n|^2 \leqslant 1$ holds.

For a function, regular and univalent in the disk $|z| < 1$, the following statement is valid (see [6]): if a function $f(z)$, regular and univalent in the disk $|z| < 1$, does not acquire in this disk some value c, then the inequality holds $|f'(0)| < 4|c|$. Then many different variants of distortion theorems were considered. Among recent research on this topic, we note, for example, works [7–9]. Also the tasks were considered where the distortion that performs the function was at several given points or with some weakening at the condition of univalence. Among the works on this direction, let us note, for example, [10–12].

In this work, we investigate the distortion that has the function of the class S^* in n points belonging to the open unit disk with the condition that no two points belong to one radius. We use proved estimates for products of inner radii of non-overlapping domains to obtain distortion theorems for univalent functions.

Let $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one point compactification, $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, U be the open unit disk centered at the origin, $\mathbb{R}^+ = (0, \infty)$.

Let $B \subset \overline{\mathbb{C}}$ be a simply connected domain and $a \in B$. According to the Riemann theorem on mapping, there exists a unique conformal mapping of the domain B onto the unit disk U at which $f(a) = 0 \in U$, $f'(a) \in \mathbb{R}^+$. Let us consider the inverse mapping φ which maps the unit disk U onto domain B such that $\varphi(0) = a$. Then

$$R(B, a) = \frac{1}{|f'(a)|} = |\varphi'(0)|$$

is called the conformal radius of the simply connected domain $B \subset \overline{\mathbb{C}}$ relative to the point $a \in B$. The conformal radius of the domain B with respect to the infinitely distant point is $R(B, \infty) = R(\varphi(B), 0)$, where $\varphi(z) = 1/z$.

A function $g_B(z, a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B \backslash \{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z, a) = -\ln|z - a| + \gamma + o(1),$$

(if $a = \infty$, then $g_B(z, \infty) = \ln|z| + \gamma + o(1)$,) is called the (classical) *Green function* of the domain B with pole at the point $a \in B$.

The *inner radius* $r(B,a)$ of the domain B with respect to the point a is the quantity e^{γ} ([2, 13, 14]).

Since the Green function is a conformal invariant, if a function f maps the domain B conformally and univalently onto a domain $f(B)$, then

$$r(B,a)|f'(a)| = r(f(B), f(a))$$

for each $a \in B$. The inner radius increases monotonically with the growth of the domain: if $B \subset B'$ then

$$r(B,a) \leqslant r(B',a), \quad a \in B.$$

It is known ([13]), that the following inequality holds

$$|f'(0)| \leqslant r(B,a).$$

An analytic function is said to be *univalent in a simply connected domain* if the images of distinct points are distinct there. A domain is called *star-shaped with respect to some point* if the segment connecting any of its points to the specified point belongs entirely to this domain. If the function $f(z)$ maps the disk $\{z\colon |z| < 1\}$ onto a star-shaped domain, then when moving z on $\{z\colon |z| = r\}$, $0 < r < 1$, in a certain direction $\arg f(z)$ changes in the same direction that is, the inequality holds

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

In [15] the following result was proved.

Theorem 1 ([15])**.** *Let $n \in \mathbb{N}$. Then for any fixed system of different points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any domains $\{B_k\}_{k=0}^{n}$ such that $a_k \in B_k \subset \overline{\mathbb{C}}$ $\big(k \in \{0, 1, \ldots, n\}\big)$, $a_0 = 0$, $B_i \cap B_j = \varnothing$ $(i \neq j)$ the following inequality holds*

$$r^n\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant n^{-\frac{n}{2}} \left(\prod_{k=1}^{n} |a_k|\right)^2. \tag{2}$$

Inequality (2) will be referred to as the *generalized Lavrentiev's inequality* because, for $n = 1$, we obtain the result of the paper [1].

Denote $a_{n+1} := 0$ and, respectively, $B_{n+1} := B_0$, the inequality (2) we can rewrite in the followimg form

$$r^n\left(B_{n+1}, a_{n+1}\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant n^{-\frac{n}{2}} \left(\prod_{k=1}^{n} |a_k|\right)^2. \tag{3}$$

For each fixed $p \in \{1, 2, \ldots, n\}$ consider the transformation $\omega^{(p)} = z - a_p$ to the system of points $\{a_k\}_{k=1}^{n+1}$ and domains $\{B_k\}_{k=1}^{n+1}$ in the inequality (3). Let $B_k^{(p)}$ and $a_k^{(p)}$ are images of the domain B_k and the point a_k , respectively, by this transformation. Since $a_p^{(p)} = 0$ and $a_k^{(p)} = a_k - a_p$, $k \neq p$, we can use the inequality (3) to the system of points $\{a_k^{(p)}\}_{k=1}^{n+1}$ and domains $\{B_k^{(p)}\}_{k=1}^{n+1}$ and obtain the next inequality

$$r^n\left(B_p^{(p)}, 0\right) \prod_{k=1,\, k\neq p}^{n+1} r\left(B_k^{(p)}, a_k - a_p\right) \leqslant n^{-\frac{n}{2}} \left(\prod_{k=1,\, k\neq p}^{n+1} |a_k - a_p|\right)^2.$$

Since the inner radius of domain is the invariant for a parallel transport then $r(B_p^{(p)},0) = r\left(B_p,a_p\right)$ and $r(B_k^{(p)},a_k-a_p) = r\left(B_k,a_k\right)$. Thus the following relationship holds

$$r^n(B_p,a_p)\prod_{k=1,\,k\neq p}^{n+1} r(B_k,a_k) \leqslant n^{-\frac{n}{2}}\left(\prod_{k=1,\,k\neq p}^{n+1}|a_k-a_p|\right)^2. \tag{4}$$

By multiplying $n+1$ inequalities of the form (4), we obtain

$$\left[\prod_{k=1}^{n+1} r\left(B_k,a_k\right)\right]^{2n} \leqslant n^{-\frac{n(n+1)}{2}}\left(\prod_{1\leqslant k<p\leqslant n+1}|a_k-a_p|\right)^4.$$

Taking the $2n$-th root of the left-hand and right-hand sides of this inequality, we have

$$\prod_{k=1}^{n+1} r\left(B_k,a_k\right) \leqslant n^{-\frac{n+1}{4}}\left(\prod_{1\leqslant k<p\leqslant n+1}|a_k-a_p|\right)^{\frac{2}{n}}.$$

Now, substituting $n+1$ by n and n by $n-1$, we obtain

$$\prod_{k=1}^{n} r\left(B_k,a_k\right) \leqslant (n-1)^{-\frac{n}{4}}\left(\prod_{1\leqslant k<p\leqslant n}|a_k-a_p|\right)^{\frac{2}{n-1}}. \tag{5}$$

The condition $a_k=0$ for some k can be relaxed because the inner radius of domain is the invariant for a parallel transport. In [4] inequality (5) was obtained by another method. In [16] the following relationship was established.

Theorem 2. *Let $n\in\mathbb{N}$, $n\geqslant 3$, a_k $\big(k\in\{1,2,\dots,n\}\big)$ be some fixed points of the complex plane and γ_k $\big(k\in\{1,2,\dots,n\}\big)$ be some positive real numbers such that $\gamma_k>\frac{1}{2n-2}\sum_{s=1}^{n}\gamma_s$ for all $k\in\{1,2,\dots,n\}$. Then for any set of domains $B_k\subset\overline{\mathbb{C}}$ such that $a_k\in B_k$ $(k\in\{1,2,\dots,n\})$, $B_i\cap B_j=\varnothing$ $(i\neq j)$ the following inequality holds*

$$\prod_{k=1}^{n} r^{\gamma_k}(B_k,a_k) \leqslant (n-1)^{-\frac{1}{4}\sum\limits_{k=1}^{n}\gamma_k}\prod_{i,j=1,i<j}^{n}|a_j-a_i|^{\frac{2}{n-2}\left(\gamma_i+\gamma_j-\frac{1}{n-1}\sum\limits_{k=1}^{n}\gamma_k\right)}. \tag{6}$$

2. Results. Denote by S^* the class of all univalent and regular functions $f(z)$ in the disk U, $f(0)=1-f'(0)=0$, which map U onto a star-shaped domain relative to the origin.

Let $\theta_k\in\mathbb{R}$, $k\in\{1,2,\dots,n\}$, besides $0\leqslant\theta_1<\theta_2<\ldots<\theta_n<2\pi$, $\theta_0:=\theta_n-2\pi$, $\theta_{n+1}:=\theta_1+2\pi$. For each $k\in\{1,2,\dots,n\}$ denote by m_k the least integer positive number for which

$$\frac{\pi}{m_k}\leqslant\min\left\{\frac{\theta_{k+1}-\theta_k}{2},\ \frac{\theta_k-\theta_{k-1}}{2}\right\}.$$

Theorem 3. *Let $n\in\mathbb{N}$, $\rho_k\in(0,1)$, $\rho_0:=\rho_n$, $\rho_{n+1}:=\rho_1$, $z_k=\rho_k\exp\{i\theta_k\}$, $k\in\{1,2,\dots,n\}$. Then for each function $f(z)\in S^*$ the following inequality holds*

$$\prod_{k=1}^{n}|f'(z_k)| \leqslant (n-1)^{-\frac{n}{4}}\prod_{k=1}^{n}\left(\frac{4\rho_k}{m_k}\cdot\frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}}\right)^{-1}\left(\prod_{1\leqslant p<k\leqslant n}|f(z_p)-f(z_k)|\right)^{\frac{2}{n-1}}. \tag{7}$$

Proof. Consider the set of sectors $\Delta_k = \{w \in U\colon \left|\arg w - \theta_k\right| < \frac{\pi}{m_k}\}$, $k \in \{1, 2, \dots, n\}$. It is clear that these sectors do not overlap. Then the domains $f(\Delta_k) =: B_k$, $f(z_k) \in B_k$, $k \in \{1, 2, \dots, n\}$, form a system of non-overlapping domains.

Let $z = \varphi_k(\zeta)$ be a univalent and conformal mapping U onto Δ_k, $k \in \{1, 2, \dots, n\}$, $\varphi_k(0) = z_k$, $\varphi_k'(0) > 0$. It is known ([17]), that equality

$$|\varphi_k'(0)| = \left(\frac{4\rho_k}{m_k} \cdot \frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}}\right) \tag{8}$$

is valid.

Then the functions $f(\varphi_k(\zeta))$ perform a univalent and conformal mapping of U onto B_k such that $f(\varphi_k(0)) = f(z_k)$ and

$$|f'(\varphi_k(0))| = |f'(z_k)| \cdot |\varphi_k'(0)| = r\left(B_k, f(z_k)\right). \tag{9}$$

Considering product in (9) for $k \in \{1, 2, \dots, n\}$ and taking inequalities (5), (8) into account, we get

$$\prod_{k=1}^{n} |f'(z_k)| \cdot \prod_{k=1}^{n} \left(\frac{4\rho_k}{m_k} \cdot \frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}}\right) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1 \leqslant p < k \leqslant n} |f(z_p) - f(z_k)|\right)^{\frac{2}{n-1}}.$$

By dividing both sides of the last inequality on the expression

$$\prod_{k=1}^{n} \left(\frac{4\rho_k}{m_k} \cdot \frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}}\right),$$

we obtain inequality (7). □

From Theorem 3, we obtain the following statement.

Corollary 1. *Let $n \in \mathbb{N}$, $\rho \in (0, 1)$, $z_k = \rho \exp\{i\frac{2\pi}{n}(k-1)\}$, $k \in \{1, 2, \dots, n+1\}$. Then for each function $f(z) \in S^*$ the following estimate is valid*

$$\prod_{k=1}^{n} |f'(z_k)| \leqslant (n-1)^{-\frac{n}{4}} \left(\frac{4\rho}{n} \cdot \frac{1-\rho^n}{1+\rho^n}\right)^{-n} \left(\prod_{1 \leqslant p < k \leqslant n} |f(z_p) - f(z_k)|\right)^{\frac{2}{n-1}}.$$

To prove Corollary 1, it is enough to put in equality (8) $\rho_k = \rho$ and $m_k = n$.

Theorem 4. *Let $n \in \mathbb{N}$, $\rho_k \in (0, 1)$, $\rho_0 := \rho_n$, $\rho_{n+1} := \rho_1$, $z_k = \rho_k \exp\{i\theta_k\}$, and γ_k be some positive real numbers such that $\gamma_k > \frac{1}{2n-2}\sum_{s=1}^{n}\gamma_s$ for all $k \in \{1, 2, \dots, n\}$. Then for each function $f(z) \in S^*$ the following inequality holds*

$$\prod_{k=1}^{n} |f'(z_k)|^{\gamma_k} \leqslant$$

$$\leqslant (n-1)^{-\frac{1}{4}\sum\limits_{k=1}^{n}\gamma_k} \prod_{k=1}^{n} \left(\frac{4\rho_k}{m_k} \cdot \frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}}\right)^{-\gamma_k} \prod_{i,j=1, i<j}^{n} |f(z_j) - f(z_i)|^{\frac{2}{n-2}\left(\gamma_i + \gamma_j - \frac{1}{n-1}\sum\limits_{k=1}^{n}\gamma_k\right)}. \tag{10}$$

Proof. The proof of Theorem 4 is based on constructions given above in Theorem 3. Similarly as in Theorem 3, consider the set of sectors Δ_k, domains $f(\Delta_k) =: B_k$ and functions $z = \varphi_k(\zeta)$, $k \in \{1, 2, \dots, n\}$. From inequalities (9), we get

$$\prod_{k=1}^{n} |f'(z_k)|^{\gamma_k} = \prod_{k=1}^{n} \left(\frac{r\left(B_k, f(z_k)\right)}{|\varphi_k'(0)|} \right)^{\gamma_k}. \tag{11}$$

Combining relationships (6), (8) and (11), we obtain

$$\prod_{k=1}^{n} |f'(z_k)|^{\gamma_k} \left(\frac{4\rho_k}{m_k} \frac{1-(\rho_k)^{m_k}}{1+(\rho_k)^{m_k}} \right)^{\gamma_k} \leqslant (n-1)^{-\frac{1}{4}\sum\limits_{k=1}^{n}\gamma_k} \prod_{i,j=1, i<j}^{n} |f(z_j) - f(z_i)|^{\frac{2}{n-2}\left(\gamma_i+\gamma_j-\frac{1}{n-1}\sum\limits_{k=1}^{n}\gamma_k\right)}.$$

Hence, we have inequality (10). □

If in equality (8) $\rho_k = \rho$ and $m_k = n$, then the following result is also true.

Corollary 2. *Let $n \in \mathbb{N}$, $\rho \in (0,1)$, $z_k = \rho \exp\{i\frac{2\pi}{n}(k-1)\}$, $k \in \{1, 2, \dots, n+1\}$. If γ_k ($k \in \{1, 2, \dots, n\}$), are some positive real numbers such that $\gamma_k > \frac{1}{2n-2}\sum_{s=1}^{n}\gamma_s$ for all $k \in \{1, 2, \dots, n\}$, then for each function $f(z) \in S^*$ the following estimate is valid*

$$\prod_{k=1}^{n} |f'(z_k)|^{\gamma_k} \leqslant (n-1)^{-\frac{1}{4}\sum\limits_{k=1}^{n}\gamma_k} \left(\frac{4\rho}{n} \frac{1-\rho^n}{1+\rho^n} \right)^{-\gamma_k n} \prod_{i,j=1, i<j}^{n} |f(z_j) - f(z_i)|^{\frac{2}{n-2}\left(\gamma_i+\gamma_j-\frac{1}{n-1}\sum\limits_{k=1}^{n}\gamma_k\right)}.$$

Acknowledgements. The authors thank the anonymous referee for very careful analysis of this work and remarks helped improve and clarify this manuscript.

REFERENCES

1. M.A. Lavrentiev, *On the theory of conformal mappings*, Tr. Sci. Inst. An. USSR, **5** (1934), 159–245.
2. G.M. Goluzin, Geometric theory of functions of a complex variable, Amer. Math. Soc. Providence, R.I., 1969.
3. G.V. Kuz'mina, *On the problem of the maximum of the conform radien's product of non-overlapping domains*, J. Soviet Math., **19** (1982), №6, 1715–1726.
4. A.K. Bakhtin, Y.V. Zabolotnii, *Estimates of the products of inner radii for multiconnected domains*, Ukr. Mat. Zhurnal, **73** (2021), №1, 9–22.
5. T.H. Gronwall, *Some remarks on conformal representation*, Annals Math., **16** (1914–1915), №1, 72–76.
6. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S. B. Preuss. Acad. Wiss., **138** (1916), 940–955.
7. O. Roth, *A distortion theorem for bounded univalent functions*, Ann. Acad. Sci. Fenn. Math., **27** (2002), 257–272.
8. V.N. Dubinin, V.Y. Kim, *Distortion theorems for functions regular and bounded in the disk*, J. Math. Sci., **150** (2008), 2018–2026.
9. V.N. Dubinin, *Distortion theorem for complex polynomials*, Sib. Elektron. Mat. Izv., **15** (2018), 1410–1415.
10. Seong-A. Kim, D. Minda, *Two-point distortion theorems for univalent functions*, Pacific J. Math., **163** (1994), №1, 137–157.

11. J.A. Jenkins, *On two-point distortion theorems for bounded univalent regular functions*, Kodai. Math. J., **24** (2001), №3, 329–338.
12. S.I. Kalmykov, *About multipoint distortion theorems for rational functions*, Siberian Math. J., **61** (2020), №1, 85–94.
13. W.K. Hayman, Multivalent functions, Cambridge Univ. Press, 1994.
14. A.K. Bakhtin, G.P. Bakhtina, Yu.B. Zelinskii, Topological-algebraic structures and geometric methods in complex analysis, Zb. prats of the Inst. of Math. of NASU, 2008.
15. A.K. Bakhtin, I.V. Denega, *Generalized M.A. Lavrentiev's inequality*, J. Math. Sci., **262** (2022), №2, 138–153.
16. A.K. Bakhtin, Y.V. Zabolotnii, *Estimates of products of some powers of inner radii for multiconnected domains*, Ukr. Mat. Zhurn., **73** (2021), №9, 1155–1169.
17. E.G. Emelyanov, *On the connection of two problems on extremal partitioning*, Zap. Nauchn. Sem. LOMI. **160** (1987), 91–98.

Institute of Mathematics NAS of Ukraine
Department of complex analysis and potential theory
Kyiv, Ukraine
iradenega@gmail.com
yaroslavzabolotnii@gmail.com

Received 01.06.2023
Revised 18.10.2023